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Max/Min : a novel rank order based filter family

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Abstract

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In the early development of signal and image processing, linear filters were the primary tools. Their mathematical simplicity made them easy to design and implement. They also offered satisfactory performance in many applications. However, they show poor performance in other cases, such as image processing.

Nonlinear filtering offered a solution to these problems. Nonlinear filtering has undergone a lot development for a long time. Among the first known nonlinear filters was the median filter, which was proposed by Tukey. Median filter preserves signal or image details better compared to a linear filter. Later other nonlinear filters were developed, including median based filters, rank-order based filter family etc.

A new filter family the Max/Min filter is proposed here. This was studied in this thesis and was found to offer better filtering compared to the classical median filter together with a reduction in processing time. The filter family has interesting operational properties, these were also investigated. Some of these properties are unique to the family while others are typical for rank order based filters. The statistical and the spectral properties for the filter output were studied with white noise input. A number of modifications on the basic filter are also suggested.

2) Max/Min: A Novel Rank Order Based Filter Family

by

1) Raafat E. Kamel

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To my Parents,
my Brother and Sister and their Families.

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Chapter 1

INTRODUCTION

1.1 Overview

Whenever a signal, such as speech, image or other data, is transmitted over a communication medium it undergoes some changes. Some of these are unintentional, like white noise added from the receiver's or the transmitter's electronics. Other types of changes may be due to the scheme used in the pre-transmission stages *e.g.* modulation, compression *etc.*

Whatever the cause or the source of these alterations might be, the ultimate goal of the reception is to recover the original signal. It is these kinds of problems that gave rise to what is now known, in the area of signal processing, as signal recovery.

Among the first tools used for signal recovery was the linear filter. Linear filters were known for a long time and have a rich literature. Linear systems are characterized by the superposition property, which facilitated the study of their performance. With the help of the superposition principle the filter's response to a signal corrupted with additive noise can be determined by considering the filter's response to the signal and noise, each applied separately. The concept of system transfer function was also developed for linear filters. The system transfer function determines the system's response to different types of excitations. Application of system transfer function greatly simplifies sysytem analysis.

Nevertheless, linear filters were found to be optimal for some but not all applications. When applied to spiky noise removal in images, which contain complex signal structures such as edges, lines *etc*, linear filters blurred the edges, removed the lines and reduced the image sharpness and clarity.

In the early seventies Tukey [1] suggested median processing as a scheme for smoothing out noise while preserving important signal details like edges. The nonlinear behavior of the median filter attracted many scholars and has been under a lot of research since its early days [2-4]. The research on the median filter led to the development of new types of nonlinear filters, typical examples being the rank-order based filters. These filters proved to be useful in the task of noise suppression and signal recovery.

A new type of rank-order based filter family is introduced in chapter 2. This shares some of the properties with other rank order based filter. These properties together with others, which are unique to the new filter family, are given in chapter 3. Chapter 4 gives the statistical properties of the new filter family. A model of the new filter family is given in chapter 5 together with some modifications on the filter family. The conclusions are given in chapter 6.

Section 1.2 discusses the classical median filter as proposed by Tukey and some of its properties and applications. Section 1.3 outlines some of the median-based filters and section 1.4 discusses the more general rank-order based filter family of which the classical median filter is a member.

1.2 The Median Filter

The median filter, by definition, outputs the statistical median of a sequence of input samples inside a window or mask. The length of this sequence is known as the window size of the filter. If the input sequence is denoted by $\{\cdots x_{i-1}, x_i, x_{i+1}, \cdots\}$ and the output sequence by $\{\cdots y_{i-1}, y_i, y_{i+1}, \cdots\}$, then for a window size $2N+1$ median filter the output sample y_i is given by

$$y_i = \text{median}\,(x_{i-N}, \cdots, x_i, \cdots, x_{i+N})$$

The median filter, as defined above has been studied extensively [2-4] and most of the filter's properties have been investigated.

To define the properies of the median filter Thomas A. Nodes *et al* [4] defined the following signal structures:

- A constant neighborhood is a region of at least $N+1$ consecutive points, all of which are identically valued.

- An edge is a monotonically rising or falling set of points surrounded on both sides by constant neighborhoods.

- An impulse is a set of N or less points surrounding regions and whose surrounding regions are identically valued constant neighborhoods.

- A root is a signal which is not modified by filtering.

When used for filtering, the median filter was found to suppress impulse noise while preserving the signal's edges. Fig. 1.1 shows a noisy signal, the output of a size 15 moving average filter and the output of a size 15 median filter. It is clear from the figures that the noise suppression of the moving average is more than that of the median filter. However, the moving average filter also smeared the edge of the signal while the median filter preserved it.

Repetitive application of the median filter smooths the noise further till an invariant signal, the root signal, is produced. Depending on the input signal,

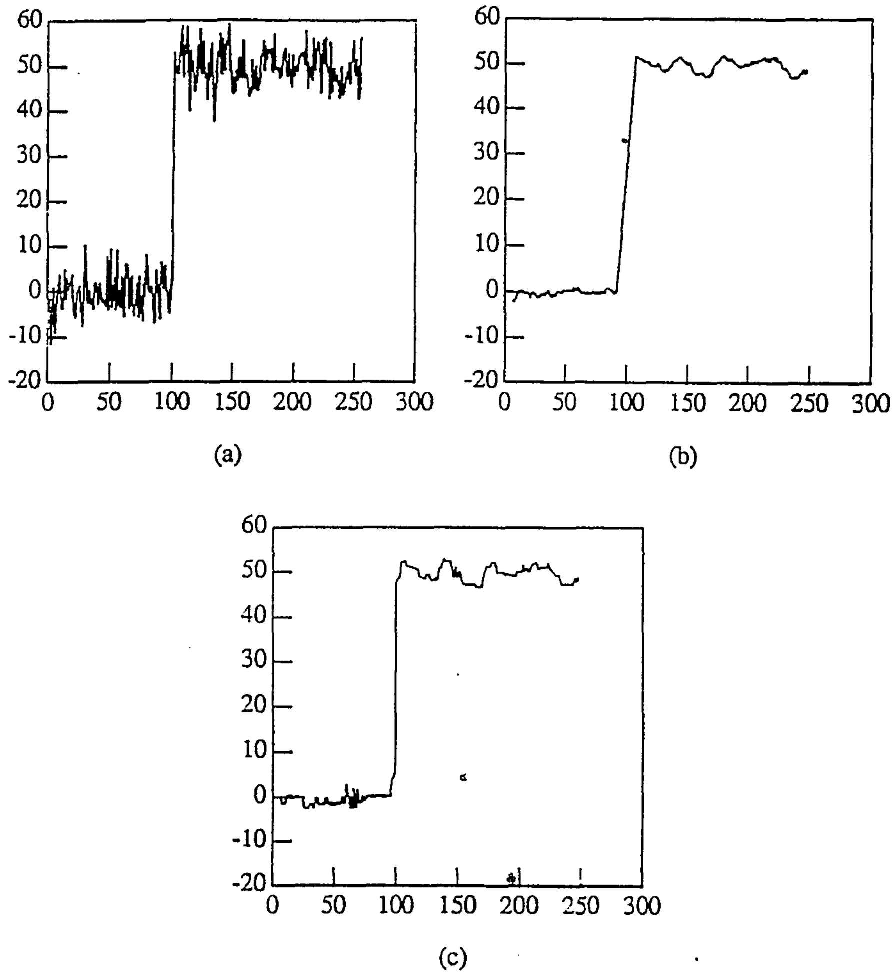

Figure 1.1: (a) A noisy signal, (b) the output of a size 15 moving average filter and (c) the output of a size 15 median filter [9]

different root signals are produced by the given median filter. As a matter of fact, the operation of the median filter can be looked at as the process that maps the input signal into the respective root signal. The structure of these signals and the rate at which the filter output signal converge to them are given in [5,6].

Among the tools which were developed for the analysis and study of median filtering is the threshold decomposition. This is used to decompose an m-ary signal into a set of binary sequences. Filtering these sequences and then reversing the decomposition is equivalent to filtering the original signal [7]. The usefulness of

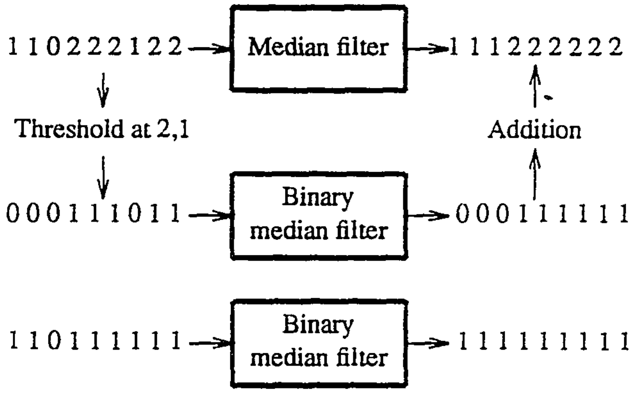

Figure 1.2: Threshold Decomposition [9].

the threshold decomposition lies in that the computation of the median for these binary sequence reduces to counting the number of 1's and comparing that to the window size. Fig. 1.2 displays this technique. Threshold decomposition was also used to study the spectral properties of the output of the median filter [8]. Further discussion of the threshold decomposition is given in appendix B.

The statistical properties of the median filter are given in [3,9]. It was found that in constant neighborhood regions, median filters have low pass characteristics.

The classical median filter, whose properties were briefly outlined above, found a number of useful applications. Rabiner *et al* used it for smoothing noise out of speech [11]. Median filtering was also used in image processing [12] and data editing [13].

Vigorous research led to the development many of new nonlinear filter structures. Stating the names and definitions of these is not feasible in this short introduction. Section 1.3 will give some typical examples.

1.3 Median Based Filters

A typical example of a median based filter is the recursive median filter, in which the output of previous samples are fedback into the current window. Using the above notation, the output y_i of the recursive median filter is given by,

$$y_i = \text{median}\,(y_{i-N}, y_{i-N-1}, \cdots, y_{i-1}, x_i, x_{i+1}, \cdots, x_{i+N})$$

The recursive median filter achieves a greater degree of noise smoothing compared to the classical one. It also generates its root signal after a single pass [4].

Median-based filtering was also applied to multi-dimensional applications such as image processing. An example of this is the separable median filter [14]. This is a two dimensional filter which uses one dimensional median filter along the rows of the image and then along its columns. The separable median filter is not identical to the two dimensional non-separable median filter [12], but gives comparable results in the smoothing of spiky image noise and edge preserving. On the other hand the separable median filter is simpler from the point of view of computation and implemention.

A third example of median-based filter is the selective median filter which was proposed by S.J. Ko *et al* [15]. This filter uses two internal windows beside the traditional one. One of these windows is oriented to the left of the output sample and the other to the right of it. The selective median filter computes the medians of the samples in each of the two windows and compares these with the mid sample. It then outputs the median with the smaller absolute difference from the mid sample. Using the above notation, the output sample y_i is given by

$$y_i = \begin{cases} M_1(N) & \text{if } |M_1(N) - x_i| \leq |M_2(N) - x_i| \\ M_2(N) & \text{otherwise} \end{cases}$$

where, $M_1(N) = \text{median}\,(x_{i-N}, \cdots, x_{i-1})$ and, $M_2(N) = \text{median}\,(x_{i+1}, \cdots, x_{i+N})$ with N being an odd integer.

The selective median filter was found to be computational economical compared to the classical median filter. In addition it is more effective in enhancing edges and suppressing noise. The selective median filter generates its root signal after a number of passes which is more than that required by the median filter.

Other examples of nonlinear filters that use the median operator include FIR-median hybrid filter, multistage median filters [16] and max/median filters [17]. Median filters and median-based filters are typical examples of rank-order based filters. These use rank order operators. Section 1.4 introduces other types of rank-order based filters.

1.4 Rank-Order Based Filters

There are different types of nonlinear filters incorporating rank-order operations. Only a few of these are discussed here. These filters have proved to be very useful in the task of noise suppression and detail preservation.

The L Filter

An L-filter [18] uses a symmetrical window about the output sample position and sorts the contents of the window. The output of this filter is the linear combination of the rank-ordered samples in the window.

Consider the input samples $\{x_{i-N}, x_{i-N+1}, \cdots, x_i, \cdots, x_{i+N-1}, x_{i+N}\}$. Denote the ordered samples by $x_{(j)}, j = 1, \cdots, 2N+1$ where $2N+1$ is the window size. The output y_i of the L filter is then given by

$$y_i = \sum_{j=1}^{2N+1} a_j x_{(j)}$$

where the coefficients a_j are functions of sample's rank. These coefficients are optimized for a given noise statistics according to a given criterion, minimum mean square error for example.

The median filter and the $\alpha-$trimmed mean [19] are special cases of the L filter with,

$$a_j = \begin{cases} 1 & \text{j=N+1} \\ 0 & \text{otherwise} \end{cases}$$

for the median filter and

$$a_j = \begin{cases} \frac{1}{2N+1}(1-2\alpha) & j = (2N+1)\alpha + 1, \cdots, (2N+1)(1-\alpha) \\ 0 & \text{otherwise} \end{cases}$$

for the $\alpha-$trimmed mean. For the $\alpha-$trimmed mean filter, the value of α must satisfy $0 \leq \alpha < 0.5$ with $(2N+1)\alpha$ being an integer [18].

The C filter

The L and other median-based filters have a drawback. They destroy the time-order information of the signal through their ordering procedure. For constant level signals time-ordering is not vital, but for non-constant signals this is not the case [20]. This drawback becomes more noticeable as the window size increases, as a result the performance of the L filter deteriorates.

To rectify the time-ordering problem the combination filter (C filter) was developed [21]. The output of the C filter is also given by a linear combination of order statistics, but the filter coefficients are now functions of both the rank and time index of the sample. This is represented by,

$$y_i = \sum_{j=i-N}^{i+N} a\left(R(x_j), j\right) x_j$$

where $\{x_{i-N}, \cdots, x_i, \cdots, x_{i+N}\}$ are the input data samples in the window, $R(x_i)$ and i are the rank and the time index of x_i sample respectively. The C filter can thus be looked to as a time varying L filter.

The Stack Filter

The threshold decomposition property for median filter was discussed in section 1.2 and appendix B. It was pointed out that the process of median filtering binary sequences generated from an m-ary level signal, by threshold decomposition, and then reversing the decomposition is equivalent to median filtering the original signal.

If the filtered binary sequences are piled at any instance of time, each pile will consist of a column of 1's having a column of 0's on its top, see fig. 1.2. The above property is known as the stacking property. These properties, threshold decomposition and stacking, favored the VLSI implementation of the median filter and other rank-order based filters in general [23]. They also gave rise to a new type of nonlinear filters known as stack filters [22]. Rank-order based filters are special cases of stack filters.

A close examination of the median operation on binary sequences will reveal that it can be represented by a boolean function. For example, consider determining the median y_i of three binary samples x_{i-1}, x_i and x_{i+1}. This can be represented by the following boolean equation

$$y_i = x_{i-1} \cdot x_i + x_i \cdot x_{i+1} + x_{i+1} \cdot x_{i-1}$$

where $\cdot$ and $+$ are the AND and OR boolean operations. A number of operators, for the binary input case, can be implemented using boolean logic.

The stack filters are implemented by using a threshold decomposition stage, a boolean function, which characterizes the filter, and a stacking stage. Since the threshold decomposition and stacking stages are common for all stack filters, the design of a stack filter reduces to determining the appropriate boolean function. For example, to implement a filter that outputs the maximum of three input samples, the following boolean equation may be used,

$$y_i = x_{i-1} + x_i + x_{i+1}$$

Through the use of boolean operations new filters were realized. The stack filter is still under active research and a number of developments are taking place to optimize it [24,25].

The next chapter will discuss a new type of rank-order based filter family, Max/Min filters.

Chapter 2

MAX/MIN FILTER FAMILY

2.1 Introduction

The Max/Min filter family is a rank order based family of filters [26]. These use an asymmetric window about the output sample, as opposed to the symmetric window traditionally used. The Max filter outputs the larger of the two samples positioned at each end of the window, while the Min outputs the smaller.

Depending on the direction to which the window is skewed relative to the input sample, two types of Max and Min filters are defined. If the input sequence to the filter by is denoted $\{\cdots, x_{i-1}, x_i, x_{i+1}, \cdots\}$, the output of right skewed window filter by $\{\cdots, y_{i-1}, y_i, y_{i+1}, \cdots\}$ and the output of left skewed window filter by $\{\cdots, z_{i-1}, z_i, z_{i+1}, \cdots\}$.

For Max filter with the right skewed window, y_i is related to x_i by

$$y_i = \max(x_i, x_{i+n}) \tag{2.1}$$

where n, known as the filter parameter, is the window size less one.

For the corresponding Min filter, y_i is given by

$$y_i = \min(x_i, x_{i+n}) \tag{2.2}$$

The right skewed Max filter is similar to the left skewed one other than for some delay. Consider a left skewed size $n+1$ Max filter, where the output z_i is given by

$$\begin{aligned} z_i &= \max(x_i, x_{i-n}) \\ &= \max(x_{i-n}, x_i) \\ &= y_{i-n} \end{aligned}$$

The left skewed and the right skewed Max filters differ only on where the output is taken relative to the window. For the left skewed filter the output is taken to the right end of the window, while for the right skewed one it is the opposite. The above equivalence is also true for the two types of the Min filter.

As a result of the above equivalence, the discussion will address only one of these and the right skewed types are chosen.

2.2 Max/Min Filter Characteristics

The signal shown in fig. 2.1 was processed using a size 2 Max filter. Figs. 2.2, 2.3 and 2.4 show the output of 2, 4 and 7 passes of the filter respectively, superimposed with the original noise free signal. Comparing fig. 2.1 with 2.2, 2.3 and 2.4 reveals the following characteristics of the Max filter:

(i) Noise smoothing: the Max filter smoothed the noise in the signal substantially.

(ii) Edge preserving: examining figs. 2.2, 2.3 and 2.4 shows that the Max filter, like other rank order based filters, preserves edges.

(iii) Edge shifting: the Max filter also shifted the rising edges of the signal to the left, opposite to the direction of the asymmetry, and kept the falling edge intact.

(iv) Repetitive processing of the Max filter smooths the noise more and introduces more edge shifting. The amount of shifting per sample is proportional to the number of passes.

Fig. 2.5 shows the output of 7 passes of a size 2 Min filter. The noise smoothing and edge preserving characteristics of the Min filter are clearly demonstrated. However, notice the Min filter shifted the falling edge while preserved the rising in position.

To study the relationship between edge shifting and window size for fixed number of passes, the test waveform shown in fig. 2.6 is used. Figs. 2.7 and 2.8 show the outputs of single pass size 5 and size 7 Max filters respectively. The amount of edge shifting is proportional to the window size. Thus one can conclude that the shifting of the rising edge produced by the Max filter is proportional to the number of passes and the window size.

In order to achieve greater amount of noise smoothing more passes and/or larger window sizes of Max filters are required. This in turn produces more edge shifting and after finite number of passes the edges merge together, fig. 2.9. Further processing will wipe out the signal. This tends to limit the use of the filter. Next sections will discuss possible applications of the filter family.

2.3 Edge Preserving and Noise Smoothing

A possible way to compensate for the above effect is to cascade equal numbers of Max and Min filters. The former shifts the rising edge while the latter shifts the falling, their cascade will shift both of them by equal amounts, thus finally preserving the pulse duration.

Fig. 2.10 shows a signal corrupted with noise, the signal to noise ratio of which is 0 dB, superimposed with the original signal. Use of the above arrangement of Max and Min filter with 25 passes of each, smooths the noise and restores the shape of the signal, fig. 2.11. Fig. 2.12 shows the output the classical median filter with window size 15, the performance of the Max/Min filter is apparently superior.

The Max/Min filter also requires fewer number of computations. A size N median filter requires $N \log_2 N$ [27] comparisons per output sample, while the Max/Min requires one comparison. In the above case, fig. 2.11, 25 passes of Min filter followed by 25 passes of Max filter require 50 comparisons per output sample, while 3 passes of the size 15 median filter requires 180 comparisons. Thus the above arrangement of Max/Min requires less than 30% of the compaisons nedded by the median filter. The average signal level at the output of the Max/Min filter is higher than that of the input fig. 2.11. This is because Max filter tends to propagate the value of the local maximum sample over its neighborhood, thereby increasing the average signal level. If Min filter was used in the above arrangement prior to the Max, the average signal level at the output would drop below that of the input for a similar reason.

2.4 Max/Min based edge detector

The edge response of the Max and Min filters can be utilized in edge detection. It was pointed out earlier that the Max filter preserves the rising edge while the Min filter preserves the falling one. Both of them can be used in a scheme to detect the falling and rising edges. The figure below shows the block diagram of a possible scheme. The scheme consists of three stages: the pre-filtering stage, the edge-conversion and enhancement stage and finally the differentiation and thresholding stage. The first stage utilizes two branches, the upper branch contains a combination of Max filters, 25 in cascade, which preserves the rising edge, while the lower has a cascade of 25 Min filters whose purpose is to preserve the falling edge. Each branch is terminated by a median filter, this is used for further smoothing.

The second stage consists of a subtracter, a median filter and a Max filter. The subtracter converts the preserved rising and falling edges from the prior stage into rising edges. The median filter is installed after the subtracter for further smoothing. Finally a Max filter is used to enhance the rising edges further and smooth out any noise left by prior stages.

The output of the second stage is then fed into a differentiation and a threshold stage. Fig. 2.14 shows a signal corrupted with noise, the SNR of which is 7dB. The edges generated by the scheme in fig. 2.13 and the original signal are shown in fig. 2.15.

