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A PC based computer model with a postprocessor for the movement of groundwater

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2) A PC BASED COMPUTER MODEL WITH A POSTPROCESSOR FOR THE MOVEMENT OF GROUNDWATER

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ABSTRACT

A PC BASED COMPUTER MODEL WITH A POSTPROCESSOR FOR THE MOVEMENT OF GROUNDWATER

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Master of Science in Civil Engineering, December 1990

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Currently available computer programs that model groundwater flow are inherently limited in their capabilities, in one way or another. Additionally, many of these computer programs are relatively inaccessible to the general user because they operate off of mainframe computers and sophisticated plotters. As a result, a simple to use computer program called 2DFLOW has been modified so it can be run on a personal IBM compatible computer. It models steady-state fluid flow through saturated porous media. In addition, a postprocessor called 2DPLOT has been developed. The postprocessor generates the equipotential contours and the flux vectors that describe the physics of the flow, as well as the flow geometry. 2DPLOT can be used with most laser or dot matrix printers that support graphics.

In the computer program 2DFLOW, Laplace's equation is solved by using the finite element method. The domain of the problem is divided into elements, and the program calculates the total hydraulic head at the element nodes and the flow velocities at the center of each element.

The program is capable of generating flow nets for any saturated flow domain, regardless of its shape or geometry. It is also capable of modeling flow under any boundary condition, ie. constant hydraulic head or varying hydraulic head along any boundary. Other features of the program are mentioned in the user manual provided within the contents of this treatise. Also included, are several examples of flow nets generated by the program package 2DFLOW/2DPLOT.

INTRODUCTION

There are several computer programs available to the general user for solving problems of steady state fluid flow through porous media. Most of these computer programs use the finite difference method to solve Laplace's equation in two dimensions. As a result, they are inherently limited in one way or another in their capabilities. Most finite difference programs can only solve the simplest groundwater flow systems, such as the computer program FLOWNS (5). FLOWNS can only generate flow nets for regular rectangular areas and can not readily solve problems involving phreatic surfaces.

In general, the limitations of the finite difference method, as utilized in computer programs like FLOWNS, are the following:

1) It is applicable only to square elements. That is, the problem domain must be defined by a mesh of square elements. Consequently, the finite element method works best with regular boundaries.
2) The porous media must be homogeneous. It can not have any irregularities in it.
3) It can not solve problems with boundaries that are curved or that are defined on a circular arc. An example of such a boundary would be the phreatic

surface in an earth dam.

4) It does not have the capability to directly solve problem domains containing anisotropic media. In the finite difference method, anisotropic media can be handled only indirectly by scaling down the domain into an equivalent isotropic domain. However, this results in a distorted view of the problem at hand.

The above conditions, if encountered, can create a real problem in the finite difference method. Conversely, these same factors can easily be accounted for in the finite element method. As a result, the finite element method appears to be the best numerical method to solve Laplace's equation for steady state fluid flow through porous media. However, there are very few computer programs available that utilize the finite element method to generate computer drawn flow nets. Also, most of these computer programs operate off of mainframe computers like the VAX and UNIVAX systems and require sophisticated plotters. Therefore, they are relatively inaccessable to the general user.

As a result, a finite element computer program that can be run on a personal microcomputer and a conventional laser or dot matrix printer would surely be welcomed. Today in the 1990's, advancing computer technology has made the personal microcomputer relatively cheap and readily available. They are compact and as a result, can be conveniently used at home, at the office, or at school without taking up much space or resources.

For the very reasons outlined above, a computer program called 2DFLOW has been downgraded from a mainframe computer system to work on a personal IBM compatible computer. In addition, a postprocessor and a plotter program called 2DPLOT has been developed to generate the mesh for a given problem domain, as well as the finished flow net. 2DPLOT does not generate a traditional flow net consisting of equipotential lines and flow lines. Instead, 2DPLOT generates separate plots of the equipotential contours and flux vectors. The advantage of using flux vectors is that they show the relative magnitude and direction of the flow in any given element of the problem domain. 2DPLOT can be used with most laser or dot matrix printers that support graphics.

The uses of the computer package 2DFLOW/2DPLOT are numerous. It can be an invaluable tool in analyzing, predicting, and managing groundwater flow. The software package 2DFLOW/2DPLOT can be used to help design dams, cofferdams, concrete curtains, contaminant containment facilities, and any other structure whose design depends on the flow of water through it or around it. Flow nets generated by the program package 2DFLOW/2DPLOT can help determine the following:

1) Predict the flow of water through a dam or any structure built out of porous material.
2) Predict the flow of water through each layer of a multi-layer aquifer system.
3) Predict the regional flow of water, as well as the local flow of water through a given cross section.
4) Predict the extent of contamination downstream of a pollution source.

Another valuable use of 2DFLOW/2DPLOT is as an academic aid. By studying different examples and cases, the student can gain a better understanding and an appreciation of the complex nature of groundwater flow and the factors that influence it.

BACKGROUND ON GROUNDWATER MODELING

For many years, engineers, hydrogeologists, and groundwater hydrologists have used various types of models to simulate how groundwater moves in the subsurface environment. A model is a tool designed to represent a simplified verson of reality. Consequently, groundwater models can be a valuable tool in the management of groundwater resources. Driscoll (12) states that models are useful 1) for studies that precede field investigations, 2) for the interpretive analysis following the field program, and 3) as a predictive tool to estimate how water quality may change in the aquifer.

Several types of models have been used to study groundwater flow systems. There are: 1) hand drawn models, 2) sand tank models, 3) analog models, and 4) analytical and numerical types of mathematical models.

Flow nets are painstakingly difficult to construct by hand because the equipotential lines and the flow lines must be drawn so that they are perpendicular to each other and so that they form curvilinear squares. Additional requirements that must be met to produce a fairly reasonable model of groundwater flow include the following:

1) The same quantity of seepage must flow between adja-

cent pairs of flow lines.

2) Adjacent equipotential lines must have equal head loss.

3) The quantity of seepage flowing through an element must be constant throughout the element.

4) A deflection rule must be followed in passing from a material of one hydraulic conductivity to a material of a different hydraulic conductivity.

Additionally, the complexity of the problem increases when irregular boundaries and two or more porous materials of different hydraulic conductivity are to be considered. Drawing by hand an accurate model of the groundwater flow of such a complex system is challenging and time consuming, to say the very least.

A sand model consists of a tank filled with an unconsolidated porous material through which water is induced to flow. A major drawback of sand tank models is the problem of scaling down a field situation to the dimensions of a laboratory model. Phenomena measured in a sand tank model are often different from conditions observed in the field. As a result, the sand tank model is best applicable to problems with simple and well defined boundary conditions.

The two types of analog models used most frequently in groundwater modeling are viscous fluid anolog models and electrical analog models. Viscous fluid models are known as Hele-Shaw models or parallel plate models. In these models, a viscous fluid such as oil is made to flow between two closely spaced parallel plates, which may be oriented either vertically or horizontally. These viscous models represent a vertical or horizontal cross section through an aquifer. As with the sand tank model, only simple problems can be modeled using this technique.

Electrical anolog models consist of boards wired with electrical networks of resistors and capacitors. They work according to the principle that flow of groundwater is analogous to the flow of electricity. Darcy's law for groundwater flow and Ohm's law for electricity are similar. Changes in voltage in an electrical analog are similar to changes in hydraulic head. The major drawback of the electrical anolog model is that a specific model must be designed for each aquifer system to be studied.

Mathematical models consist of a solution to a set of differential equations that govern the flow of groundwater. However, assumptions that simplify the problem must be made in order to construct a mathematical model because the field situation is normally too complicated to be simulated exactly. Analytical solutions require prior knowledge of any nonhomogeneity and anisotropy in the porous medium.

To deal with more realistic situations, it is usually necessary to solve the mathematical model approximately using numerical methods. However, numerical solutions to practical multi-layer cross sections can be extremely complex and difficult to solve by hand. Luckily, today's personal computers are capable of solving these numerous complex equations quickly and easily.

HISTORY OF NUMERICAL METHODS

Numerical solutions involve approximating a continuous partial differential equation with a set of discrete equations. In other words, continuous variables are replaced with discrete variables that are defined at a selected, finite number of points. These variables are defined by a finite number of algebraic equations that describe a certain parameter, such as hydraulic head.

As the number of points increase in any given problem domain, the numerical solution approaches the analytical solution to the problem. Two general types of numerical models exist: 1) finite difference method (FDM), and 2) finite element method (FEM).

The finite difference method was the first used. The essential feature of the finite difference method is that a derivative is replaced by an algebraic expression relating the differences between values located a finite distance apart. These finite difference approximations are then substituted into the governing Laplace differential equation. This leads to a set of simultaneous equations, with the values of dependent variables as unknowns.

The finite difference method utilizes a grid system of points separated from each other by a finite distance. Each point is specified by an integer pair (i,j) to represent its location on the grid system. The ordered pair (1,1) is located in the upper left hand corner as can be seen in Illustration 1 below. Values of i increase in the positive x-direction and values of j increase in the negative y-direction.