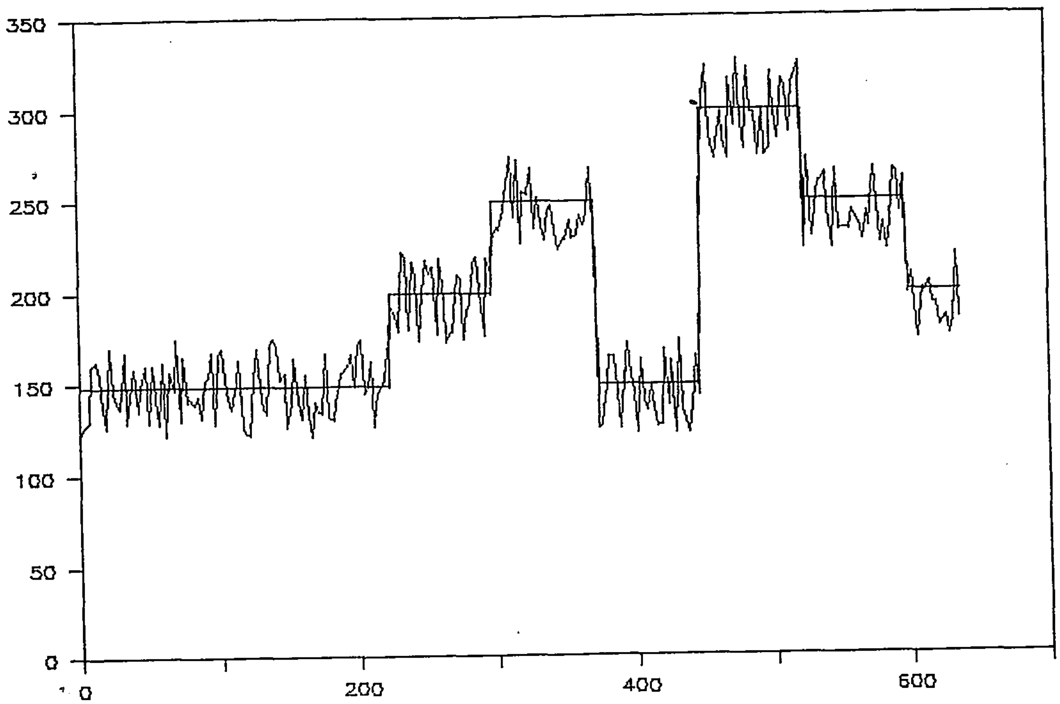

Figure 2.1: A input signal, SNR 7 dB, superimposed with the noise free signal.

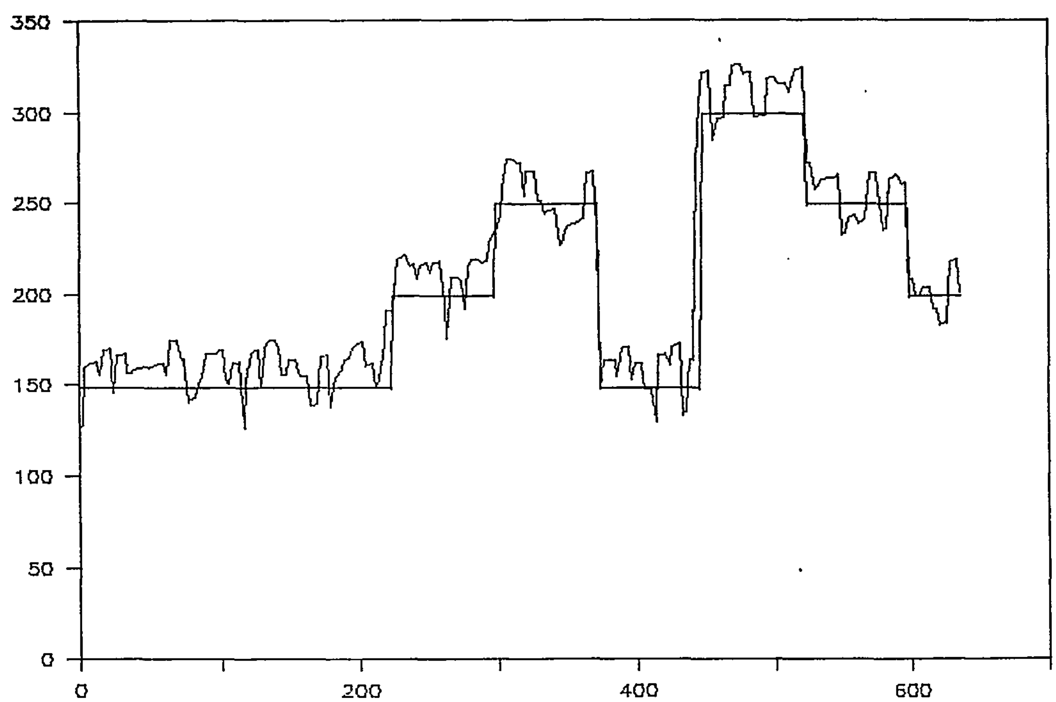

Figure 2.2: The output of 2 passes of a size 2 Max filter.

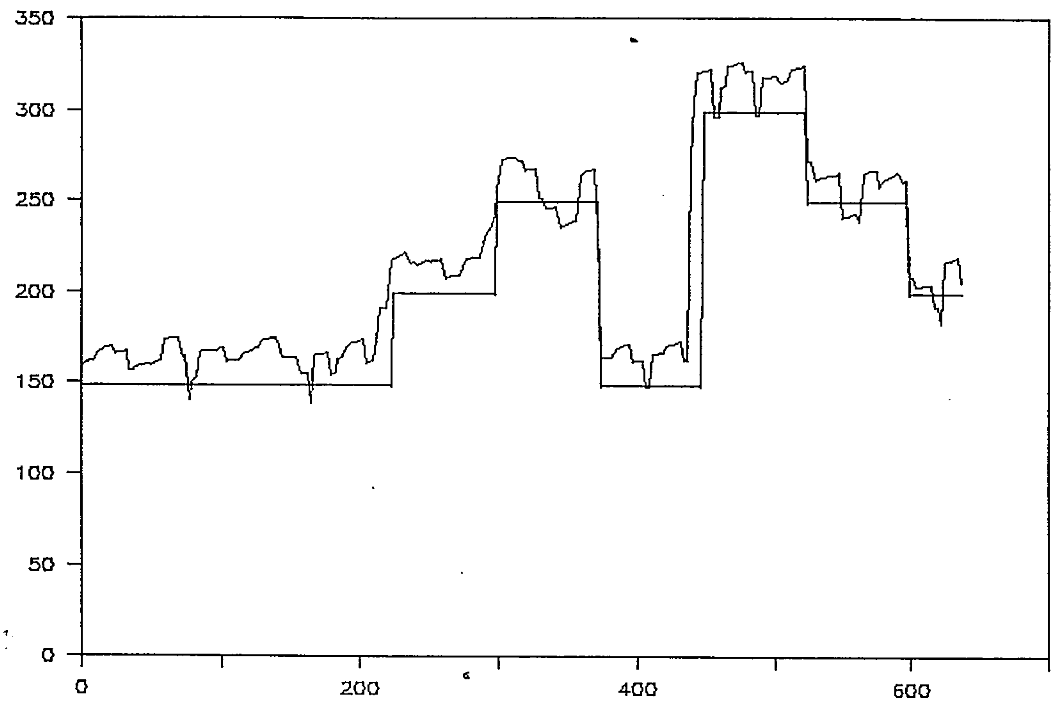

Figure 2.3: The output of 4 passes of a size 2 Max filter.

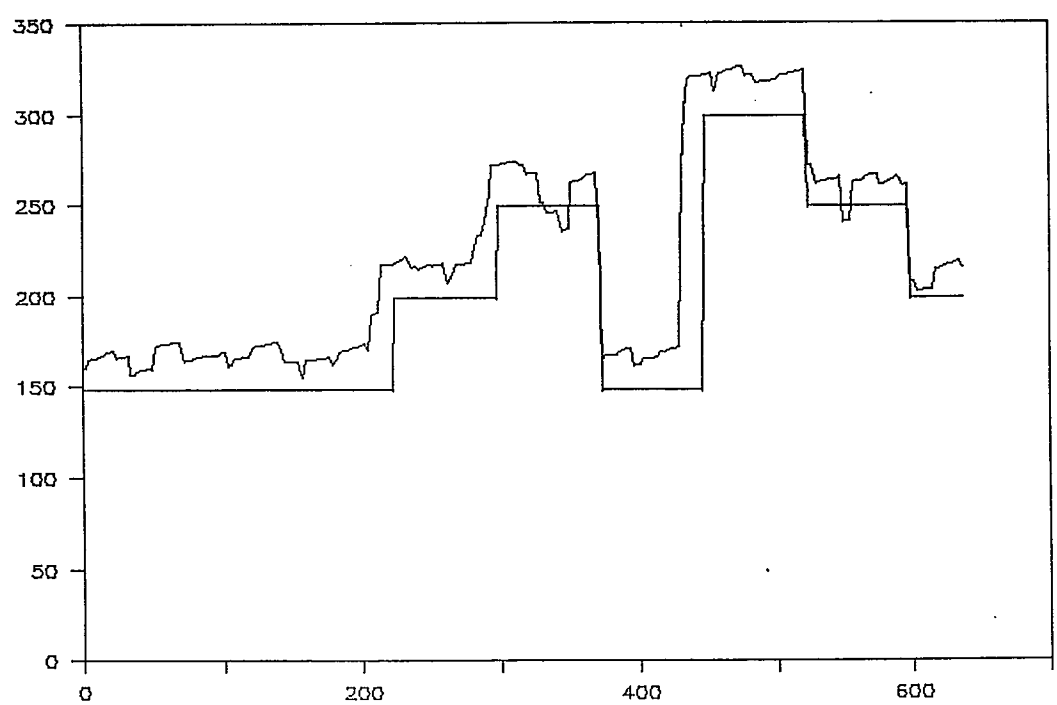

Figure 2.4: The output of 7 passes of a size 2 Max filter.

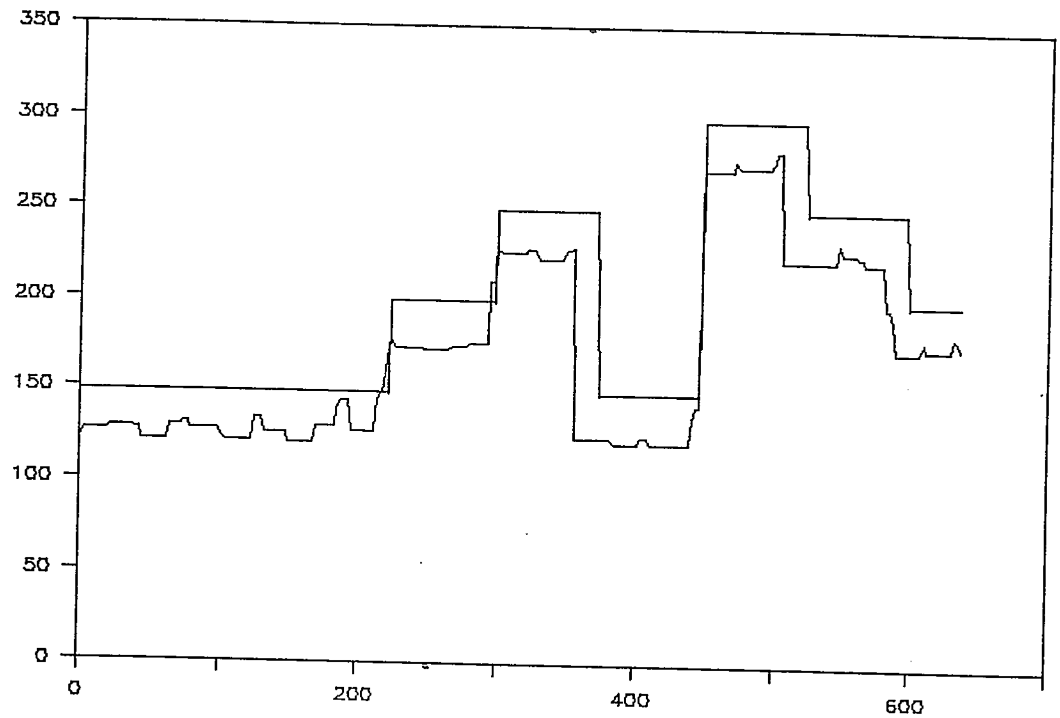

Figure 2.5: The output of 7 passes of a size 2 Min filter.

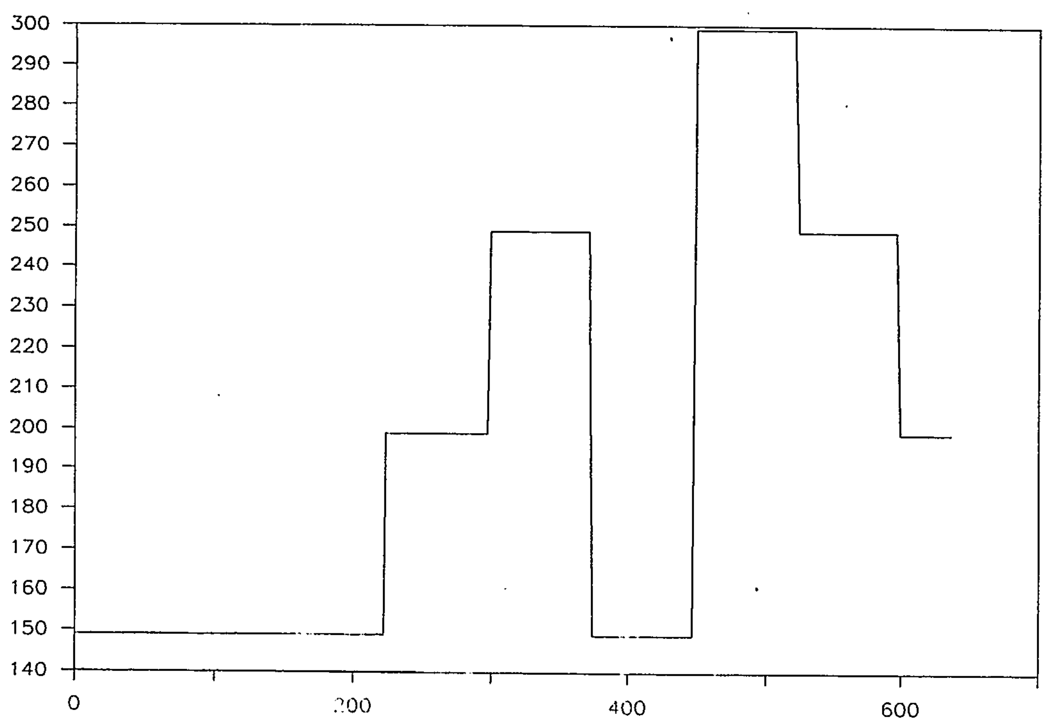

Figure 2.6: A noise free signal to study the edge shifting effect.

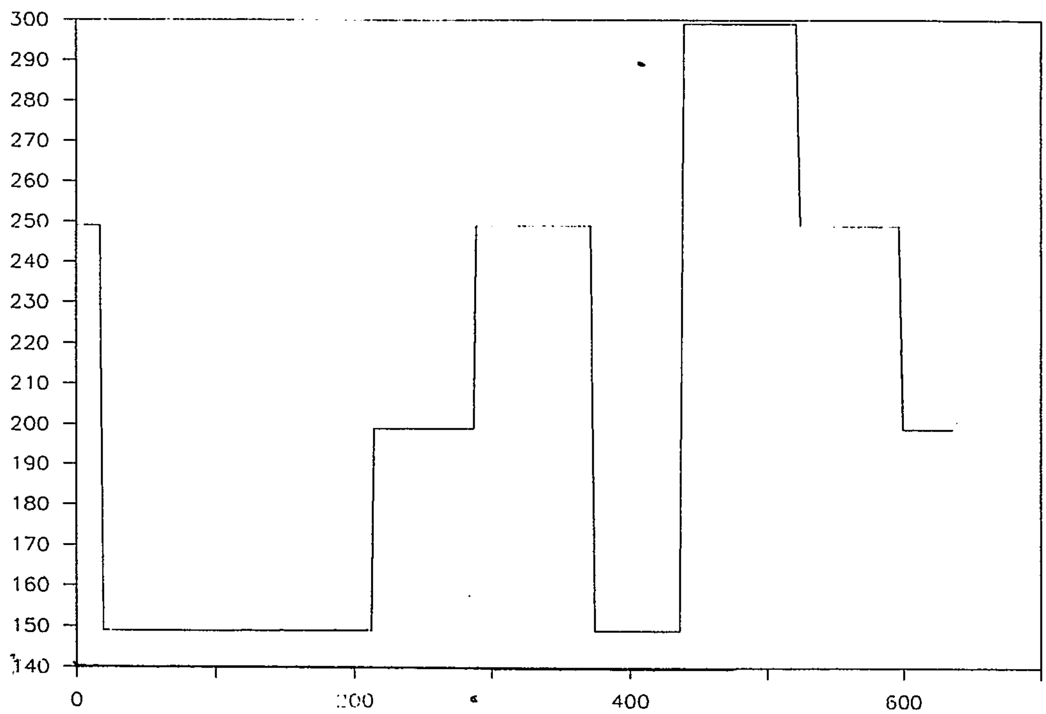

Figure 2.7: The output of a single pass of size 5 Max filter.

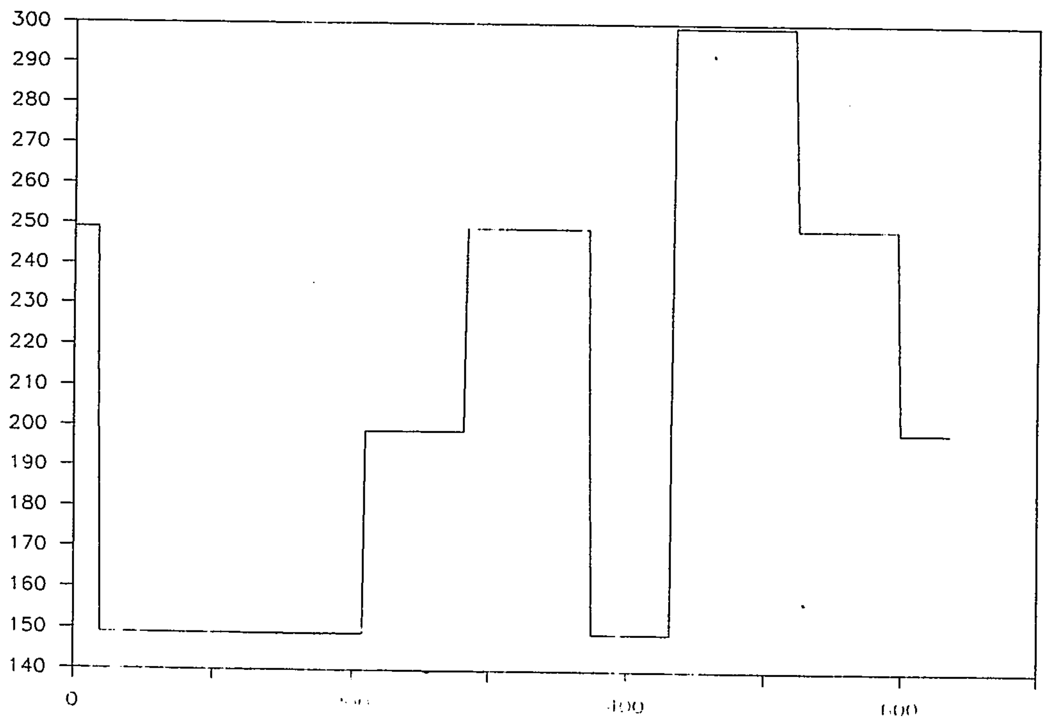

Figure 2.8: The output of a single pass of size 7 Max filter.

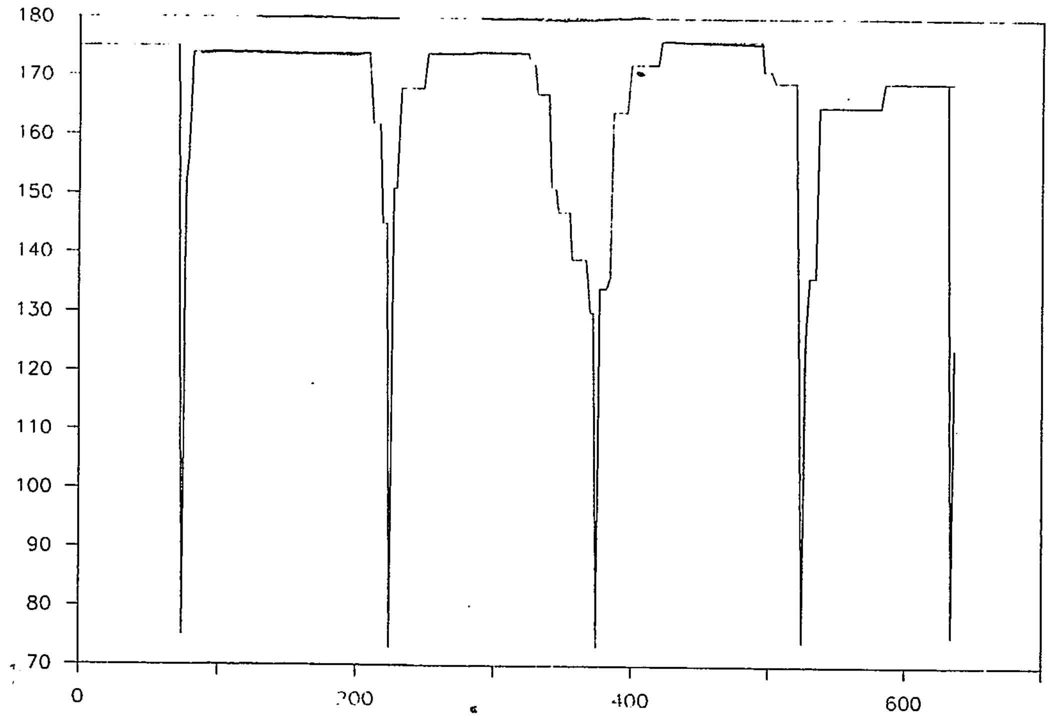

Figure 2.9: Edges merging after successive application of Max filter.

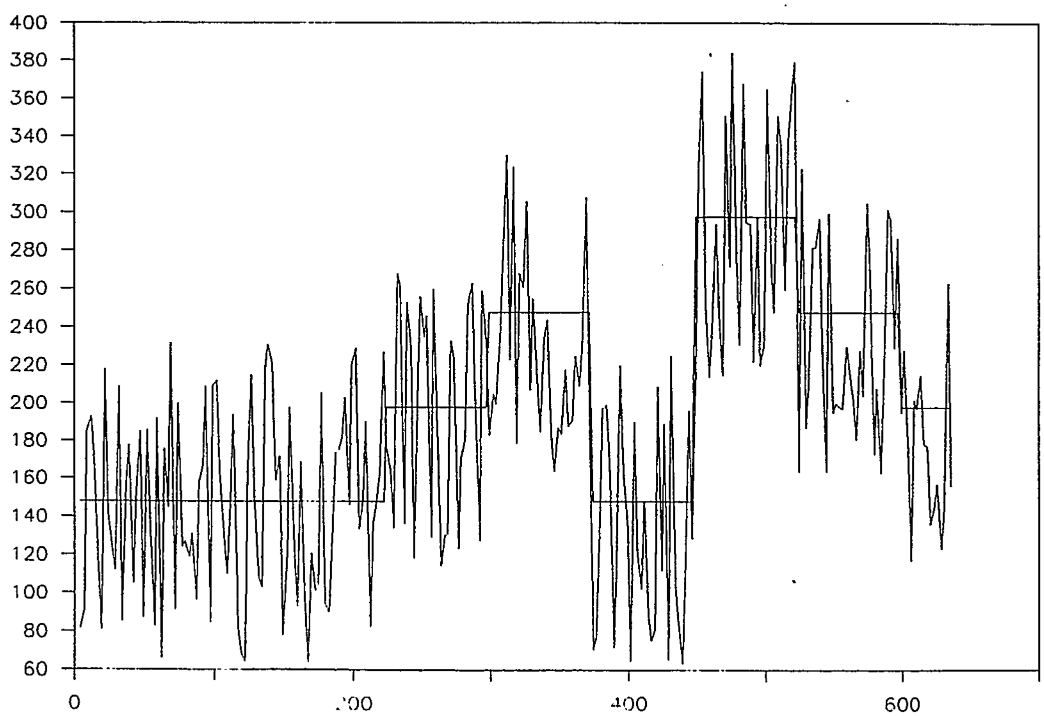

Figure 2.10: A signal corrupted with noise with SNR of 0 dB.