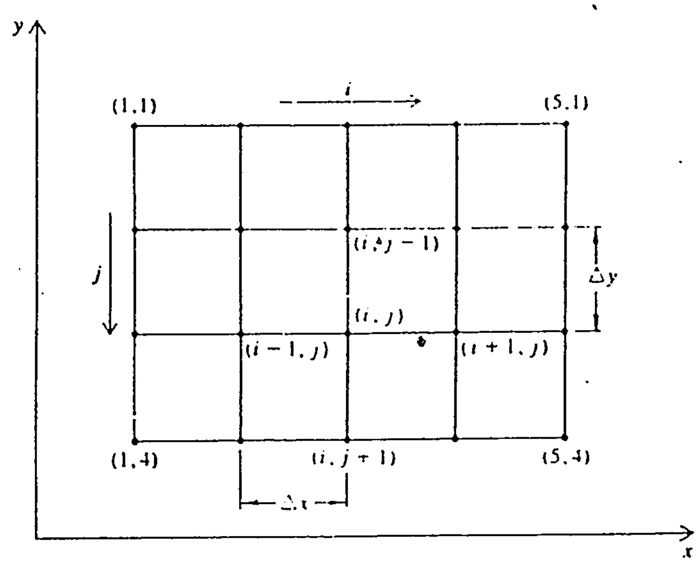


Illustration 1: Finite difference grid showing index numbering convention

The value of the hydraulic head at any point (i,j) is indicated as $h_{i,j}$. In the finite difference approximation, derivatives are replaced by differences taken between two node points. Therefore, an approximation to the partial derivative $\partial^2 h/\partial x^2$ at a point (x,y) is obtained by approximating the first derivative at (x+dx/2, y) and at (x-dx/2, y). The second derivative is obtained by taking the difference between the first derivatives at these two points. That is:

at (x+dx/2, y)

$$\frac{dh}{dx} = \frac{h(i+1,j)-h(i,j)}{dx} \qquad (1)$$

at (x-dx/2, y)

$$\frac{dh}{dx} = \frac{h(i-1,j)-h(i,j)}{dx} \qquad (2)$$

Therefore, at (x,y)

$$\frac{\partial^2 h}{\partial x^2} = \frac{\frac{h(i+1,j)-h(i,j)}{dx} - \frac{h(i,j)-h(i-1,j)}{dx}}{dx} \qquad (3)$$

which simplifies to

$$\frac{\partial^2 h}{\partial x^2} = \frac{h(i-1,j)-2h(i,j)+h(i+1,j)}{(dx)^2} \qquad (4)$$

Similarly

$$\frac{\partial^2 h}{\partial y^2} = \frac{h(i,j-1)-2h(i,j)+h(i,j+1)}{(dy)^2} \qquad (5)$$

Equations (4) and (5) are then substituted into Laplace's equation which takes the form:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \qquad (6)$$

As a result, if dx = dy, the finite difference approximation for the hydraulic head at a point (i,j) is:

$$h(i,j) = \frac{h(i-1,j)+h(i,j-1)+h(i+1,j)+h(i,j+1)}{4} \qquad (7)$$

The advantage of the finite difference method is that its resulting equations require less time to assemble and solve. As a result, the computation time is greatly reduced. However, the finite difference method works best with a regular rectangular spacing of nodes. Therefore, it is less able to deal with irregular geometries than is the finite element method. In comparison, the finite element method can easily handle triangular and equilateral elements, whereas the finite difference method can not. As a result, the mesh generated by the finite element method will more closely define the actual problem domain than will the mesh generated by the finite difference method. See Illustration 2 on the following page. In general, finite difference methods have the advantage of speed and the disadvantage of limited versatility.

As a result, the best numerical method appears to be the finite element method. In this method, both the total hydraulic head and the stream function are formulated so as to satisfy Laplace's equation in the x and y direction.

Further explanation of this method will be given in the sections that follow.