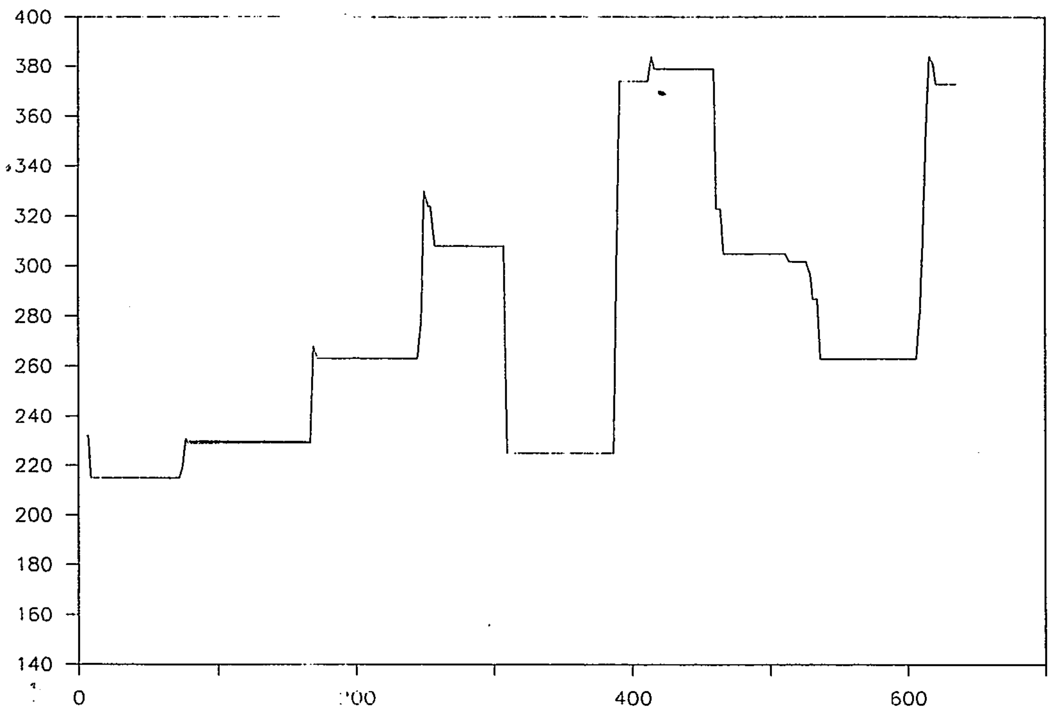

Figure 2.11: The output of the Max/Min filters arrangement.

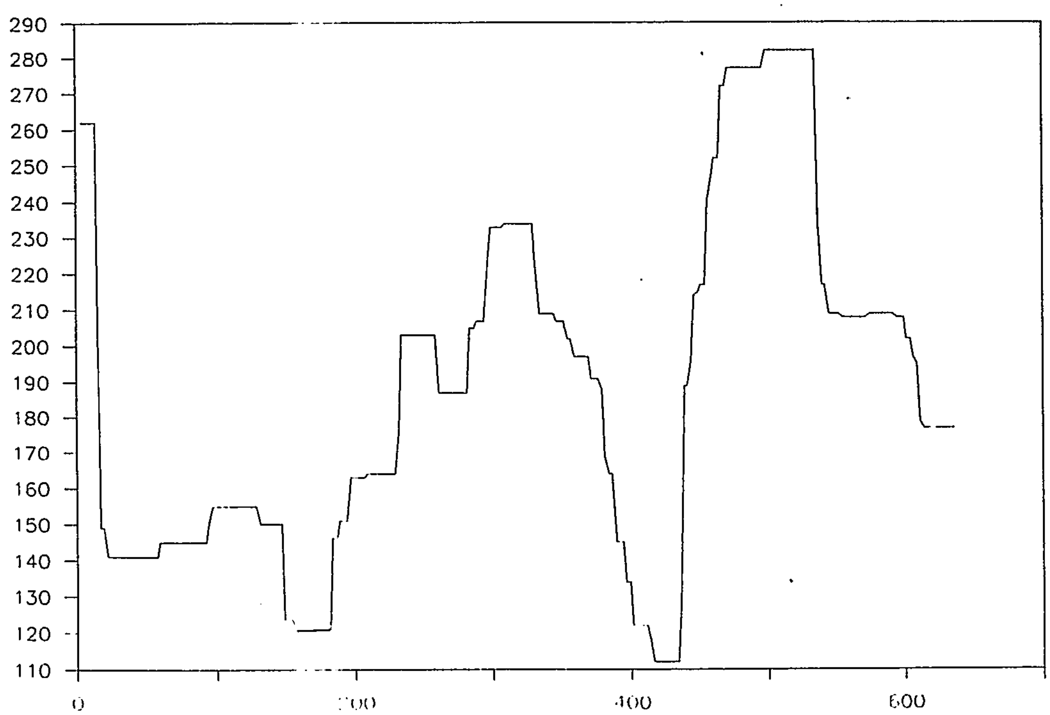

Figure 2.12: The output of 3 passes of a size 15 median filter.

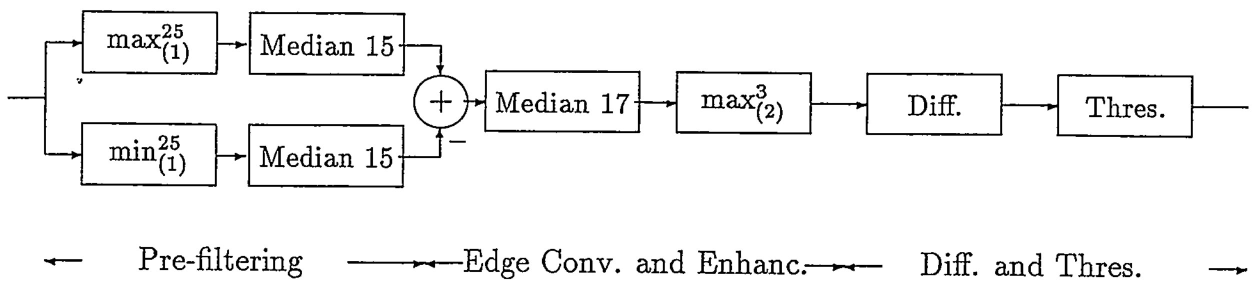

Figure 2.13: An edge detector.

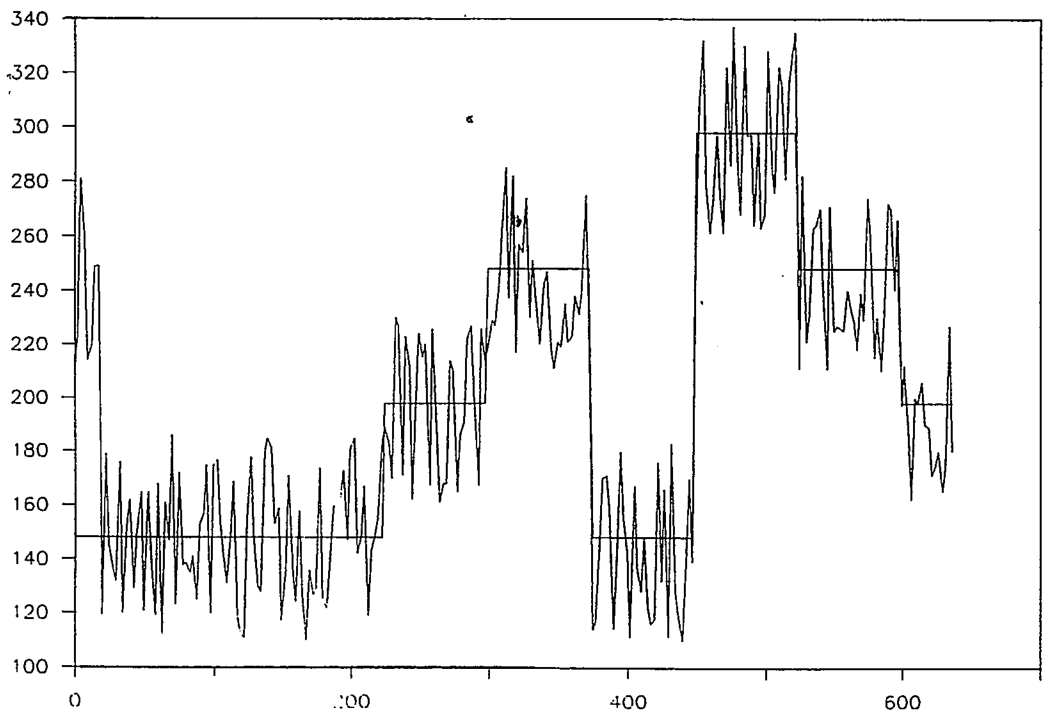

Figure 2.14: A signal corrupted with noise with SNR of 7 dB.

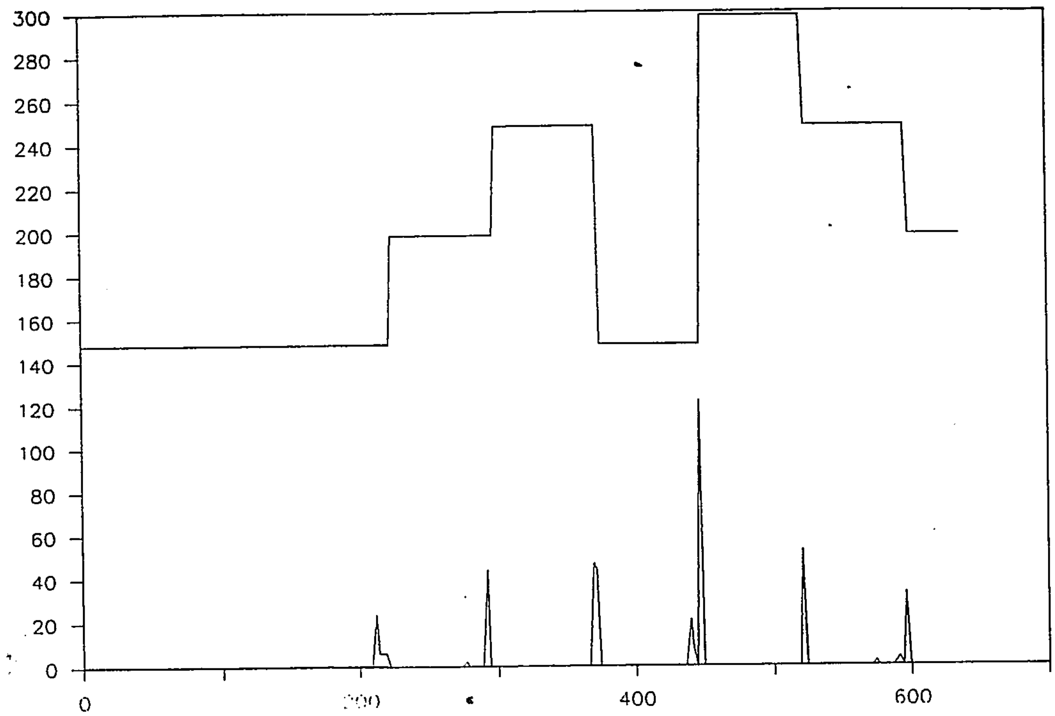

Figure 2.15: Output of the edge detector.

Chapter 3

OPERATIONAL PROPERTIES

3.1 Introduction

The previous chapter introduced the Max/Min filter family and some of its properties *viz*, noise reduction, edge preserving and shifting properties. The Max/Min filter family also has interesting operational properties [28] some of which are unique to the family. These properties are useful for system analysis and for understanding the mechanism of the filters. System implementation with Max/Min filters can be also facilitated with these properties.

Before stating these properties and providing their proofs, it is while worth to introduce the notation that was developed to present and prove these properties. Section 3.2 introduces this notation and draws the equivalence between Max and Min filters and section 3.3 discusses the properties.

3.2 Definition and Notation

A data sequence or time series is denoted by X and Y, *etc.* Specifically, let

$$X \triangleq \{x_i\} = \{\cdots, x_{i-1}, x_i, x_{i+1}, \cdots\}$$

$$Y \triangleq \{y_i\} = \{\cdots, y_{i-1}, y_i, y_{i+1}, \cdots\}$$

A Max filter of window size $n+1$ is denoted by the operator

$$\max_{(n)} \text{ or } \max_{(n)}$$

where the subscript (n) indicates the filter's parameter, window size less one. Thus if Y is the output of the filter to an input X. Then X and Y are related by

$$Y = \max_{(n)}(X) \tag{3.1}$$

or

$$\{y_i\} = \max_{(n)}(\{x_i\}) \tag{3.2}$$

The output samples y_i's are related to the input samples x_i's by

$$y_i \triangleq \max(x_i, x_{i+n})$$

where max is the ordinary maximum operator which outputs the largest member of the elements in the argument. For ease of mathematical manipulation, the output sample is also denoted by $x_i^{(n)}$. Thus

$$y_i = x_i^{(n)} \triangleq \max(x_i, x_{i+n}) \tag{3.3}$$

where the superscript (n) is the filter parameter.

The cascade of two filters is represented by

$$Y = \max_{(m)}\left(\max_{(n)}(X)\right)$$

which indicates that the input sequence X is operated on by the filter $\max_{(n)}$ of window size $n+1$ followed by another, $\max_{(m)}$, of window size $m+1$. For brevity, the parentheses between cascaded filters can be omitted. Thus,

$$Y = \max_{(m)}\max_{(n)}(X)$$

If the cascaded filters are of the same window size, a superscript is used to indicate the number of passes. Thus

$$Y = \max_{(m)}\left(\max_{(m)}(X)\right) = \max_{(m)}\max_{(m)}(X) = \overset{2}{\max_{(m)}}(X) = \max^2_{(m)}(X)$$

The output Y, to an input X, of a p-pass Max filter of window size $m+1$ is given by

$$Y = \overset{p}{\max_{(m)}}(X)$$

For the above case the output sample y_i is also represented by $x_i^{(m)(p)}$. The first superscript (m) indicates the window size less one, the second superscript (p) indicates the number of passes the filter is applied. Therefore

$$\{y_i\} = \left\{x_i^{(m)(p)}\right\} = \overset{p}{\max_{(m)}}\left(\{x_i\}\right)$$

With zero pass, there is no filtering. Thus

$$x_i^{(m)(0)} = x_i \text{ and } \overset{0}{\max_{(m)}}(X) = X$$

In practice, the smallest window size is two $(m = 1)$. A window size of one $(m = 0)$ corresponds to the degenerate situation of no filtering. Thus,

$$x_i^{(0)(p)} = x_i \text{ and } \overset{p}{\max_{(0)}}(X) = X$$

With a single pass, the second superscript is omitted for brevity.

$$x_i^{(m)(1)} = x_i^{(m)}$$

Relation (3.1) is generalized, for the multipass Max filter, to

$$x_i^{(m)(k+1)} = \max\left(x_i^{(m)(k)}, x_{i+m}^{(m)(k)}\right) \tag{3.4}$$

which reduces to (1) when $k = 0$. The above notations and relations will be used in the next section. The same notations apply for the Min filter with operator "min".

3.3 Filter Properties

This section discusses four properties of the Max filter. Min filter has the same properties. Note that Max and Min filters are related by the following,

$$\max(x,y) = -\min(-x,-y)$$

Property 1: Commutative Property

Max filters can be cascaded. The output of the composite filter is independent of the order in which the Max filters are cascaded.

Consider two filters $\max_{(m)}$ and $\max_{(n)}$. Let $X = \{x_i\}$ be the input sequence and denote the output sequence of the first and second filters by $Y = \{y_i\}$ and $Z = \{z_i\}$ respectively, *i.e.*

$$Y = \max_{(n)}(X) \text{ and } Z = \max_{(m)}(Y).$$

Then,

$$\begin{aligned}
z_i &= y_i^{(m)} = \max(y_i, y_{i+m}) \\
&= \max\left(x_i^{(n)}, x_{i+m}^{(n)}\right) \\
&= \max\left(\max(x_i, x_{i+n}), \max(x_{i+m}, x_{i+n+m})\right) \\
&= \max(x_i, x_{i+n}, x_{i+m}, x_{i+n+m}) \qquad (3.5)
\end{aligned}$$

If the order of the filter is changed with the output signals being $W = \{w_i\}$ and $U = \{u_i\}$,

$$U = \max_{(m)}(X) \text{ and } W = \max_{(n)}(U)$$

Then,

$$w_i = u_i^{(n)} = \max(u_i, u_{i+n})$$

$$
\begin{aligned}
&= \max\left(x_i^{(m)}, x_{i+n}^{(m)}\right) \\
&= \max\left(\max(x_i, x_{i+m}), \max(x_{i+n}, x_{i+m+n})\right) \\
&= \max(x_i, x_{i+m}, x_{i+n}, x_{i+m+n}) \qquad (3.6)
\end{aligned}
$$

Comparison of (3.5) and (3.6) shows that

$$\max_{(m)} \max_{(n)} = \max_{(n)} \max_{(m)}$$

The above can be extended to more than two filters and it can be shown that

$$\max_{(m_1)} \max_{(m_2)} \cdots \max_{(m_k)} = \max_{(n_1)} \max_{(n_2)} \cdots \max_{(n_k)} \qquad (3.7)$$

where $(n_1, n_2, \cdots, n_k)$ is any permutation of $(m_1, m_2, \cdots, m_k)$.

A special case of the above is

$$\overset{p_1}{\max_{(m_1)}} \overset{p_2}{\max_{(m_2)}} = \overset{p_2}{\max_{(m_2)}} \overset{p_1}{\max_{(m_1)}}$$

which is useful for later development. Nonlinear filters are in general not commutative. In this sense the commutative property of the Max/Min filters is very unique. The commutative property does not hold for the composite filter of Max and Min.

It can be noted from relations (3.5) and (3.6) that the composite filter made by cascading size m+1 Max filter with another of size n+1 is equivalent to a **quaternary** maximum filter of window size $m+n+1$. A generalization of this observation is given in the following property.

Property 2:

A p-pass Max filter of window size $m+1$ is equivalent to a $(p+1)$-ary maximum filter of window size $pm+1$. Using the notation developed earlier, this is represented by

$$x_i^{(m)(p)} = \max(x_i, x_{i+m}, \cdots, x_{i+pm}) \qquad (3.8)$$

Proof:

$$
\begin{aligned}
x_i^{(m)(0)} &= x_i && \text{/* zero pass, no filtering */}\\
x_i^{(m)(1)} &= \max(x_i, x_{i+m}) && \text{/* one pass */}\\
x_i^{(m)(2)} &= \max\left(x_i^{(m)(1)}, x_{i+m}^{(m)(1)}\right)\\
&= \max(\max(x_i, x_{i+m}), \max(x_{i+m}, x_{i+2m}))\\
&= \max(x_i, x_{i+m}, x_{i+2m})\\
x_i^{(m)(3)} &= \max\left(x_i^{(m)(2)}, x_{i+m}^{(m)(2)}\right)\\
&= \max(\max(x_i, x_{i+m}, x_{i+2m}), \max(x_{i+m}, x_{i+2m}, x_{i+3m}))\\
&= \max(x_i, x_{i+m}, x_{i+2m}, x_{i+3m})
\end{aligned}
$$

Assume (3.8) is true for p equal to any integer k,

$$x_i^{(m)(k)} = \max(x_i, x_{i+m}, x_{i+2m}, \cdots, x_{i+km})$$

Then, for $p = k + 1$,

$$
\begin{aligned}
x_i^{(m)(k+1)} &= \max\left(x_i^{(m)(k)}, x_{i+m}^{(m)(k)}\right)\\
&= \max(\max(x_i, x_{i+m}, x_{i+2m}, \cdots, x_{i+km}), \max(x_{i+m}, x_{i+2m}, \cdots, x_{i+(k+1)m}))\\
&= \max(x_i, x_{i+m}, \cdots, x_{i+(k+1)m})
\end{aligned}
$$

By mathematical induction, the proof is complete.

For the special case of p-pass Max filter of window size two ($m = 1$),

$$x_i^{(1)(p)} = \max(x_i, x_{i+1}, x_{i+2}, \cdots, x_{i+p})$$

Thus the cascade of p Max filters of window size 2 is equivalent to a $(p+1)$-ary "maximum" filter of window size $p+1$. Note that the Max filter $\max_{(m)}$ is a **binary** "maximum" filter of window size $m+1$. A p-pass Max filter and the equivalent $(p+1)$-ary "maximum" filter both require p comparisons. The following relation

$$x_i^{(m)(p)} = \max\left(x_i^{(m)(p_1)}, x_{i+m}^{(m)(p_1)}, \cdots, x_{i+(p-p_1)m}^{(m)(p_1)}\right)$$

where m, p, p_1 are non-negative integers and $p > p_1$, follows from Property 2 and the relation

$$\max_{(m)}^{p_1} \max_{(m)}^{p_2} = \max_{(m)}^{p_1+p_2}$$

Property 2 demonstrates the edge shifting property of a Max filter. The filter tends to propagate to the left the local maximum sample, thereby shifting the rising edge to the left while maintaining the falling edge in position.

Property 3:

The cascade of k Max filters whose window sizes are consecutive multiples of a base number n, is equivalent to a p-pass Max filter of window size n.

$$\max_{(n)} \max_{(2n)} \cdots \max_{(kn)} = \max_{(n)}^{p} \tag{3.9}$$

where $p = 1 + 2 + 3 + \cdots + k$.

Proof:

(i) For $k = 1, p = 1$, $\max_{(n)} = \max^1_{(n)}$. This is a trivial case.

(ii) For $k = 2$,

$$\begin{aligned}
\max_{(n)} \max_{(2n)}(\{x_i\}) &= \max_{(n)} \left(\left\{x_i^{(2n)}\right\}\right) \\
&= \left\{\max\left(x_i^{(2n)}, x_{i+n}^{(2n)}\right)\right\} \\
&= \{\max(\max(x_i, x_{i+2n}), \max(x_{i+n}, x_{i+3n}))\} \\
&= \{\max(x_i, x_{i+n}, x_{i+2n}, x_{i+3n})\} \\
&= \max_{(n)}^{3}(\{x_i\})
\end{aligned}$$

Thus the relation is true for $k = 2$.

(iii) Assume the relation is true for k equal to a positive integer $m > 2$, *i.e.*,

$$\max_{(n)} \max_{(2n)} \cdots \max_{(mn)} = \max_{(n)}^{q}$$

where $q = 1 + 2 + 3 + ... + m = \frac{m(m+1)}{2}$. Then for $k = m + 1$,

$$\max_{(n)} \max_{(2n)} \cdots \max_{(mn)} \max_{((m+1)n)} = \overset{q}{\max_{(n)}} \max_{((m+1)n)} = \max_{((m+1)n)} \overset{q}{\max_{(n)}}$$

From Property 2

$$\overset{q}{\max_{(n)}}(\{x_i\}) = \left\{x_i^{(n)(q)}\right\} = \{\max(x_i, x_{i+n}, \cdots, x_{i+qn})\}$$

Now, noting that $q > m + 1$, then

$$\begin{aligned}
\max_{(m+1)n} \overset{q}{\max_{(n)}}(\{x_i\}) &= \left\{\max\left(x_i^{(n)(q)}, x_{i+(m+1)n}^{(n)(q)}\right)\right\} \\
&= \left\{\max\left(\max(x_i, \cdots, x_{i+qn}), \max(x_{i+(m+1)n}, \cdots, x_{i+qn+(m+1)n})\right)\right\} \\
&= \left\{\max\left(x_i, x_{i+n}, \cdots, x_{i+qn}, \cdots, x_{i+qn+(m+1)n}\right)\right\} \\
&= \overset{\frac{(m+1)(m+2)}{2}}{\max_{(n)}}(\{x_i\})
\end{aligned}$$

Thus (3.9) is also true for $k = m + 1$.

(iv) By mathematical induction, relation (3.9) is true for any positive integer k and the proof is complete.

Alternatively, using the following property,

$$\max_{(m)} \max_{(n)}(\{x_i\}) = \{\max(x_i, x_{i+m}, x_{i+n}, x_{i+m+n})\}$$

it can be shown directly that

$$\begin{aligned}
\max_{(n)} \max_{(2n)} \cdots \max_{(kn)}(\{x_i\}) &= \{\max(x_i, x_{i+n}, x_{i+2n}, \cdots, x_{i+pn})\} \\
&= \overset{p}{\max_{(n)}}(\{x_i\})
\end{aligned}$$

The above property can be utilized to reduce computation time. Suppose it is desired to process a given signal using six passes of a Max filter of window size 2. This can be implemented either by cascading six filters each with window size 2 or cascading three Max filters with window sizes of 2, 3 and 4. The latter arrangement requires one half of the computation needed by the former.

The next property provides an algorithm for decomposing a p-pass Max filter to a cascade of smaller number of single-pass Max filters with different window sizes.

Property 4: Decomposition Property

A p-pass Max filter of window size $m+1$ is equivalent to a cascade of Max filters with window sizes $b_j m + 1$.

$$\max_{(m)}^{p} = \max_{(b_1 m)} \max_{(b_2 m)} \cdots \max_{(b_k m)} \tag{3.10}$$

where the numbers $b_1, b_2, \ldots\ldots, b_k$ are determined with the following rules,

(i) $b_1 = \lceil p \div 2 \rceil$.

(ii) $b_j = \lceil (p - \sum_{i=1}^{j-1} b_i) \div 2 \rceil$ for $j > 1$.

and satisfy the relations

$$\sum_{i=1}^{k} b_i = p, \qquad b_k = 1$$

where $\lceil x \rceil$ is the integer greater or equal to x.

Before a formal proof is given, two simple examples are examined.

- Example 1:

$$\max_{(m)}^{5} = \max_{(3m)} \max_{(m)} \max_{(m)}$$

The integers b_i's are computed as follows.

$$\begin{aligned} b_1 &= \lceil 5/2 \rceil = 3 \\ b_2 &= \lceil (5-3)/2 \rceil = 1 \\ b_3 &= \lceil (5-3-1)/2 \rceil = 1 \end{aligned}$$

$$b_4 = b_5 = \cdots = 0$$

To verify,

$$\begin{aligned}
\overset{5}{\underset{(m)}{\max}}(\{x_i\}) &= \{\max(x_i, x_{i+m}, x_{i+2m}, x_{i+3m}, x_{i+4m}, x_{i+5m})\} \\
&= \{\max(\max(x_i, x_{i+m}, x_{i+2m}), \max(x_{i+3m}, x_{i+4m}, x_{i+5m}))\} \\
&= \underset{(3m)}{\max}(\{\max(x_i, x_{i+m}, x_{i+2m})\}) \\
&= \underset{(3m)}{\max}(\{\max(x_i, x_{i+m}), \max(x_{i+m}, x_{i+2m})\}) \\
&= \underset{(3m)}{\max}\,\underset{(m)}{\max}(\{\max(x_i, x_{i+m})\}) \\
&= \underset{(3m)}{\max}\,\underset{(m)}{\max}\,\overset{1}{\underset{(m)}{\max}}(\{x_i\})
\end{aligned}$$

- Example 2:

$$\overset{6}{\underset{(m)}{\max}} = \underset{(3m)}{\max}\,\underset{(2m)}{\max}\,\underset{(m)}{\max}$$

The integers b_i's are computed as follows.

$$\begin{aligned}
b_1 &= \lceil 6/2 \rceil = 3 \\
b_2 &= \lceil (6-3)/2 \rceil = 2 \\
b_3 &= \lceil (6-3-2)/2 \rceil = 1 \\
b_4 &= b_5 = \cdots = 0
\end{aligned}$$

To verify,

$$\begin{aligned}
\overset{6}{\underset{(m)}{\max}}(\{x_i\}) &= \{\max(\max(x_i, x_{i+m}, \cdots, x_{i+3m}), \max(x_{i+3m}, x_{i+4m}, \cdots, x_{i+6m}))\} \\
&= \underset{(3m)}{\max}\,\overset{3}{\underset{(m)}{\max}}(\{x_i\}) \\
&= \underset{(3m)}{\max}(\{\max(x_i, x_{i+m}), \max(x_{i+2m}, x_{i+3m})\}) \\
&= \underset{(3m)}{\max}\,\underset{(2m)}{\max}\,\overset{1}{\underset{(m)}{\max}}(\{x_i\})
\end{aligned}$$

Proof: The property is proved by construction. From Property 2,

$$\overset{p}{\underset{(m)}{\max}}(\{x_i\}) = \{\max(x_i, x_{i+m}, \cdots, x_{i+pm})\}$$

The objective is to split the argument of the operator max into two parts of equal number of entries. If there are odd number of entries, the middle entry is counted twice. For an odd number of passes, there are an even number $(p+1)$ entries in the argument of the max operator. Thus

$$\begin{aligned}
\overset{p}{\underset{(m)}{\max}}(\{x_i\}) &= \left\{\max\left(x_i, x_{i+m}, \cdots, x_{i+\frac{p-1}{2}m}, x_{i+\frac{p+1}{2}m}, \cdots, x_{i+pm}\right)\right\} \\
&= \underset{(\frac{p+1}{2}m)}{\max}\left(\left\{\max\left(x_i, x_{i+m}, \cdots, x_{i+\frac{p-1}{2}m}\right)\right\}\right) \\
&= \underset{(\frac{p+1}{2}m)}{\max}\ \overset{\frac{p-1}{2}}{\underset{(m)}{\max}}(\{x_i\})
\end{aligned}$$

For an even number of passes, there are an odd number of entries in the argument of the max operator. So

$$\begin{aligned}
\overset{p}{\underset{(m)}{\max}}(\{x_i\}) &= \left\{\max\left(x_i, x_{i+m}, \cdots, x_{i+\frac{p}{2}m}, x_{i+\frac{p}{2}m}, \cdots, x_{i+pm}\right)\right\} \\
&= \underset{(\frac{p}{2})m)}{\max}\left(\left\{\max(x_i, x_{i+m}, \cdots, x_{i+\frac{p}{2}m})\right\}\right) \\
&= \underset{(\frac{p}{2}m)}{\max}\,\overset{\frac{p}{2}}{\underset{(m)}{\max}}(\{x_i\})
\end{aligned}$$

Therefore,

$$\overset{p}{\underset{(m)}{\max}} = \underset{(b_1 m)}{\max}\,\overset{p-b_1}{\underset{(m)}{\max}}$$

where

$$b_1 = \lceil p \div 2 \rceil = \begin{cases} p/2 & \text{for } p \text{ even} \\ (p+1)/2 & \text{for } p \text{ odd} \end{cases}$$

If this procedure is repeated once the following will be obtained,

$$\overset{p}{\underset{(m)}{\max}} = \underset{(b_1 m)}{\max}\,\underset{(b_2 m)}{\max}\,\overset{p-b_1-b_2}{\underset{(m)}{\max}}$$

where

$$b_2 = \lceil (p-b_1) \div 2 \rceil = \begin{cases} (p-b_1)/2 & \text{for } (p-b_1) \text{ even} \\ (p-b_1+1)/2 & \text{for } (p-b_1) \text{ odd} \end{cases}$$

This procedure is iteratively repeated until $b_{k+1} = 0$ for some integer k. All subsequent values of the integers b_j will be zero and the procedure terminates. As can be noted from the above two examples, the last nonzero integer b_k is always equal to one and, the corresponding Max filter has a single pass.

$$\begin{aligned}
\max_{(m)}^{p} &= \max_{(b_1m)} \max_{(b_2m)} \max_{(m)}^{p-b_1-b_2} \\
&= \max_{(b_1m)} \max_{(b_2m)} \max_{(b_3m)} \max_{(m)}^{p-b_1-b_2-b_3} \\
&\vdots \\
&= \max_{(b_1m)} \max_{(b_2m)} \cdots \max_{(b_{k-1}m)} \max_{(m)}^{p-b_1-b_2-\cdots-b_{k-1}} \\
&= \max_{(b_1m)} \max_{(b_2m)} \cdots \max_{(b_{k-1}m)} \max_{(b_km)}^{1}
\end{aligned}$$

It follows that $b_k = 1$, $p - b_1 - b_2 - \cdots - b_{k-1} = 1$. Therefore, $b_1 + b_2 + \cdots + b_k = p$.

Property 5: Threshold Decomposition and Stacking Property

This is a typical property for ranked order operations, the proof for threshold decomposition for rank order operators are given in [29]. Being a rank-order based filter, the threshold decomposition applies for the Max filter. This was stated here for the sake of completeness

Chapter 4

STATISTICAL AND SPECTRAL PROPERTIES

4.1 Introduction

In this chapter the statistical and spectral properties are investigated. The bivariate distribution function of the Max filter output is derived, the autocorrelation function is then determined and finally power spectral density is given. The complete derivation for the Min filter is given in appendix A.

4.2 Distribution Function

For two independent random variables U and V respectively with cummulative distrinbution functions, *CDFs* $F_U(u)$ and $F_V(v)$ and probability density functions, *pdfs* $f_U(u)$ and $f_V(v)$, the CDF of a random variable Z given by

$$Z \triangleq \max(U, V) \tag{4.1}$$

is

$$F_Z(z) = F_U(z)F_V(z)$$

and the pdf is

$$f_Z(z) = F_U(z)f_V(z) + f_U(z)F_V(z)$$

If U and V are independent identically distributed random variables, then

$$F_Z(z) = F_U^2(z)$$

and

$$f_Z(z) = 2f_U(z)F_U(z)$$

Let $X = \{\{\cdots, X_{i-2}, X_{i-1}, X_i, X_{i+1}, X_{i+2}, X_{i+3}, \cdots\}$[1] be a sequence of identically distributed random variables. Input X to a Max filter of window size $(n+1)$ and denote the output sequence by $Y = \{Y_i; i = \cdots, -2, -1, 0, 1, 2, \cdots\}$, where

$$Y_i \triangleq \max(X_i, X_{i+n})$$

Joint Probability Density Function

Consider samples Y_i and Y_{i+n}. By definition, Y_i and Y_{i+n} are given by

$$\begin{aligned} Y_i &= \max(X_i, X_{i+n}) \\ Y_{i+n} &= \max(X_{i+n}, X_{i+2n}) \end{aligned}$$

It is apparent that both Y_i and Y_{i+n} depend on X_{i+n}. As a result Y_i and Y_{i+n} are dependent. By a similar reasoning one can deduce that other samples in Y are independent. Thus for $|j| \neq n$ and $j \neq 0$

$$\begin{aligned} P\{Y_i \leq \alpha, Y_{i+j} \leq \beta\} &= P\{Y_i \leq \alpha\}P\{Y_{i+j} \leq \beta\} \\ &= F_X^2(\alpha)F_X^2(\beta) \end{aligned} \tag{4.2}$$

[1]The upper case letters X_i and Y_i are used here for signal samples to indicate that each sample is a random variable.

For $j = n$, the joint probability is given by

$$\begin{aligned} P\{Y_i \le \alpha, Y_{i+n} \le \beta\} &= P\{\max(X_i, X_{i+n}) \le \alpha, \max(X_{i+n}, X_{i+2n}) \le \beta\} \\ &= P\{X_i \le \alpha, X_{i+n} \le \alpha, X_{i+n} \le \beta, X_{i+2n} \le \beta\} \quad (4.3) \end{aligned}$$

For $\alpha \le \beta$, the above equation becomes

$$\begin{aligned} P\{Y_i \le \alpha, Y_{i+n} \le \beta\} &= P\{X_i \le \alpha, X_{i+n} \le \alpha, X_{i+2n} \le \beta\} \\ &= P\{X_i \le \alpha\}P\{X_{i+n} \le \alpha\}P\{X_{i+2n} \le \beta\} \\ &= F_X^2(\alpha)F_X(\beta) \quad (4.4) \end{aligned}$$

Similarly, for $\alpha > \beta$

$$P\{Y_i \le \alpha, Y_{i+n} \le \beta\} = F_X(\alpha)F_X^2(\beta) \quad (4.5)$$

For $|k| = n$, the joint pdf $f_{Y_iY_{i+n}}$is given by

$$\begin{aligned} & f_{Y_iY_{i+n}}(\alpha, \beta) \quad (4.6) \\ =& \frac{\partial^2}{\partial\alpha\partial\beta} F_{Y_iY_{i+n}}(\alpha, \beta) \\ =& \begin{cases} \frac{\partial^2}{\partial\alpha\partial\beta} F_X^2(\alpha)F_X(\beta) & \text{for} \quad \alpha \le \beta \\ \frac{\partial^2}{\partial\alpha\partial\beta} F_X(\alpha)F_X^2(\beta) & \text{for} \quad \alpha > \beta \end{cases} \\ =& \begin{cases} 2F_X(\alpha)f_X(\alpha)f_X(\beta) & \text{for} \quad \alpha < \beta \\ 2F_X(\beta)f_X(\beta)f_X(\alpha) & \text{for} \quad \alpha > \beta \\ f_X(\alpha)F_X(\beta)[2F_X(\alpha) - F_X(\beta)]\delta(\alpha - \beta) & \text{for} \quad \alpha = \beta \end{cases} \end{aligned}$$

Therefore the joint pdf is given by

$$\begin{aligned} & f_{Y_iY_{i+j}}(\alpha, \beta) \\ =& \begin{cases} \left\{ \begin{array}{lll} 2F_X(\alpha)f_X(\alpha)f_X(\beta) & \text{for} & \alpha < \beta \\ 2F_X(\beta)f_X(\beta)f_X(\alpha) & \text{for} & \alpha > \beta \\ f_X(\alpha)F_X(\beta)[2F_X(\alpha) - F_X(\beta)]\delta(\alpha - \beta) & \text{for} & \alpha = \beta \end{array} \right\} & \text{for } j = n \\ 4f_X(\alpha)f_X(\beta)F_X(\alpha)F_X(\beta) & \text{for } j \ne n \text{ and } j \ne 0 \end{cases} \end{aligned}$$

For $j = 0$, the pdf is given by

$$f_{Y_i}(\alpha) = 2f_X(\alpha)F_X(\alpha)$$

Assume each sample X_i of the signal sequence $\{\cdots, X_{i-1}, X_i, X_{i+1}, \cdots\}$ has a value from the m-ary alphabet $\{\omega_0, \omega_1, \cdots \omega_{m-1}\}$ with $\omega_0 < \omega_1 < \cdots < \omega_{m-1}$. Further assume that each of the symbols ω_j are equally likely. Now, the pdf and the CDF for the case of equally likely discrete input alphabets case are given by

$$\begin{aligned} f_X(x) &= \frac{1}{m}\sum_{i=0}^{m-1}\delta(x-\omega_i) \\ F_X(x) &= \frac{1}{m}\sum_{i=0}^{m-1}u(x-\omega_i) \end{aligned}$$

respectively. Assume X_i's are independent, the CDF of the output samples of the Max filter is

$$\begin{aligned} F_Y(y) &= F_X^2(y) \\ &= \frac{1}{m^2}\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}u(y-\omega_i)u(y-\omega_j) \\ &= \frac{1}{m^2}\sum_{i=0}^{m-1}(2i+1)u(y-\omega_i) \end{aligned} \tag{4.7}$$

and the corresponding pdf is given by

$$f_Y(y) = \frac{1}{m^2}\sum_{i=0}^{m-1}(2i+1)\delta(y-\omega_i) \tag{4.8}$$

4.3 Autocorrelation Function

Assume X_i's are independent identically distributed random variables. The autocorrelation of the Max filter output for the case of equally likely input alphabet is derived below. The autocorrelation function R(k) of Y is given by

$$R(k) \triangleq E\{Y_iY_{i+k}\}$$

Next consider three cases (i)$k=0$, (ii)$|k| \neq n$ and $|k| \neq 0$ and (iii) $|k| = n$.

Case I $K = 0$,

the autocorrelation function is equal to the second order moment, *i.e.*

$$\begin{aligned} R(0) &= E\{Y_i^2\} \\ &= \int_{-\infty}^{\infty} y^2 f_Y(y)dy \\ &= \int_{-\infty}^{\infty} y^2 \frac{1}{m^2} \sum_{i=0}^{m-1}(2i+1)\delta(y-\omega_i)dy \\ &= \frac{1}{m^2}\sum_{i=0}^{m-1}(2i+1)\omega_i^2 \end{aligned}$$

Case II $|k| \neq n$,

the autocorrelation function is given by

$$\begin{aligned} R(k) &= E\{Y_i\}E\{Y_{i+k}\} \qquad /*\, Y_i \text{ and } Y_{i+k} \text{ being independent } */ \\ &= \left[\int_{-\infty}^{\infty} \alpha f_{Y_i}(\alpha)d\alpha\right]^2 \\ &= \left[2\int_{-\infty}^{\infty} \alpha f_{x_i}(\alpha)F_{x_i}(\alpha)d\alpha\right]^2 \end{aligned}$$

the autocorrelation function is thus given by

$$\begin{aligned} R(k) &= \left[\frac{1}{m^2}\sum_{i=0}^{m-1}(2i+1)\int_{-\infty}^{\infty}\alpha\delta(\alpha-\omega_i)d\alpha\right]^2 \\ &= \left[\frac{1}{m^2}\sum_{i=0}^{m-1}(2i+1)\omega_i\right]^2 \end{aligned}$$

Case III $|k| = n$,

the autocorrelation function is given by

$$\begin{aligned} R(n) &= E\{Y_iY_{i+n}\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha\beta f_{Y_iY_{i+n}}(\alpha,\beta)d\alpha d\beta \end{aligned} \tag{4.9}$$

where $f_{Y_iY_{i+n}}(\alpha,\beta)$ is given in equation (4.6). Now, substituting equation(4.6) in the above

$$R(n) = \int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha^+}^{\infty} \alpha\beta f_{Y_iY_{i+n}}(\alpha,\beta)d\beta d\alpha$$

$$
\begin{aligned}
& + \int_{\alpha=-\infty}^{\infty} \int_{\beta=-\infty}^{\alpha^-} \alpha\beta f_{Y_iY_{i+n}}(\alpha,\beta) d\beta d\alpha \\
& + \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta f_{Y_iY_{i+n}}(\alpha,\beta) d\beta d\alpha \\
= & \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^+}^{\infty} 2\alpha\beta F_X(\alpha) f_X(\alpha) f_X(\beta) d\beta d\alpha \\
& + \int_{\alpha=-\infty}^{\infty} \int_{\beta=-\infty}^{\alpha^-} 2\alpha\beta F_X(\beta) f_X(\beta) f_X(\alpha) d\beta d\alpha + \\
& \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta f_X(\alpha) F_X(\beta)[2F_X(\alpha) - F_X(\beta)]\delta(\alpha-\beta) d\beta d\alpha \\
= & \; I_1 + I_2 + I_3 \qquad (4.10)
\end{aligned}
$$

where the terms I_1, I_2 and I_3 are given by

$$
\begin{aligned}
I_1 & \triangleq \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^+}^{\infty} 2\alpha\beta F_X(\alpha) f_X(\alpha) f_X(\beta) d\beta d\alpha \\
I_2 & \triangleq \int_{\alpha=-\infty}^{\infty} \int_{\beta=-\infty}^{\alpha} 2\alpha\beta F_X(\beta) f_X(\beta) f_X(\alpha) d\beta d\alpha \\
I_3 & \triangleq \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta f_X(\alpha) F_X(\beta)[2F_X(\alpha) - F_X(\beta)]\delta(\alpha-\beta) d\beta d\alpha
\end{aligned}
$$

Next consider each of the terms individually. For I_1,

$$
\begin{aligned}
I_1 & = \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^+}^{\infty} \alpha\beta \frac{2}{m^3} \sum_{j=0}^{m-1} u(\alpha-\omega_j) \sum_{k=0}^{m-1} \delta(\alpha-\omega_k) \sum_{l=0}^{m-1} \delta(\beta-\omega_l) d\beta d\alpha \\
& = \frac{2}{m^3} \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^+}^{\infty} \alpha\beta \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} u(\alpha-\omega_j)\delta(\alpha-\omega_k) \sum_{l=0}^{m-1} \delta(\beta-\omega_l) d\beta d\alpha \\
& = \frac{2}{m^3} \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^+}^{\infty} \alpha\beta \sum_{k=0}^{m-1} (k+\frac{1}{2})\delta(\alpha-\omega_k) \sum_{l=0}^{m-1} \delta(\beta-\omega_l) d\beta d\alpha \\
& = \frac{2}{m^3} \int_{\alpha=-\infty}^{\infty} \alpha \sum_{k=0}^{m-1} (k+\frac{1}{2})\delta(\alpha-\omega_k) \int_{\beta=\alpha}^{\infty} \beta \sum_{l=0}^{m-1} \delta(\beta-\omega_l) d\beta d\alpha \\
& = \frac{2}{m^3} \int_{\alpha=-\infty}^{\infty} \alpha g(\alpha) \sum_{k=0}^{m-1} (k+\frac{1}{2})\delta(\alpha-\omega_k) d\alpha \\
& = \frac{2}{m^3} \sum_{k=0}^{m-1} (k+\frac{1}{2})\omega_k g(\omega_k)
\end{aligned}
$$

where,

$$
g(\alpha^+) = \int_{\beta=\alpha}^{\infty} \beta \sum_{l=0}^{m-1} \delta(\beta-\omega_l) d\beta
$$

$$
\begin{aligned}
&= \sum_{l=0}^{m-1} \omega_l \int_{\beta=\alpha}^{\infty} \delta(\beta - \omega_l) d\beta \\
g(\omega_k) &= \sum_{l=0}^{m-1} \omega_l \int_{\beta=\omega_k}^{\infty} \delta(\beta - \omega_l) d\beta \\
&= \frac{1}{2}\omega_k + \sum_{l=k+1}^{m-1} \omega_l
\end{aligned}
$$

Therefore

$$
I_1 = \frac{2}{m^3} \sum_{k=0}^{m-1} \left(k + \frac{1}{2}\right) \omega_k \left[\frac{1}{2}\omega_k + \sum_{l=k+1}^{m-1} \omega_l\right] \tag{4.11}
$$

The second term I_2 is given by

$$
\begin{aligned}
I_2 &= \frac{2}{m^3} \int_{\alpha=-\infty}^{\infty} \int_{\beta=-\infty}^{\alpha} \alpha\beta \sum_{j=0}^{m-1} u(\beta - \omega_j) \sum_{k=0}^{m-1} \delta(\beta - \omega_k) \sum_{l=0}^{m-1} \delta(\alpha - \omega_l) d\beta d\alpha \\
&= \frac{2}{m^3} \int_{\alpha=-\infty}^{\infty} \int_{\beta=-\infty}^{\alpha} \alpha\beta \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} u(\beta - \omega_j)\delta(\beta - \omega_k) \sum_{l=0}^{m-1} \delta(\alpha - \omega_l) d\beta d\alpha \\
&= \frac{2}{m^3} \int_{\alpha=-\infty}^{\infty} \int_{\beta=-\infty}^{\alpha} \alpha\beta \sum_{k=0}^{m-1} (k + \frac{1}{2})\delta(\beta - \omega_k) \sum_{l=0}^{m-1} \delta(\alpha - \omega_l) d\beta d\alpha \\
&= \frac{2}{m^3} \int_{\alpha=-\infty}^{\infty} \alpha \sum_{l=0}^{m-1} \delta(\alpha - \omega_l) \int_{\beta=-\infty}^{\alpha} \sum_{k=0}^{m-1} (k + \frac{1}{2})\omega_k \delta(\beta - \omega_k) d\beta d\alpha \\
&= \frac{2}{m^3} \int_{\alpha=-\infty}^{\infty} \alpha \sum_{l=0}^{m-1} \delta(\alpha - \omega_l) h(\alpha) d\alpha \\
&= \frac{2}{m^3} \sum_{l=0}^{m-1} \omega_l h(\omega_l)
\end{aligned}
$$

where,

$$
h(\alpha) = \sum_{k=0}^{m-1} (k + \frac{1}{2})\omega_k \int_{\beta=-\infty}^{\alpha} \delta(\beta - \omega_k) d\beta
$$

$$
\begin{aligned}
h(\omega_l) &= \sum_{k=0}^{m-1} (k + \frac{1}{2})\omega_k \int_{\beta=-\infty}^{\omega_l} \delta(\beta - \omega_k) d\beta \\
&= \frac{1}{2}(l + \frac{1}{2})\omega_l + \sum_{k=0}^{l-1} (k + \frac{1}{2})\omega_k
\end{aligned}
$$

Therefore

$$
I_2 = \frac{2}{m^3} \sum_{l=0}^{m-1} \omega_l \left[\frac{1}{2}(l + \frac{1}{2})\omega_l + \sum_{k=0}^{l-1} (k + \frac{1}{2})\omega_k\right] \tag{4.12}
$$

Similarly the third term, I_3

$$
\begin{aligned}
I_3 &= 2\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta F_X(\alpha) f_X(\alpha) F_X(\beta)\delta(\alpha-\beta)d\beta d\alpha \\
&\quad -\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta f_X(\alpha) F_X^2(\beta)\delta(\alpha-\beta)d\beta d\alpha \qquad (4.13)
\end{aligned}
$$

Now consider the first term in the right hand side of the above equation,

$$
\begin{aligned}
&\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha^-}^{\alpha^+} 2\alpha\beta F_X(\alpha) f_X(\alpha) F_X(\beta)\delta(\alpha-\beta)d\beta d\alpha \\
&= \frac{2}{m^3}\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta \sum_{j=0}^{m-1} u(\alpha-\omega_j) \sum_{k=0}^{m-1}\delta(\alpha-\omega_k) \sum_{l=0}^{m-1} u(\beta-\omega_l)\delta(\beta-\alpha)d\beta d\alpha \\
&= \frac{2}{m^3}\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta \sum_{k=0}^{m-1}(k+\frac{1}{2})\delta(\alpha-\omega_k) \sum_{l=0}^{m-1} u(\beta-\omega_l)\delta(\beta-\alpha)d\beta d\alpha \\
&= \frac{2}{m^3}\int_{\alpha=-\infty}^{\infty} \alpha \sum_{k=0}^{m-1}(k+\frac{1}{2})\delta(\alpha-\omega_k) \int_{\beta=\alpha^-}^{\alpha^+} \sum_{l=0}^{m-1} u(\beta-\omega_l)\beta\delta(\beta-\alpha)d\beta d\alpha \\
&= \frac{2}{m^3}\int_{\alpha=-\infty}^{\infty} \alpha \sum_{k=0}^{m-1}(k+\frac{1}{2})\delta(\alpha-\omega_k) a(\alpha) d\alpha \\
&= \frac{2}{m^3}\sum_{k=0}^{m-1}(k+\frac{1}{2})\omega_k a(\omega_k)
\end{aligned}
$$

where

$$
a(\alpha) = \int_{\beta=\alpha^-}^{\alpha^+} \sum_{l=0}^{m-1} u(\beta-\omega_l)\beta\delta(\beta-\alpha)d\beta
$$

and

$$
\begin{aligned}
a(\omega_k) &= \int_{\beta=\omega_k^-}^{\omega_k^+} \sum_{l=0}^{m-1} u(\beta-\omega_l)\beta\delta(\beta-\omega_k)d\beta \\
&= \sum_{l=0}^{k-1} u(\omega_k-\omega_l)\omega_k + \frac{1}{2}\omega_k \\
&= \left(k+\frac{1}{2}\right)\omega_k
\end{aligned}
$$

Thus

$$
\begin{aligned}
&2\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta F_X(\alpha) f_X(\alpha) F_X(\beta)\delta(\alpha-\beta)d\beta d\alpha \\
&= \frac{2}{m^3}\sum_{k=0}^{m-1}\left[k+\frac{1}{2}\right]^2 \omega_k^2 \qquad (4.14)
\end{aligned}
$$

Similarly for the second term of equation(4.13)

$$\begin{aligned}
&\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta f_X(\alpha)F_X^2(\beta)\delta(\alpha-\beta)d\beta d\alpha \\
&= \frac{1}{m^3}\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta\sum_{j=0}^{m-1}\delta(\alpha-\omega_j)\sum_{k=0}^{m-1}(2k+1)u(\beta-\omega_k)\delta(\beta-\alpha)d\beta d\alpha \\
&= \frac{1}{m^3}\int_{\alpha=-\infty}^{\infty} \alpha\sum_{j=0}^{m-1}\delta(\alpha-\omega_j)b(\alpha)d\alpha \\
&= \frac{1}{m^3}\sum_{j=0}^{m-1}\omega_j b(\omega_j)
\end{aligned}$$

where,

$$b(\alpha) = \int_{\beta=\alpha^-}^{\alpha^+} \beta\sum_{k=0}^{m-1}(2k+1)u(\beta-\omega_k)\delta(\beta-\alpha)d\beta$$

and

$$\begin{aligned}
b(\omega_j) &= \int_{\omega_j^-}^{\omega_j^+}\sum_{k=0}^{m-1}(2k+1)u(\beta-\omega_k)\delta(\beta-\omega_j)\beta d\beta \\
&= \sum_{k=0}^{j-1}(2k+1)\omega_j + \frac{2j+1}{2}\omega_j \\
&= \left[j^2+j+\frac{1}{2}\right]\omega_j
\end{aligned}$$

Hence,

$$\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta f_X(\alpha)F_X^2(\beta)\delta(\alpha-\beta)d\beta d\alpha = \frac{1}{m^3}\sum_{j=0}^{m-1}\left[j^2+j+\frac{1}{2}\right]\omega_j^2 \tag{4.15}$$

Substituting equations (4.11), (4.12), (4.14) and (4.15) in (4.16)

$$\begin{aligned}
R(n) &= \frac{2}{m^3}\left\{\sum_{k=0}^{m-1}\left(k+\frac{1}{2}\right)\omega_k\left[\sum_{l=k+1}^{m-1}\omega_l+\frac{1}{2}\omega_k\right]+\right. \\
&+ \sum_{l=0}^{m-1}\omega_l\left[\sum_{k=0}^{l-1}\left(k+\frac{1}{2}\right)\omega_k+\frac{1}{2}\left(l+\frac{1}{2}\right)\omega_l\right] \\
&+ \left.\sum_{j=0}^{m-1}\frac{1}{2}j(j+1)\omega_j^2\right\}
\end{aligned} \tag{4.16}$$

For the Min filter the autocorrelation function $R(k)$ is given by, for $k = 0$,

$$R(0) = \frac{1}{m^2}\sum_{i=0}^{m-1}(2m - 2i - 1)x_i^2 \tag{4.17}$$

For $k \neq n$ and $k \neq 0$

$$R(k) = \left[\frac{1}{m^2}\sum_{i=0}^{m-1}(2m - 2i - 1)x_i\right]^2 \tag{4.18}$$

For $k = n$ the autocorelation function is given by

$$\begin{aligned} R(n) &= \frac{2}{m^3}\sum_{i=0}^{m-1}\left(\sum_{j=i+1}^{m-1} x_j\left(1 - \frac{2j+1}{2m}\right) + \frac{1}{2}\left(1 - \frac{2i+1}{2m}\right)x_i\right) \\ &+ \frac{2}{m^2}\sum_{i=0}^{m-1} x_i\left(\sum_{j=0}^{i-1} x_j + \frac{1}{2}x_i\right) - \frac{2}{m^3}\sum_{i=0}^{m-1}\left(i + \frac{1}{2}\right)x_i\left(\sum_{j=0}^{i-1} x_j + \frac{1}{2}x_i\right) \\ &+ \frac{1}{m^3}\sum_{i=0}^{m-1}\left(i(i+1) - 2m\left(i + \frac{1}{2}\right) + m^2\right)x_i^2 \end{aligned} \tag{4.19}$$

The values for the Max filter with binary input, $\{-1, 1\}$, and ternary input, $\{-1, 0, 1\}$, are given in tables 4.1 and 4.2 below.

The figures below show the the autocorrelation function for the Max filter with window sizes 2 and 5 for the binary and the ternary input cases.

4.4 Power Spectral Density

The spectral power density $\phi(\omega)$ is related to the autocorrelation function $R(k)$ by,

$$\phi(\omega) = \sum_{k=-\infty}^{\infty} R(k)e^{-jk\omega} \tag{4.20}$$

Using the above equation and the results of the previous section the power spectral density function for the different cases can be determined. For size 2 Max filter with a binary input, this is given by

$$\phi(\omega) = 0.75 + \sum_{k=-\infty}^{\infty} 0.25e^{-jk\omega} + 0.50\cos(\omega)$$

For size 5 Max filter the power spectral density is given by,

$$\phi(\omega) = 0.75 + \sum_{k=-\infty}^{\infty} 0.25e^{-jk\omega} + 0.50\cos(4\omega)$$

Similarly for the ternary input, the power spectral densities for window sizes 2 and 5 are given by

$$\phi(\omega) = 0.469 + \sum_{k=-\infty}^{\infty} 0.198e^{-jk\omega} + 0.344\cos(\omega)$$
$$\phi(\omega) = 0.469 + \sum_{k=-\infty}^{\infty} 0.198e^{-jk\omega} + 0.344\cos(4\omega)$$

The four power spectral densities are shown in figs. 4.5, 4.6, 4.7 and 4.8. All of them are characterized by an impulse at zero frequency and a sinusoid. The input to the Max filter has an average d.c. level of zero. Due to the Max filter this d.c. is increased, which explains the impulse at zero frequency in the power spectral density function.

Next chapter discusses a model of the Max/Min filter. This model, for Max filter, is based on the following relation,

$$\max(x_i, x_{i+n}) = \frac{1}{2}\left(x_i + x_{i+n} + |x_i - x_{i+n}|\right) \tag{4.21}$$

The model thus includes two comb filters, $\frac{x_i + x_{i+n}}{2}$ and $\frac{x_i - x_{i+n}}{2}$. The presence of these comb filters explains the sinusoidal function in the power spectral density. Increasing the window size $n+1$ will result in sinusoid of higher frequency. This can be explained also by the means of the model. As the window size increases the value of n increases as a results the number of the delay elements in the comb filters increase. This in turn increase the number of teeth (lobes) in the comb filter.

Index	Window size				
k	2	3	5	7	9
0	1.000	1.000	1.000	1.000	1.000
1	0.500	0.250	0.250	0.250	0.250
2	0.250	0.500	0.250	0.250	0.250
3	0.250	0.250	0.250	0.250	0.250
4	0.250	0.250	0.500	0.250	0.250
5	0.250	0.250	0.250	0.250	0.250
6	0.250	0.250	0.250	0.500	0.250
7	0.250	0.250	0.250	0.250	0.250
8	0.250	0.250	0.250	0.250	0.500
9	0.250	0.250	0.250	0.250	0.250

Table 4.1: The autocorrelation function $R(k)$ for binary input.

Index	Max filter				
k	2	3	5	7	9
0	0.667	0.667	0.667	0.667	0.667
1	0.370	0.198	0.198	0.198	0.198
2	0.198	0.370	0.198	0.198	0.198
3	0.198	0.198	0.198	0.198	0.198
4	0.198	0.198	0.370	0.198	0.198
5	0.198	0.198	0.198	0.198	0.198
6	0.198	0.198	0.198	0.370	0.198
7	0.198	0.198	0.198	0.198	0.198
8	0.198	0.198	0.198	0.198	0.370
9	0.198	0.198	0.198	0.198	0.198

Table 4.2: The autocorrelation function $R(k)$ for ternary input.

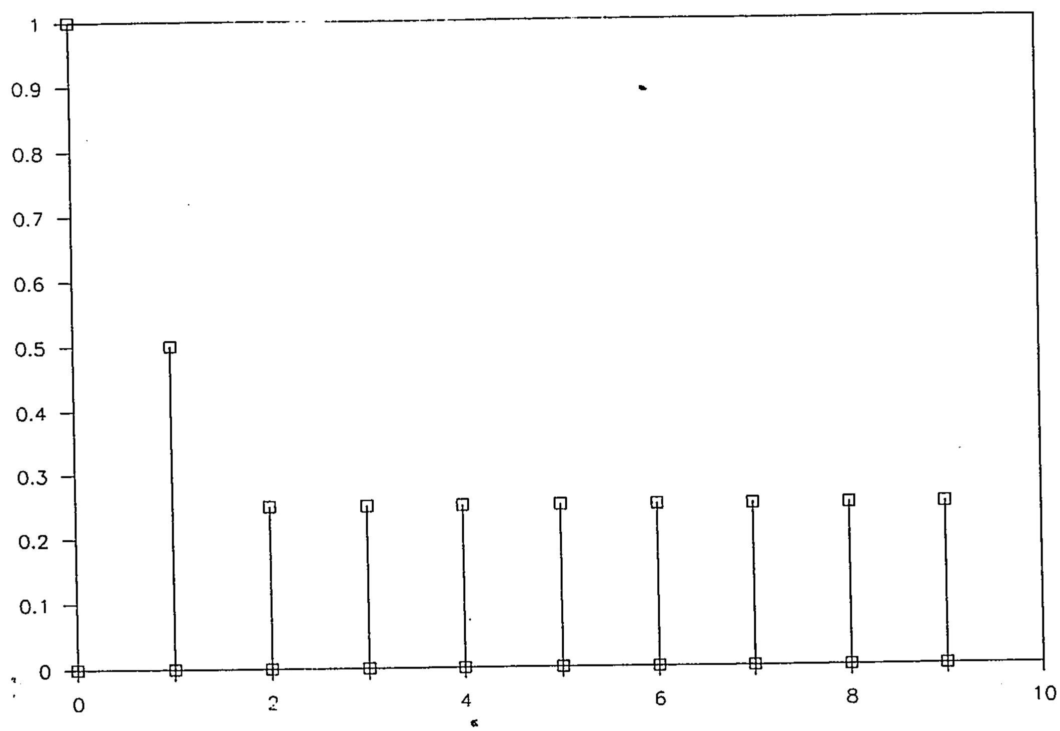

Figure 4.1: Autocorrelation function for size-2 Max filter with a binary input.

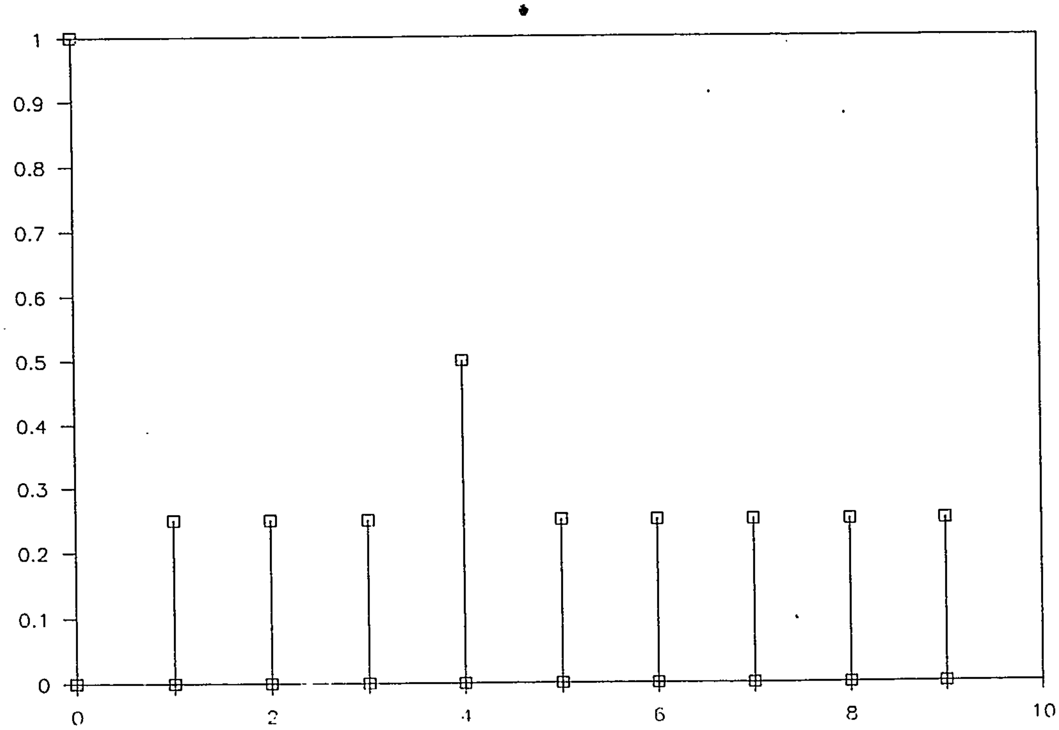

Figure 4.2: Autocorrelation function for size-5 Max filter with a binary Input.

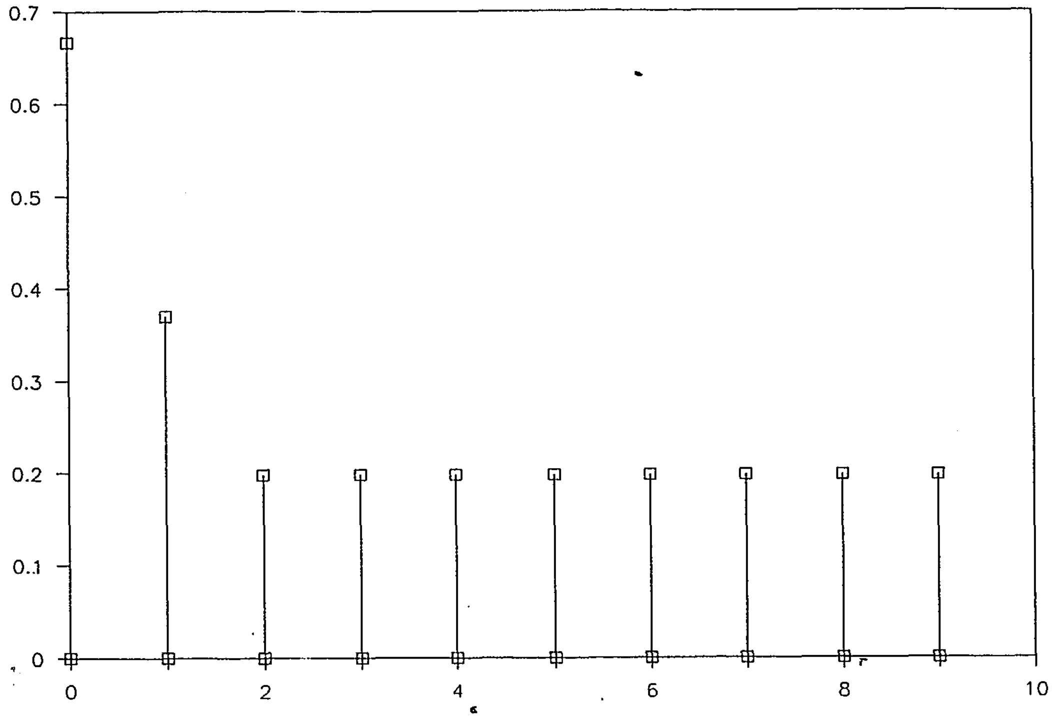

Figure 4.3: Autocorrelation function for size-2 Max filter with a ternary input.

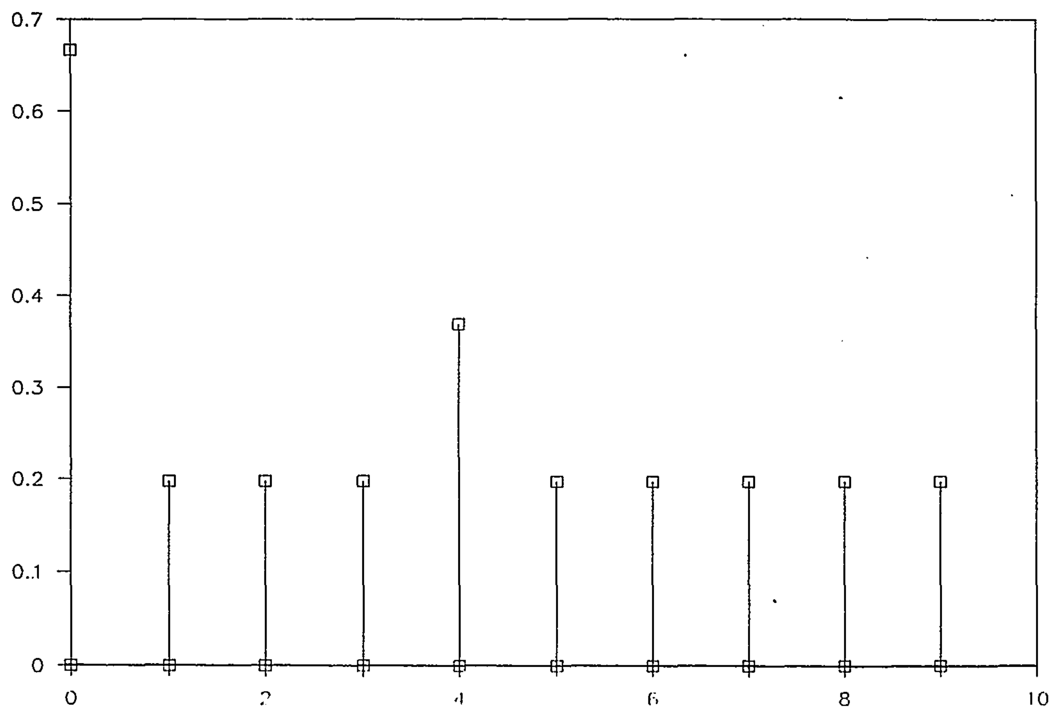

Figure 4.4: Autocorrelation function for size-5 Max filter with a ternary input.

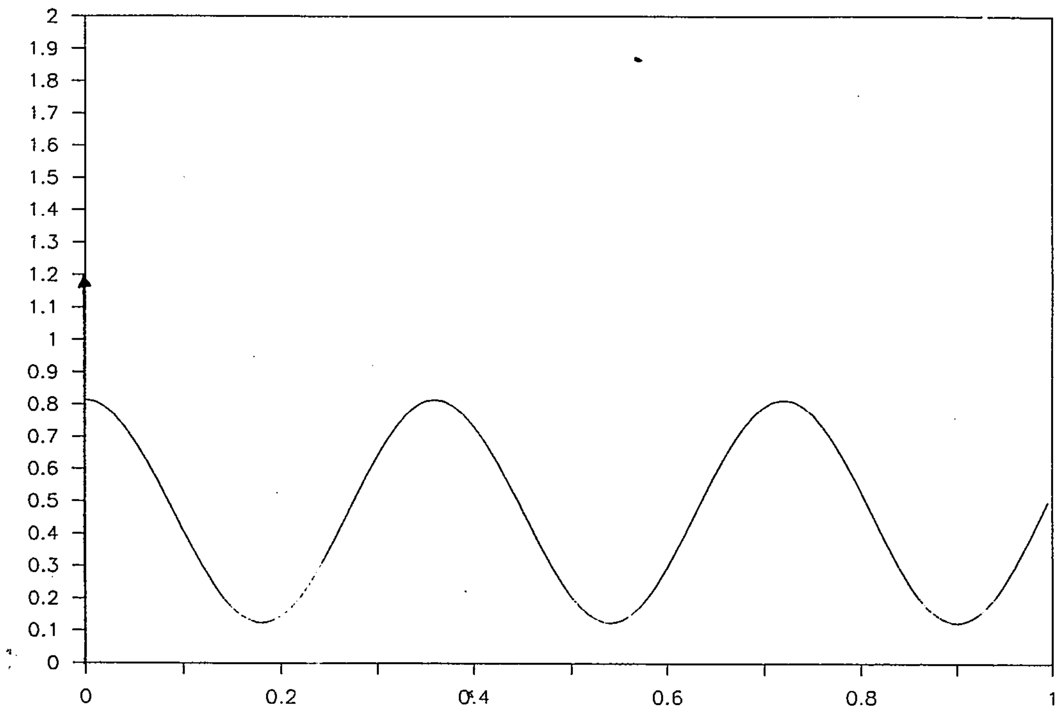

Figure 4.5: Spectral power density of size-2 Max filter for a binary input.

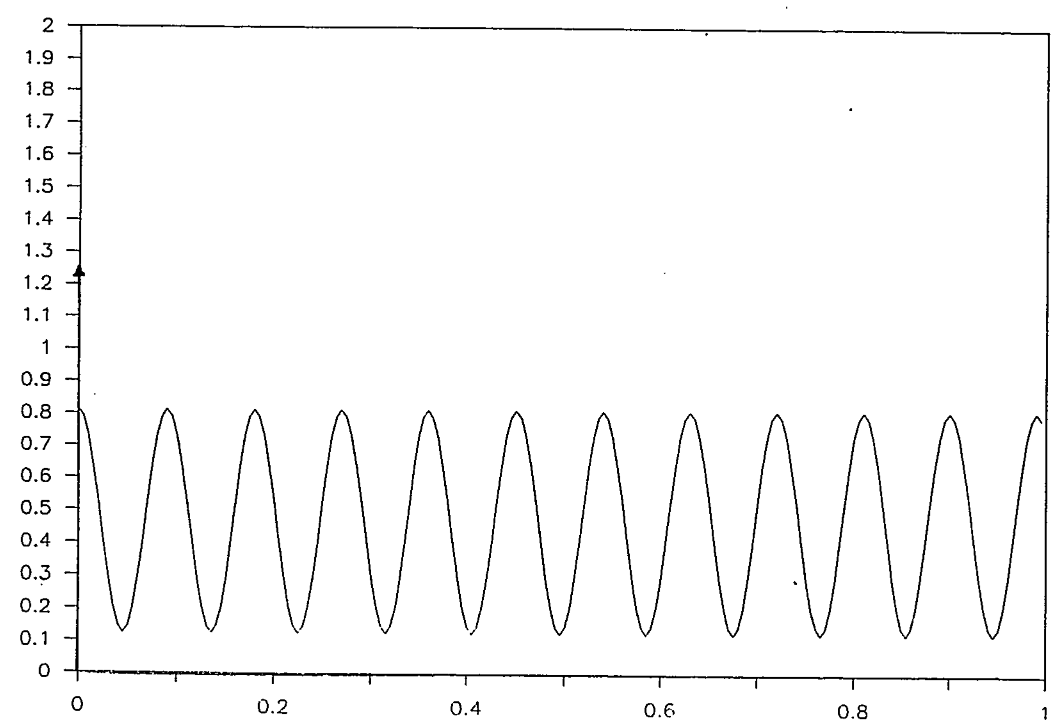

Figure 4.6: Spectral power density of size-5 Max filter for a binary input.

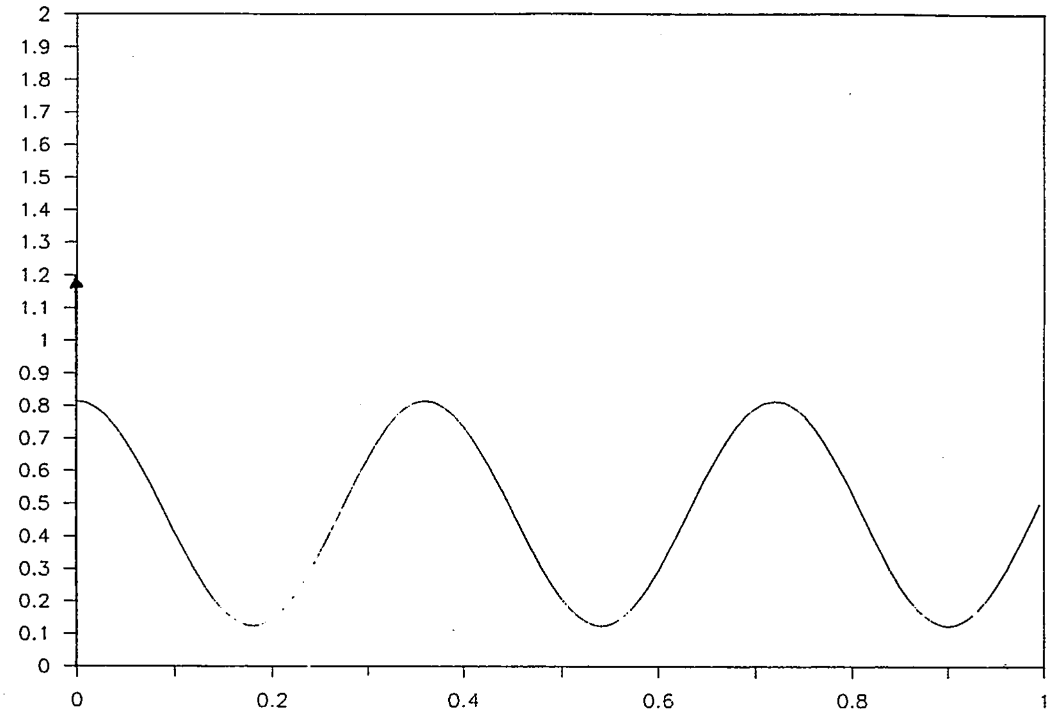

Figure 4.7: Spectral power density of size-2 Max filter for a ternary input.

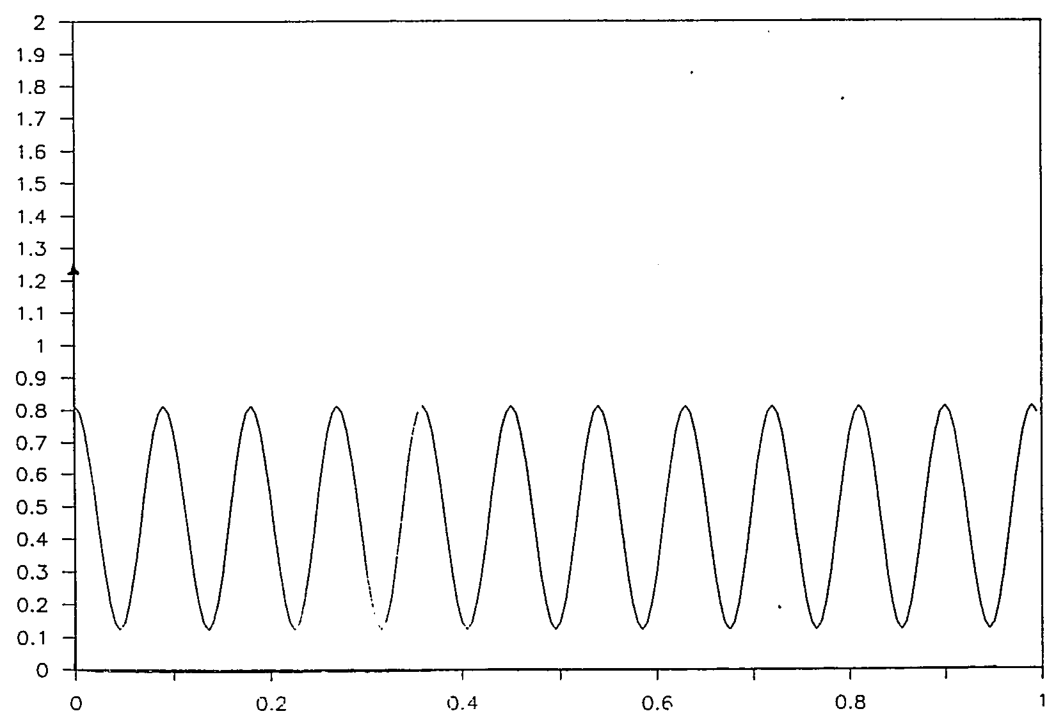

Figure 4.8: Spectral power density of size-5 Max filter for a ternary input.

Chapter 5

A MODEL OF MAX/MIN FILTER

5.1 Introduction

In this chapter a model for the Max and Min filter is presented. This model provides one possible way for the implementation of the Max/Min filter. It also portrays the nonlinear nature of the filter. With the help of this model new modifications on the Max/Min filter were made possible. Only a few of these are introduced here. It is the belief of the author that a thorough investigation of this model would lead to new filter structures and would open new research areas in the field of signal processing.

5.2 Max/Min Filter Model

It can be shown that max and min operations can be expressed mathematically as,

$$\max(x,y) = \frac{1}{2}(x+y+|y-x|)$$

and

$$\min(x,y) = \frac{1}{2}(x+y-|y-x|)$$

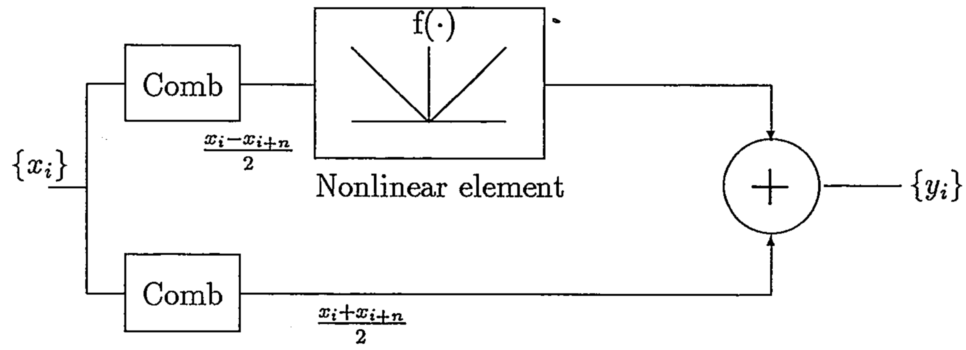

Figure 5.1: Max filter model.

Examining the above equations, the Max or Min filter can be implemented using an adder and full wave rectifier.

The above scheme can be used for the hardware implementation of the Max filter. Min filter can be implemented using a similar scheme with the upper branch being inverted in polarity. The nonlinear nature of the Max filter is apparent in the full-wave rectifier of the upper branch. Changing the nonlinear element would change the scheme to a different filter. As an example, setting to zero the upper branch will reduce the filter to an average filter. The nonlinear element f(Δ) for the Max filter is given by

$$\begin{aligned} f(\Delta) &= |\Delta| \\ &= \begin{cases} \Delta & \text{for } \Delta \geq 0 \\ -\Delta & \text{for } \Delta < 0 \end{cases} \end{aligned}$$

For positive values of Δ the nonlinear element is given by

$$f(\Delta) = \Delta$$

and

$$y_i = x_i$$

thus there will be no filtering. For negative values of Δ the nonlinear element is given by

$$f(\Delta) = -\Delta$$

and

$$y_i = x_{i+n}$$

i.e. the filter output is the input delay by n samples.

The edge shifting effect of the Max filter can be explained by considering different values of Δ. For falling edges Δ is positive, hence they will pass unaffected, while for the rising edges Δ is negative hence the filter will output the input delay, resulting in the shifting of the rising edge.

The next section will present some modifications on the nonlinear element. These brought about new filters, whose simulations were carried out.

5.3 Max/Min Filter Modifications

The modifications are based on the model shown in fig. 5.1. Four of these are described below. A common test signal was used to investigate the performance of each of these modifications. The test signal, whose SNR is 10 dB, is shown in fig. 5.2 below.

The first modification is based on the nonlinear element shown in the fig. 5.3. The output of the resulting filter, with **a** being set 60, is shown in fig. 5.4.

The edge shifting, which is inherent in the original Max/Min filter, is absent in this modified filter. This can be explained by noting that the element shown in fig. 5.3 can be divided into three regions of operation, *viz*

(1) Region II, for $x_i - x_{i+1}$ greater than $\mathbf{a}$,

(2) Region III, for $x_i - x_{i+1}$ less than $-\mathbf{a}$ and

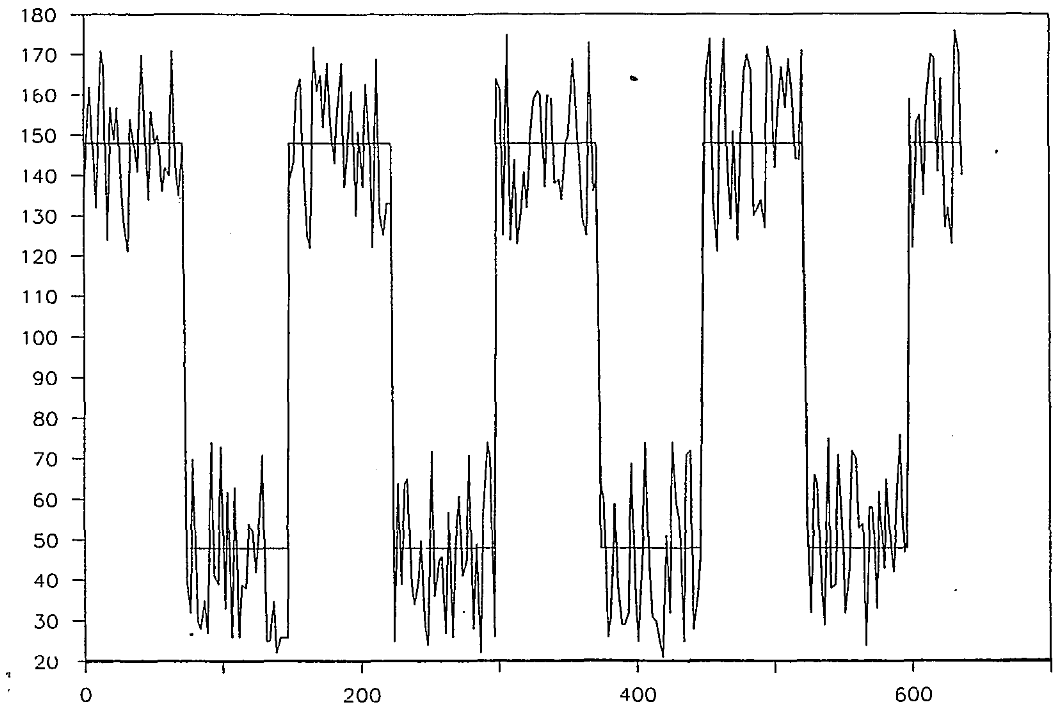

Figure 5.2: Test signal.

(3) Region I, the region in between.

Referring to the model in fig. 5.1, the input to the nonlinear element is the difference between the extreme samples in the window. With the regions described for the nonlinear element in fig. 5.3, it follows that regions II and III respectively correspond to samples lying in the falling and rising edges whose transitions are greater than **a**. Other samples will lie in region I.

Next, to understand the edge response of this filter, consider the nonlinear element in each of the edge regions *i.e.* regions II and III. In the falling edge region, II, the nonlinear element is positive with unit gradient, thus is identical to the Max filter nonlinear element, full-wave rectifier. The same is true for region II, rising edge region, the nonlinear element shown in fig. 5.3 corresponds to that of the Min filter. Hence for the rising edges the new filter behaves as a Min filter and for

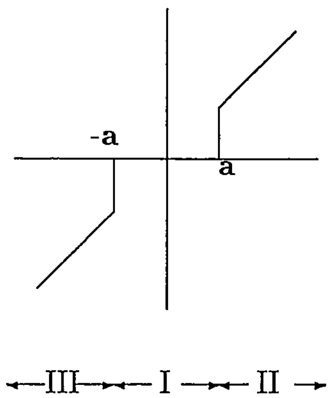

Figure 5.3: A Nonlinear element

the falling edges as a Max filter, as a result it preserves both edges in shape and position.

In region I the filter reduces to a moving average filter, giving better noise smoothing performance there. Fig. 5.4 shows the output of this filter with a window size of 2. The output of the filter is smooth in some parts of the signal. This is explained by the linear behavior of the filter.

The choice of the parameter **a** is critical. Setting **a** to a larger value will reduce the system to a moving average filter in a wider range, thus smoothing edges out. A small value for **a**, on the other hand, would pass the signal unchanged as the element will then tend to the unit gradient straight line

$$f(\Delta) = \Delta$$

In this case there will be no filtering.

The multilevel signal case will put more stringent conditions on **a**. The choice of **a** must take into consideration the different edges. It must be set to a value smaller than the smallest edge transition. A better filter will be the one in which the parameter **a** is adaptive to the different signal and noise levels of the

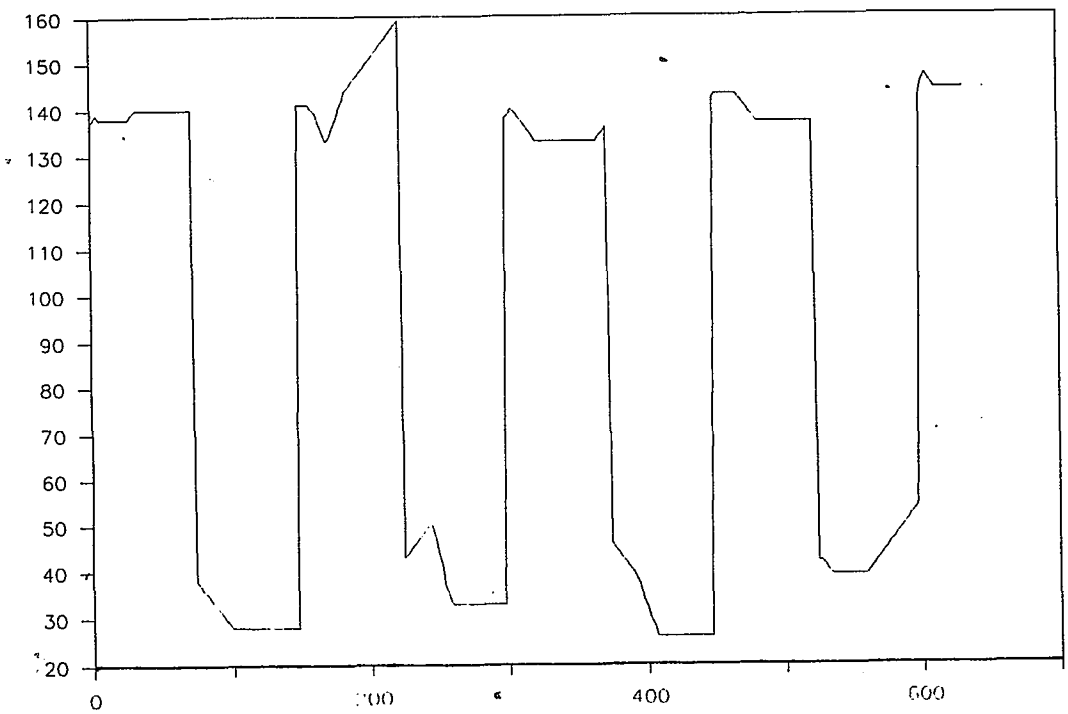

Figure 5.4: The output of the filter whose nonlinear element is given in fig. 5.3

input signal. In the following, another variation in the nonlinear element is given. Understanding of the mechanism of these filter is based on the previous argument of dividing the element into three regions. Also, these filters depend on some parameter whose value must be optimized relative to the input signal to obtain the best result.

Another possible nonlinear element that can be incorporated in the model is shown in fig. 5.5 below. This filter smooths out noise while shifting both edges by the same amounts.

Following a similar argument, this filter has a behavior similar to that of the Max filter in region III. This explains the shifting of the falling edge. In region II the filter behaves like the Min filter, thus preserving the falling edge in position while shifting the rising edge. The net result of the filter is the shifting of both edges in the same direction, opposite to that of the window skewness, thus preserving the

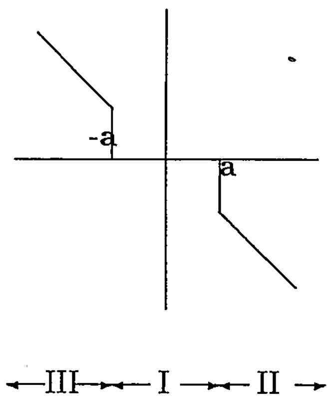

Figure 5.5: A Nonlinear element

pulse shape. In region I the filter response is the same as the one above, moving average filter. The output of this filter to the input shown in fig. 5.2 is given in fig. 5.6. The parameter **a** for the nonlinear element shown in fig. 5.5 was taken as 50.

A possible modification can be made in region I of figs. 5.3 and 5.5. Instead of using a linear filter in that region a nonlinear Max or Min can be used. To preserve the pulse shape the type of the filter to be used in region I should be complementary to that in the neighboring region. That is if region II or III is using a Max filter then region I should use a Min filter and vice versa. The only possible way to achieve this is to divide region I into two, one next to II and other next to III, and then use different filter types in these regions to satisfy the above criterion. The possible nonlinear elements are shown below.

The filter utilizing the nonlinear element shown in fig. 5.7 has been simulated, with the value of **a** set to 60. This filter did not introduce any edge shifting. This property can be interpreted by considering the nonlinear element in regions II and III. The corresponding output signal is shown in fig. 5.9.

On the other hand, the filter incorporating the element in fig. 5.8 shifts

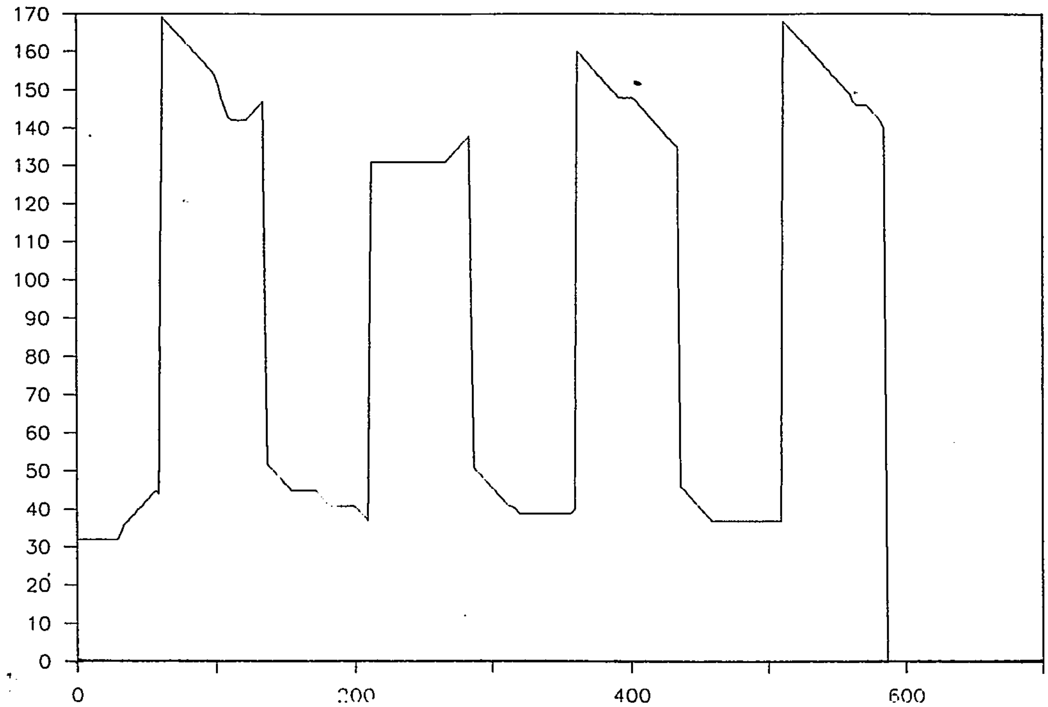

Figure 5.6: Output of the filter whose nonlinear element is shown in fig. 5.5.

both the rising and falling edges in the same direction. This is because the filter behaves like a Max filter region III and like a Min filter in II. The output signal for the above filter is shown in fig. 5.10 below. Both the filters using the elements shown in figs. 5.7 and 5.8, were simulated with the parameter a set to 60.

The author believes that a lot can be done in the area of optimization and development of new filter structures based on the model in fig. 5.1. A thorough study of this model could lead to new research projects in the area.

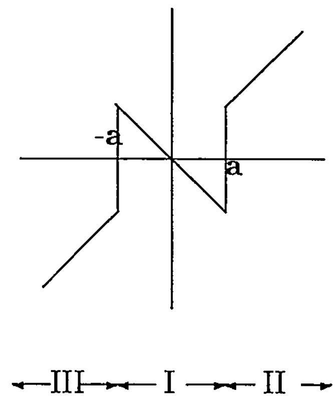

Figure 5.7: A nonlinear element based on fig. 5.3.

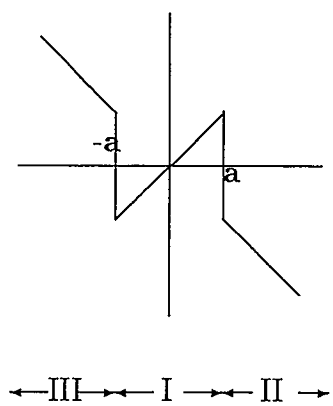

Figure 5.8: A nonlinear element based on fig. 5.5

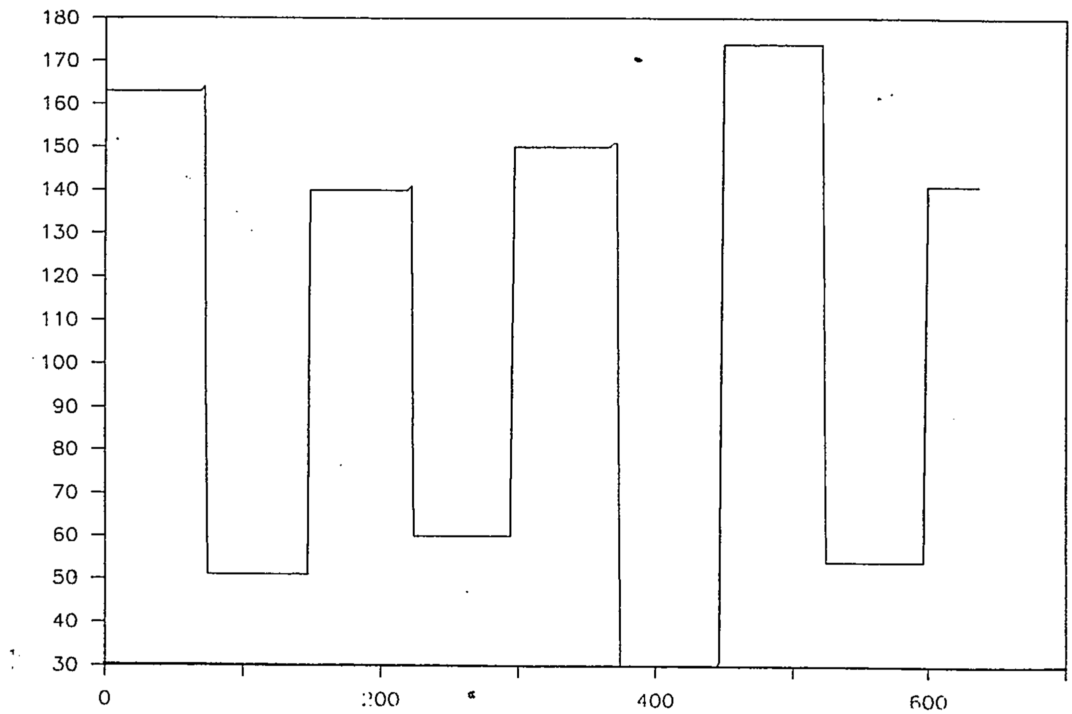

Figure 5.9: Output of filter whose nonlinear element is shown in fig. 5.7

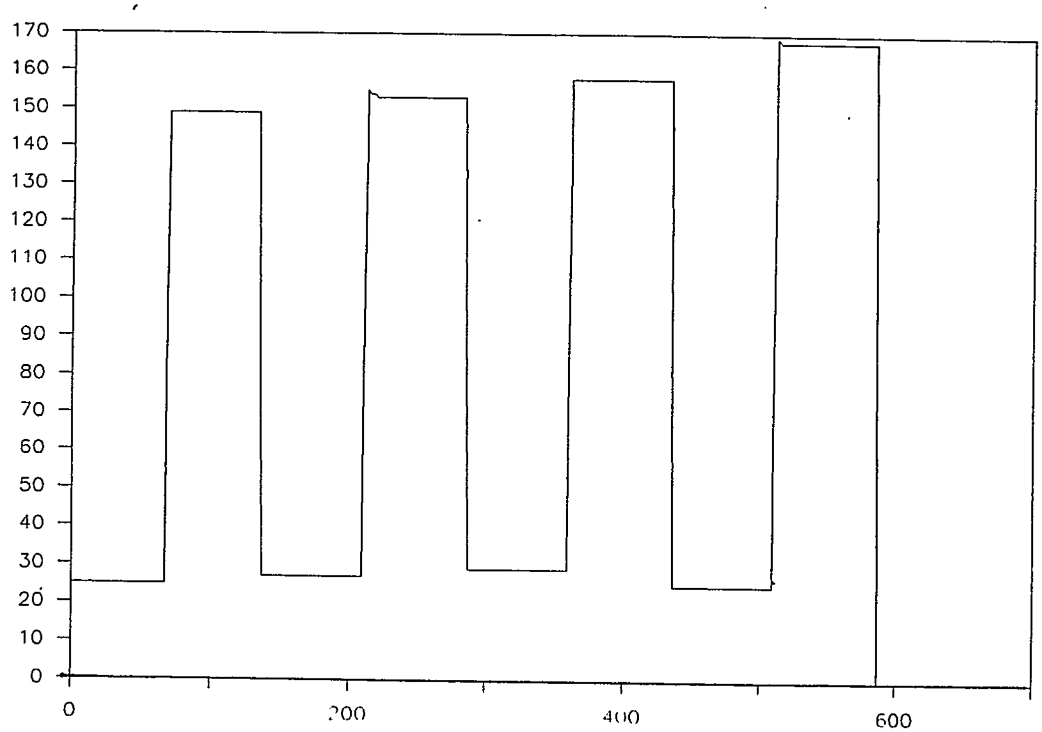

Figure 5.10: Output of filter whose nonlinear element is given in fig. 5.8

Chapter 6

CONCLUSION

A new nonlinear filter family, Max/Min, was developed. The properties of this filter family were investigated and its mechanism is now understood. Some of these properties are unique to the family, while others are typical for rank order based filters. An algorithm was developed to reduce a given combination of Max or Min filter with the same window size into a fewer number of filters with different window sizes. This algorithm can be used to enhance the processing time of any scheme using this family. The statistical and spectral properties of the Max/Min were also studied. It was found that they have a combing effect.

Two applications for the Max/Min filters were proposed, one in filtering and another as an edge detector. As a noise reduction filter, the Max/Min filter was found to have better performance than the classical median filter under severe noise conditions. The Max/Min filter also requires less number of comparisons compared to the median filter.

A model for the Max filter was introduced in chapter 5. This provides one possible way for the implementation of the filter. It also reveals the nonlinear nature of the filter. Based on this model, a number of modifications on the Max/Min were suggested. These were examined and did not have the edge shifting effect present in the Max/Min filter. These modifications were not studied extensively.

The Max/Min filter can be extended to two-dimensional signal case. Direct application of the Max/Min filter to images is not expected to be successful, as its edge shifting effect can remove important image details. Modifications are needed. For example, the modifications described in chapter 5 are expected to work better for images

The author believes that more research can be done on the Max/Min filter. The model given in chapter 5 is promising, as new nonlinear functions can be incorporated in it. Efforts should also be made towards the adaptivity of these new filters. With further study, more useful applications can be developed for the Max/Min filter family.

Appendix A

Distribution and Autocorrelation Function of Min Filter

A.1 Distribution Function

For two independent random variables U and V with CDF $F_U(u)$ and $F_V(v)$ respectively and pdf $f_U(u)$ and $f_V(v)$, the CDF of a random variable Z given by

$$Z = \min(U, V) \tag{A.1}$$

is

$$F_Z(z) = F_U(z) + F_V(z) - F_U(z)F_V(z)$$

and the pdf is

$$f_Z(z) = f_U(z) + f_V(z) - F_U(z)f_V(z) - f_U(z)F_V(z)$$

If U and V are independent identically distributed random variables, then

$$F_Z(z) = 2F_U(z) - F_U^2(z)$$

and

$$f_Z(z) = 2f_U(z) - 2f_U(z)F_U(z)$$

Let $X = \{X_i; i = \cdots . -2, -1, 0, 1, 2, \cdots\}$ be a sequence of identically distributed random variables. Input X to a Min filter window size $n+1$ and the output sequence be $Y = \{Y_i; i = \cdots, -2, -1, 0, 1, 2, \cdots\}$,where

$$Y_i \triangleq \min(X_i, X_{i+n})$$

Joint Probability Density Function

Following a similar reasoning as the one in chapter 4, it is readily seen that only the elements in Y on steps of n are dependent, while the others are independent, *i.e.* for $|j| \neq n$ and $|j| \neq n$

$$\begin{aligned} P\{Y_i \leq \alpha, Y_{i+j} \leq \beta\} &= P\{Y_i \leq \alpha\}P\{Y_{i+j} \leq \beta\} \\ &= F_X(\alpha)F_X(\beta)(F_X(\alpha)-2)(F_X(\beta)-2) \end{aligned} \tag{A.2}$$

For j=n, the joint probability is given by

$$\begin{aligned} & P\{Y_i \leq \alpha, Y_{i+n} \leq \beta\} \qquad\qquad (A.3) \\ = \;& 1 - P\{Y_i > \alpha\} - P\{Y_{i+n} > \beta\} + P\{Y_i > \alpha, Y_{i+n} > \beta\} \\ = \;& F_Y(\alpha) + F_Y(\beta) - 1 + P\{X_i > \alpha, X_{i+n} > \alpha, X_{i+n} > \beta, X_{i+2n} > \beta\} \\ = \;& F_X(\alpha)(F_X(\alpha)-2) + F_X(\beta)(F_X(\beta)-2) - 1 \\ & + P\{X_i > \alpha, X_{i+n} > \max(\alpha,\beta), X_{i+2n} > \beta\} \end{aligned}$$

For $\alpha \leq \beta$, the above equation becomes

$$P\{Y_i \leq \alpha, Y_{i+n} \leq \beta\} = F_X(\alpha) - F_X^2(\alpha) + 2F_X(\alpha)F_X(\beta) - F_X^2(\beta)F_X(\alpha) \tag{A.4}$$

Similarly for $\alpha > \beta$, the above equation becomes

$$P\{Y_i \leq \alpha, Y_{i+n} \leq \beta\} = F_X(\beta) - F_X^2(\beta) + 2F_X(\alpha)F_X(\beta) - F_X^2(\alpha)F_X(\beta) \tag{A.5}$$

For $|k| = n$, the joint pdf $f_{Y_iY_{i+n}}$is given by

$$f_{Y_iY_{i+n}}(\alpha,\beta) \tag{A.6}$$

$$
\begin{aligned}
&= \frac{\partial^2}{\partial\alpha\partial\beta} F_{Y_iY_{i+n}}(\alpha,\beta) \\
&= \begin{cases} \dfrac{\partial^2}{\partial\alpha\partial\beta} F_X(\alpha) - F_X^2(\alpha) + 2F_X(\alpha)F_X(\beta) - F_X^2(\beta)F_X(\alpha) & \text{for} \quad \alpha \leq \beta \\ \dfrac{\partial^2}{\partial\alpha\partial\beta} \left(F_X(\beta) + F_X(\alpha)\right)^2 + F_X^2(\alpha)\left(1 - F_X(\beta)\right) - 4F_X(\alpha) - 3F_X(\beta) & \text{for} \quad \alpha > \beta \end{cases} \\
&= \begin{cases} 2f_X(\alpha)f_X(\beta)(1 - F_X(\beta)) & \text{for} \quad \alpha < \beta \\ F_X(\beta) - F_X^2(\beta) + 2F_X(\alpha)F_X(\beta) - F_X^2(\alpha)F_X(\beta) & \text{for} \quad \alpha > \beta \\ f_X(\alpha)\left(F_X(\beta) - 1\right)\left(2F_X(\alpha) - F_X(\beta) - 1\right)\delta(\alpha - \beta) & \text{for} \quad \alpha = \beta \end{cases}
\end{aligned}
$$

Therefore the joint pdf is given by

$$
\begin{aligned}
&f_{Y_iY_{i+j}}(\alpha,\beta) \\
&= \begin{cases} \left\{ \begin{array}{lll} 2f_X(\alpha)f_X(\beta)(1 - F_X(\beta)) & \text{for} & \alpha < \beta \\ F_X(\beta) - F_X^2(\beta) + 2F_X(\alpha)F_X(\beta) - F_X^2(\alpha)F_X(\beta) & \text{for} & \alpha > \beta \\ f_X(\alpha)\left(F_X(\beta) - 1\right)\left(2F_X(\alpha) - F_X(\beta) - 1\right)\delta(\alpha - \beta) & \text{for} & \alpha = \beta \end{array} \right\} & \text{for } j = n \\ 4f_X(\alpha)f_X(\beta)(F_X(\alpha) - 1)(F_X(\beta) - 1) & \text{for } j \neq n \text{ an} \end{cases}
\end{aligned}
$$

For $j = 0$, the pdf is given by

$$f_{Y_i}(\alpha) = 2f_X(\alpha)(1 - F_X(\alpha))$$

Assume each sample X_i of the signal sequence $\{\cdots, X_{i-1}, X_i, X_{i+1}, \cdots\}$ has a value from the m-ary alphabet $\{\omega_0, \omega_1, \cdots, \omega_{m-1}\}$. Also assume that each of the symbols ω_j are equally likely. Now, the pdf and the CDF for the case of equally likely discrete input alphabets case are given by

$$
\begin{aligned}
f_X(x) &= \frac{1}{m}\sum_{i=0}^{m-1} \delta(x - \omega_i) \\
F_X(x) &= \frac{1}{m}\sum_{i=0}^{m-1} u(x - \omega_i)
\end{aligned}
$$

respectively. The CDF of the output samples of the Min filter is

$$
\begin{aligned}
F_Y(y) &= 2F_X(y) - F_X^2(y) \\
&= \frac{2}{m}\sum_{i=0}^{m-1} u(y - \omega_i) - \frac{1}{m^2}\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} u(y - \omega_i)u(y - \omega_j) \\
&= \frac{1}{m^2}\sum_{i=0}^{m-1} (2m - 2i - 1)u(y - \omega_i) \qquad \text{(A.7)}
\end{aligned}
$$

and the corresponding pdf is given by

$$f_Y(y) = \frac{1}{m^2}\sum_{i=0}^{m-1}(2m-2i-1)\delta(y-\omega_i) \tag{A.8}$$

A.2 Autocorrelation Function

Assume X_i's are independent indentically distributed random variables. The autocorrelation of the Min filter output for the case of equally likely input alphabet is derived below. The autocorrelation function R(k) of Y is given by

$$R(k) \triangleq E\{Y_i Y_{i+k}\}$$

Next consider three cases (i)$k=0$, (ii)$|k| \neq n$ and $|k| \neq 0|$ and (iii)$|k| = n$.

Case I $K = 0$,

the autocorrelation function is equal to the second order moment *i.e.*

$$\begin{aligned} R(0) &= E\{Y_i^2\} \\ &= \int_{-\infty}^{\infty} y^2 f_Y(y)dy \\ &= \int_{-\infty}^{\infty} y^2 \frac{1}{m^2}\sum_{i=0}^{m-1}(2m-2i-1)\delta(y-x_i)dy \\ &= \frac{1}{m^2}\sum_{i=0}^{m-1}(2m-2i-1)x_i^2 \end{aligned}$$

Case II $|k| \neq n$ and $|k| \neq 0$,

the autocorrelation function is given by

$$\begin{aligned} R(k) &= E\{Y_i, Y_{i+k}\} \\ &= E\{Y_i\}E\{Y_{i+k}\} \qquad /*\, Y_i \text{ and } Y_{i+k} \text{ being independent}*/ \end{aligned}$$

The autocorrelation function is then given by

$$R(k) = \left[\frac{1}{m^2}\sum_{i=0}^{m-1}(2m-2i-1)\int_{-\infty}^{\infty}\alpha\delta(\alpha-x_i)d\alpha\right]^2$$

$$= \left[\frac{1}{m^2}\sum_{i=0}^{m-1}(2m-2i-1)x_i\right]^2$$

Case III $|k| = n$,

the autocorrelation function is given by

$$\begin{aligned}
R(n) &= \int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha}^{\infty}\alpha\beta f_{Y_iY_{i+n}}(\alpha,\beta)d\alpha d\beta \\
&\quad + \int_{\alpha=-\infty}^{\infty}\int_{\beta=-\infty}^{\alpha}\alpha\beta f_{Y_iY_{i+n}}(\alpha,\beta)d\alpha d\beta \\
&= 2\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha}^{\infty}\alpha\beta 2f_X(\alpha)f_X(\beta)(1-F_X(\beta))d\beta d\alpha + \\
&\quad 2\int_{\alpha=-\infty}^{\infty}\int_{\beta=-\infty}^{\alpha}\alpha\beta 2f_X(\beta)f_X(\alpha)(1-F_X(\alpha))d\beta d\alpha + \\
&\quad \int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha^-}^{\alpha^+}\alpha\beta f_X(\alpha)\left(F_X(\beta)-1\right)\left(2F_X(\alpha)-F_X(\beta)-1\right)\delta(\alpha-\beta)d\beta d\alpha \\
&= T_1+T_2+T_3 \qquad \text{(A.9)}
\end{aligned}$$

where the terms T_1, T_2 and T_3 are given by

$$\begin{aligned}
T_1 &\triangleq 2\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha}^{\infty}\alpha\beta 2f_X(\alpha)f_X(\beta)(1-F_X(\beta))d\beta d\alpha \\
T_2 &\triangleq 2\int_{\alpha=-\infty}^{\infty}\int_{\beta=-\infty}^{\alpha}\alpha\beta 2f_X(\beta)f_X(\alpha)(1-F_X(\alpha))d\beta d\alpha \\
T_3 &\triangleq \int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha^-}^{\alpha^+}\alpha\beta f_X(\alpha)\left(F_X(\beta)-1\right)\left(2F_X(\alpha)-F_X(\beta)-1\right)\delta(\alpha-\beta)d\beta d\alpha
\end{aligned}$$

Consider the first term T_1

$$\begin{aligned}
&2\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha}^{\infty}\alpha\beta f_X(\alpha)f_X(\beta)\left(1-F_X(\beta)\right)d\beta d\alpha \\
&= 2\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha}^{\infty}\alpha\beta f_X(\alpha)f_X(\beta)d\beta d\alpha \\
&\quad -2\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha}^{\infty}\alpha\beta f_X(\alpha)f_X(\beta)F_X(\beta)d\beta d\alpha \\
&= T_{11}-T_{12}
\end{aligned}$$

where

$$\begin{aligned}
T_{11} &\triangleq 2\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha}^{\infty}\alpha\beta f_X(\alpha)f_X(\beta)d\beta d\alpha \\
T_{12} &\triangleq 2\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha}^{\infty}\alpha\beta f_X(\alpha)f_X(\beta)F_X(\beta)d\beta d\alpha
\end{aligned}$$

Consider each of the above terms separately,

$$
\begin{aligned}
T_{11} &= 2\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha}^{\infty}\alpha\beta f_X(\alpha)f_X(\beta)d\beta d\alpha \\
&= \frac{2}{m^2}\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha}^{\infty}\alpha\beta\sum_{i=0}^{m-1}\delta(\alpha-x_i)\sum_{j=0}^{m-1}\delta(\beta-x_j)d\beta d\alpha \\
&= \frac{2}{m^2}\int_{\alpha=-\infty}^{\infty}\sum_{i=0}^{m-1}x_i\delta(\alpha-x_i)\int_{\beta=\alpha}^{\infty}\sum_{j=0}^{m-1}x_j\delta(\beta-x_j)d\beta d\alpha \\
&= \frac{2}{m^2}\int_{\alpha=-\infty}^{\infty}\alpha\sum_{i=0}^{m-1}\delta(\alpha-x_i)g(\alpha)d\beta \\
&= \frac{2}{m^2}\sum_{i=0}^{m-1}x_i g(x_i)
\end{aligned}
$$

where,

$$
\begin{aligned}
g(\alpha) &= \int_{\beta=\alpha}^{\infty}\beta\sum_{l=0}^{m-1}\delta(\beta-x_l)d\beta \\
&= \sum_{l=0}^{m-1}x_l\delta(\beta-x_l)d\beta
\end{aligned}
$$

and

$$
\begin{aligned}
g(x_i) &= \int_{\beta=x_i}^{\infty}\sum_{l=0}^{m-1}x_l\delta(\beta-x_l)d\beta \\
&= \sum_{l=i+1}^{m-1}x_j+\frac{1}{2}x_i
\end{aligned}
$$

Thus T_{11} becomes

$$
T_{11} = \frac{2}{m^2}\sum_{i=0}^{m-1}x_i\left(\sum_{j=i+1}^{m-1}x_j+\frac{1}{2}x_i\right)
$$

Next consider the other term, T_{12}

$$
\begin{aligned}
T_{12} &= 2\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha}^{\infty}\alpha\beta f_X(\alpha)f_X(\beta)F_X(\beta)d\beta d\alpha \\
&= \frac{2}{m^3}\int_{\alpha=-\infty}^{\infty}\int_{\beta=\alpha}^{\infty}\alpha\beta\sum_{i=0}^{m-1}\delta(\alpha-x_i)\sum_{j=0}^{m-1}\left(j+\frac{1}{2}\right)\delta(\beta-x_j)d\beta d\alpha \\
&= \frac{2}{m^3}\int_{\alpha=-\infty}^{\infty}\alpha\sum_{i=0}^{m-1}\delta(\alpha-x_i)h(\alpha)d\alpha \\
&= \frac{2}{m^3}\sum_{i=0}^{m-1}x_i h(x_i)
\end{aligned}
$$

where

$$
\begin{aligned}
h(\alpha) &= \int_{\beta=\alpha}^{\infty} \beta \sum_{j=0}^{m-1} \left(j+\frac{1}{2}\right) \delta(\beta - x_j) d\beta \\
h(x_i) &= \int_{\beta=x_i}^{\infty} \sum_{j=0}^{m-1} x_j \left(j+\frac{1}{2}\right) \delta(\beta - x_j) d\beta \\
&= \sum_{j=i+1}^{m-1} \left(j+\frac{1}{2}\right) x_j + \frac{1}{2}\left(i+\frac{1}{2}\right) x_i
\end{aligned}
$$

Therefore T_{12} is given by

$$
T_{12} = \frac{2}{m^3} \sum_{i=0}^{m-1} x_i \left(\sum_{j=i+1}^{m-1} \left(j+\frac{1}{2}\right) x_j + \frac{1}{2}\left(i+\frac{1}{2}\right) x_i \right)
$$

Thus the term T_1 is given by

$$
T_1 = \frac{2}{m^3} \sum_{i=0}^{m-1} \left(\sum_{j=i+1}^{m-1} x_j \left(1 - \frac{2j+1}{2m}\right) + \frac{1}{2}\left(1 - \frac{2i+1}{2m}\right) x_i \right) \qquad \text{(A.10)}
$$

Similarly for the T_2 term in equation (A.9)

$$
\begin{aligned}
& 2\int_{\alpha=-\infty}^{\infty} \int_{\beta=-\infty}^{\alpha} \alpha\beta 2 f_X(\beta) f_X(\alpha)(1 - F_X(\alpha)) d\beta d\alpha \\
&= 2\int_{\alpha=-\infty}^{\infty} \int_{\beta=-\infty}^{\alpha} \alpha\beta f_X(\alpha) f_X(\beta) d\beta d\alpha \\
&\quad -2\int_{\alpha=-\infty}^{\infty} \int_{\beta=-\infty}^{\alpha} \alpha\beta f_X(\alpha) f_X(\beta) F_X(\alpha) d\beta d\alpha \\
&= T_{21} - T_{22}
\end{aligned}
$$

where

$$
\begin{aligned}
T_{21} &\triangleq 2\int_{\alpha=-\infty}^{\infty} \int_{\beta=-\infty}^{\alpha} \alpha\beta f_X(\alpha) f_X(\beta) d\beta d\alpha \\
T_{22} &\triangleq 2\int_{\alpha=-\infty}^{\infty} \int_{\beta=-\infty}^{\alpha} \alpha\beta f_X(\alpha) f_X(\beta) F_X(\alpha) d\beta d\alpha
\end{aligned}
$$

Next consider each term separately,

$$
T_{21} = \frac{2}{m^2} \int_{\alpha=-\infty}^{\infty} \int_{\beta=-\infty}^{\alpha} \alpha\beta \sum_{i=0}^{m-1} \delta(\alpha - x_i) \sum_{j=0}^{m-1} \delta(\beta - x_j) d\beta d\alpha
$$

$$
\begin{aligned}
&= \frac{2}{m^2}\int_{\alpha=-\infty}^{\infty}\sum_{i=0}^{m-1} x_i\delta(\alpha-x_i)\int_{\beta=-\infty}^{\alpha}\sum_{j=0}^{m-1} x_j\delta(\beta-x_j)d\beta d\alpha \\
&= \frac{2}{m^2}\int_{\alpha=-\infty}^{\infty}\sum_{i=0}^{m-1} x_i\delta(\alpha-x_i)a(\alpha)d\alpha \\
&= \frac{2}{m^2}\sum_{i=0}^{m-1} x_i a(x_i)
\end{aligned}
$$

where,

$$
a(\alpha) = \int_{\beta=-\infty}^{\alpha}\sum_{j=0}^{m-1} x_j\delta(\beta-x_j)d\beta
$$

and

$$
\begin{aligned}
a(x_i) &= \int_{\beta=-\infty}^{x_i}\sum_{j=0}^{m-1} x_j\delta(\beta-x_j)d\beta \\
&= \sum_{j=0}^{i-1} x_j + \frac{1}{2}x_i
\end{aligned}
$$

Thus T_{21} becomes,

$$
T_{21} = \frac{2}{m^2}\sum_{i=0}^{m-1} x_i\left(\sum_{j=0}^{i-1} x_j + \frac{1}{2}x_i\right)
$$

For T_{22},

$$
\begin{aligned}
T_{22} &= 2\int_{\alpha=-\infty}^{\infty}\int_{\beta=-\infty}^{\alpha}\alpha\beta F_X(\alpha)f_X(\alpha)f_X(\beta)d\alpha d\beta \\
&= \frac{2}{m^3}\int_{\alpha=-\infty}^{\infty}\alpha\sum_{i=0}^{m-1}\left(i+\frac{1}{2}\right)\delta(\alpha-x_i)\int_{\beta=-\infty}^{\alpha}\beta\sum_{j=0}^{m-1}\delta(\beta-x_j)d\beta d\alpha \\
&= \frac{2}{m^3}\int_{\alpha=-\infty}^{\infty}\sum_{i=0}^{m-1} x_i\left(i+\frac{1}{2}\right)\delta(\alpha-x_i)b(\alpha)d\alpha \\
&= \frac{2}{m^3}\sum_{i=0}^{m-1} x_i\left(i+\frac{1}{2}\right)b(x_i)
\end{aligned}
$$

where,

$$
b(\beta) = \int_{\beta=-\infty}^{\alpha}\beta\sum_{j=0}^{m-1}\delta(\beta-x_j)d\beta
$$

and,

$$
b(x_i) = \int_{\beta=-\infty}^{x_i}\sum_{j=0}^{m-1} x_j\delta(\beta-x_j)d\beta
$$

$$= \sum_{j=0}^{i-1} x_i + \frac{1}{2} x_i$$

Hence T_{22} is given by

$$T_{22} = \frac{2}{m^3} \sum_{j=0}^{m-1} \left(i + \frac{1}{2}\right) x_i \left(\sum_{j=0}^{i-1} x_j + \frac{1}{2} x_i\right)$$

Therefore the T_2 term,

$$T_2 = \frac{2}{m^2} \sum_{i=0}^{m-1} x_i \left(\sum_{j=0}^{i-1} x_j + \frac{1}{2} x_i\right) - \frac{2}{m^3} \sum_{j=0}^{m-1} x_i \left(i + \frac{1}{2}\right) \left(\sum_{j=0}^{i-1} x_j + \frac{1}{2} x_i\right) \quad \text{(A.11)}$$

Finally consider T_3

$$\begin{aligned}
& \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta f_X(\alpha) \left(F_X(\beta) - 1\right) \left(2F_X(\alpha) - F_X(\beta) - 1\right) \delta(\alpha - \beta) d\beta d\alpha \\
= \; & 2 \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta f_X(\alpha) F_X(\alpha) F_X(\beta) \delta(\alpha - \beta) d\alpha d\beta \\
& - \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta f_X(\alpha) F_X^2(\beta) \delta(\beta - \alpha) d\alpha d\beta \\
& -2 \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta f_X(\alpha) F_X(\alpha) \delta(\beta - \alpha) d\beta d\alpha \\
& + \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta f_X(\alpha) \delta(\beta - \alpha) d\alpha d\beta \\
= \; & T_{31} + T_{32} + T_{33} + T_{34}
\end{aligned}$$

where,

$$\begin{aligned}
T_{31} & \triangleq 2 \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta f_X(\alpha) F_X(\alpha) F_X(\beta) \delta(\alpha - \beta) d\alpha d\beta \\
T_{32} & \triangleq \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta f_X(\alpha) F_X^2(\beta) \delta(\beta - \alpha) d\alpha d\beta \\
T_{33} & \triangleq -2 \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta f_X(\alpha) F_X(\alpha) \delta(\beta - \alpha) d\beta d\alpha \\
T_{34} & \triangleq \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta f_X(\alpha) \delta(\beta - \alpha) d\alpha d\beta
\end{aligned}$$

Consider each term separately,

$$T_{31} = \frac{2}{m^3} \int_{\alpha=-\infty}^{\infty} \int_{\alpha^-}^{\alpha^+} \alpha\beta \sum_{i=0}^{m-1} \left(i + \frac{1}{2}\right) \delta(\alpha - x_i) \sum_{j=0}^{m-1} u(\beta - x_j) \delta(\beta - \alpha) d\alpha d\beta$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\alpha=-\infty}^{\infty} \alpha \sum_{i=0}^{m-1}\left(i+\frac{1}{2}\right)\delta(\alpha-x_i)\int_{\beta=\alpha^-}^{\alpha^+}\beta\sum_{j=0}^{m-1}u(\beta-x_j)\delta(\beta-\alpha)d\beta d\alpha \\
&= \frac{1}{2}\int_{\alpha=-\infty}^{\infty}\sum_{i=0}^{m-1}\alpha\left(i+\frac{1}{2}\right)r(\alpha)d\alpha \\
&= \frac{1}{2}\sum_{i=0}^{m-1}\left(i+\frac{1}{2}\right)x_i r(x_i)
\end{aligned}
$$

where,

$$
\begin{aligned}
r(\alpha) &= \int_{\beta=\alpha^-}^{\alpha^+}\beta\sum_{j=0}^{m-1}u(\beta-x_j)\delta(\beta-\alpha)d\beta \\
r(x_i) &= \int_{x_i^-}^{x_i^+}\sum_{j=0}^{m-1}\beta u(\beta-x_j)\delta(\beta-x_i)d\beta \\
&= \sum_{j=0}^{i-1}x_i+\frac{1}{2}x_i \\
&= \left(i+\frac{1}{2}\right)x_i
\end{aligned}
$$

Therefore T_{31} is given by,

$$
T_{31} = \frac{2}{m^3}\sum_{i=0}^{m-1}\left(i+\frac{1}{2}\right)^2 x_i^2
$$

For T_{32},

$$
\begin{aligned}
T_{32} &= \frac{1}{m^3}\int_{\alpha=-\infty}^{\infty}\int_{\alpha^-}^{\alpha^+}\alpha\beta\sum_{i=0}^{m-1}\delta(\alpha-x_i)\sum_{j=0}^{m-1}(2j+1)u(\beta-x_j)\delta(\beta-\alpha)d\alpha \\
&= \frac{1}{m^3}\int_{\alpha=-\infty}^{\infty}\alpha\sum_{i=0}^{m-1}\delta(\alpha-x_i)\int_{\alpha^-}^{\alpha^+}\beta\sum_{j=0}^{m-1}(2j+1)u(\beta-x_j)\delta(\beta-\alpha)d\beta \\
&= \frac{1}{m^3}\int_{-\infty}^{\infty}\alpha\sum_{i=0}^{m-1}\delta(\alpha-x_i)p(\alpha)d\alpha \\
&= \frac{1}{m^3}\sum_{i=0}^{m-1}x_i p(x_i)
\end{aligned}
$$

where $p(\alpha)$ and $p(x_i)$

$$
\begin{aligned}
p(\alpha) &= \int_{\alpha^-}^{\alpha^+}\beta\sum_{j=0}^{m-1}(2j+1)u(\beta-x_j)\delta(\beta-\alpha)d\beta \\
p(x_i) &= \int_{x_i^-}^{x_i^+}\sum_{j=0}^{m-1}(2j+1)x_i u(\beta-x_j)\delta(\beta-x_i)d\beta
\end{aligned}
$$

$$= \left(i^2 + i + \frac{1}{2}\right) x_i$$

Thus T_{32} is given by

$$T_{32} = \frac{1}{m^3} \sum_{i=0}^{m-1} \left(i^2 + i + \frac{1}{2}\right) x_i^2$$

Next consider T_{33}

$$\begin{aligned} T_{33} &= \frac{2}{m^3} \int_{\alpha=-\infty}^{\infty} \int_{\beta=\alpha^-}^{\alpha^+} \alpha\beta \sum_{i=0}^{m-1} \delta(\alpha - x_i) \sum_{j=0}^{m-1} u(\alpha - x_j) \delta(\beta - \alpha) d\beta d\alpha \\ &= \frac{2}{m^3} \int_{-\infty}^{infty} \alpha^2 \sum_{i=0}^{m-1} \left(i + \frac{1}{2}\right) \delta(\alpha - x_i) d\alpha \\ &= \frac{2}{m^3} \sum_{i=0}^{m-1} \left(i + \frac{1}{2}\right) x_i^2 \end{aligned}$$

Finally T_{34} is given by,

$$\begin{aligned} T_{34} &= \frac{1}{m} \int_{\alpha=-\infty}^{\infty} \int_{\alpha^-}^{\alpha^+} \alpha\beta \sum_{i=0}^{m-1} \delta(\alpha - x_i) \delta(\beta - \alpha) d\beta\alpha \\ &= \frac{1}{m} \sum_{i=0}^{m-1} x_i^2 \end{aligned}$$

Combining T_{31}, T_{32}, T_{33} and T_{34} T_3 becomes

$$T_3 = \frac{1}{m^3} \sum_{i=0}^{m-1} \left(i(i+1) - 2m \left(i + \frac{1}{2}\right) + m^2 \right) x_i^2 \tag{A.12}$$

Substituting equations (A.10), (A.11) and (A.12) in (A.9)

$$R(k) = \left[\frac{1}{m^2} \sum_{i=0}^{m-1} (2m - 2i - 1) x_i \right]^2$$

For $k = n$ the autocorelation function is given by

$$\begin{aligned} R(n) &= \frac{2}{m^3} \sum_{i=0}^{m-1} \left(\sum_{j=i+1}^{m-1} x_j \left(1 - \frac{2j+1}{2m}\right) + \frac{1}{2} \left(1 - \frac{2i+1}{2m}\right) x_i \right) \\ &+ \frac{2}{m^2} \sum_{i=0}^{m-1} x_i \left(\sum_{j=0}^{i-1} x_j + \frac{1}{2} x_i \right) - \frac{2}{m^3} \sum_{i=0}^{m-1} \left(i + \frac{1}{2}\right) x_i \left(\sum_{j=0}^{i-1} x_j + \frac{1}{2} x_i \right) \\ &+ \frac{1}{m^3} \sum_{i=0}^{m-1} \left(i(i+1) - 2m \left(i + \frac{1}{2}\right) + m^2 \right) x_i^2 \end{aligned} \tag{A.13}$$

Appendix B

Threshold Decomposition for Max/Min Filter

Since Max and Min filters are related only the Max filter will be considered here. Assume input samples x_i, are quantized to one of k integer values $0, 1, \cdots, k-1$. Threshold decomposition maps the sequence $\{x_i\}$ into binary sequences $\{t^j(x_i)\}$. The elements $t^j(x_i)$ of the sequences are given by, for $1 \leq j \leq m-1$

$$t^j(x_i) = \begin{cases} 1 & \text{if } x_i \geq j \\ 0 & \text{otherwise} \end{cases}$$

where j is called the threshold level. Filtering of these binary sequences is represented by

$$\max_n t^j(x_i) = \max\left(t^j(x_i), t^j(x_{i+n})\right) = t^j_{\max}(x_i)$$

.

Three properties which were derived for the threshold decomposition of rank order operation are given in [29]. Since max is a rank order operation, then it exhibits these properties.

Property 1

If the filtered threshold value is 1 at a certain level j, $t^j_{\max}(x_i) = 1$, then it will be

1 at all levels less than j. This can be represented mathematically as follows

$$\text{If } t^j_{\max}(x_i) = 1 \text{ then}$$

$$t^j_{\max}(x_i) = 1 \text{ for all } 1 \leq k \leq j$$

Proof:

$$\begin{aligned}
t^j_{\max}(x_i) &= \max\left(t^j(x_i), t^j(x_{i+n})\right) \\
&= \begin{cases} 1 & \text{if either } t^j(x_i) \text{ or } t^j(x_{i+n}) = 1 \\ 0 & \text{if both } t^j(x_i) \text{ and } t^j(y_{i+n}) = 0 \end{cases} \\
&= I(\text{at least one of } t^j(x_i), t^j(x_{i+n}) \text{ equals } 1) \\
&= I(\text{at least one of } x_i, x_{i+n} \text{ not smaller then } j)
\end{aligned}$$

the function I() is the indicator functon.

If

$$t^j_{\max}(x_i) = 1,$$

then

$$I(\text{ at least one of } (x_i, x_{i+n}) \geq j) = 1$$

implies

$$I(\text{ at least one of } (x_i, x_{i+n}) \geq k) = 1 \text{ for } 1 \leq k \leq j.$$

That is

$$t^k_{\max}(x_i) = 1$$

Any integer X with $0 \leq X \leq m-1$ can be represented in terms of the indicator function as follow

$$X = \sum_{j=1}^{m-1} I(X \geq j)$$

This point is useful in proving the secon property **Property 2**

The set of filtered sequences $\{t^j_{\max}(x_i)\}$ is related to output sequence $\{y_i\}$ by

$$y_i = \sum_{j=1}^{k-1} t^j_{\max}(x_i)$$

Proof:

From the definition of the Max filter the output sample y_i is given by

$$\begin{aligned} y_i &= \max(x_i, x_{i+n}) \\ &= \sum_{j=1}^{k-1} I(\max(x_i, x_{i+n}) \geq j) \\ &= \sum_{j=1}^{k-1} I(\text{ at least one of } x_i, x_{i+n} \text{ not smaller than } j) \\ &= \sum_{j=1}^{k-1} t^j_{\max}(x_i) \end{aligned}$$

The above property gives the function that reverses the threshold decomposition.

Property 3

The binary sequences obtained by threshold decomposing the output sequence of a Max filter is identical to that obtained by filtering the sequences of the input. That is

$$t^j(y_i) = t^j_{\max}(x_i)$$

Proof:

$$\begin{aligned} t^j(y_i) &= \begin{cases} 1 & \text{if } y_i \geq j \\ 0 & \text{if } y_i < j \end{cases} \\ &= \begin{cases} 1 & \text{if } \sum_{j=1}^{k-1} t^j_{\max}(x_i) \geq j \text{ from property 2} \\ 0 & \text{else} \end{cases} \end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} 1 & \text{if } t^j_{\max}(x_i) = 1 \\ 0 & \text{else} \end{cases} \\
&= t^j_{\max}(x_i)
\end{aligned}
$$

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