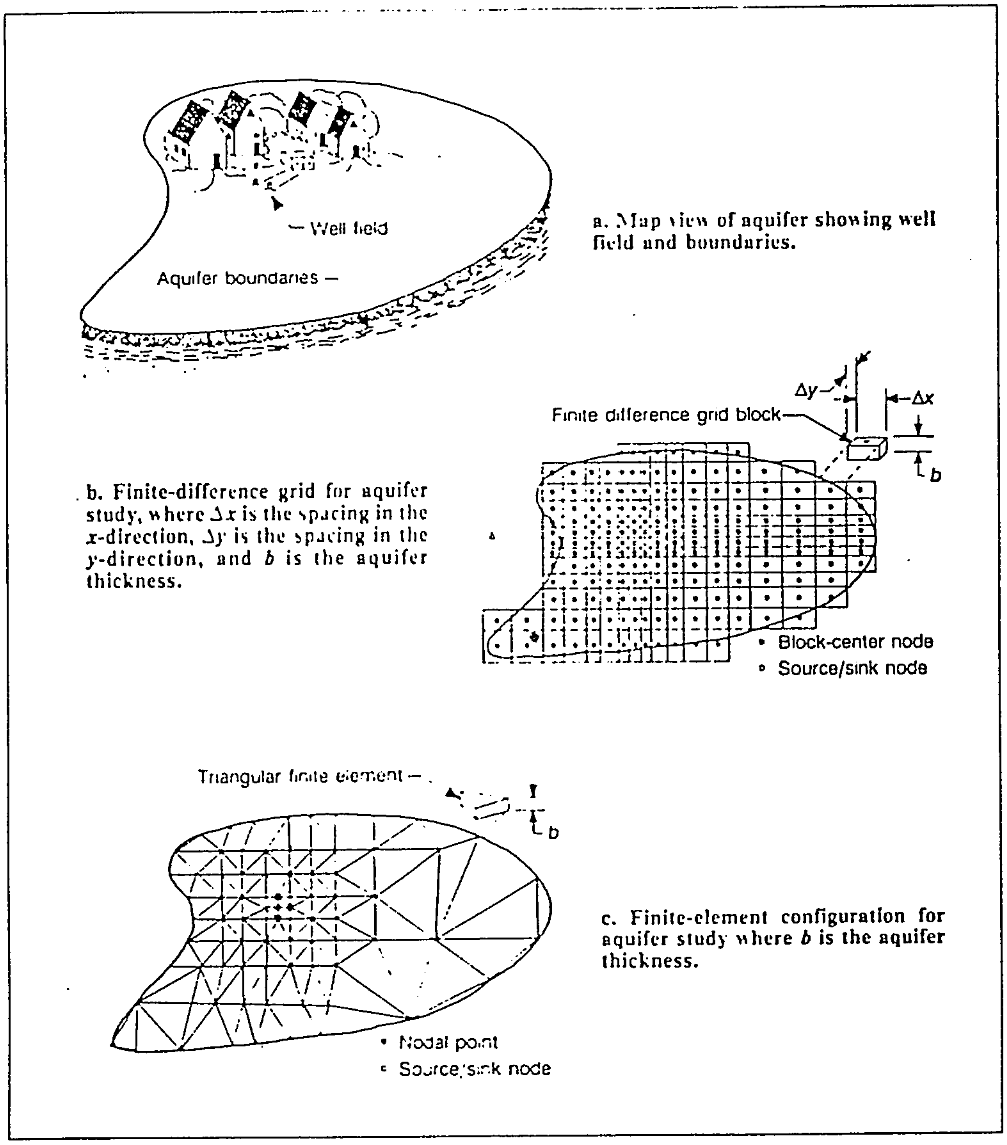


Illustration 2: Two-dimensional grid system for the finite difference and finite element methods

GOVERNING FLOW EQUATIONS

Laplace's partial differential equation describing fluid flow through porous media is the result of a combination of the continuity equation from fluid mechanics and Darcy's law in three dimensions. It is assumed that flow is occuring through an isotropic, homogeneous porous medium under steady-state conditons.

DARCY'S LAW:

The specific discharge (q), also known as the Darcy velocity, is the volume rate of flow per unit area. It is defined as:

$$q = \frac{Q}{A} \qquad (8)$$

Darcy's law is defined as:

$$Q = K\,i\,A \qquad (9)$$

where K = hydraulic conductivity

i = hydraulic gradient

A = cross sectional area

The hydraulic gradient is defined as:

$$i = \frac{dh}{dl} \qquad (10)$$

Equations (8), (9), and (10) can then be combined to write Darcy's law in the differential form:

$$q = -K \frac{dh}{dl} \qquad (11)$$

Equation (11) is true for each of the x, y, and z components of flow. As a result, Darcy's law in three dimensions can be written as:

$$q_x = -K \frac{\partial h}{\partial x} \qquad q_y = -K \frac{\partial h}{\partial y} \qquad q_z = -K \frac{\partial h}{\partial z} \qquad (12)$$

The negative sign indicates that the flow of water is in the direction of decreasing head.

CONTINUITY EQUATION:

For steady-state conditions, continuity requires that the amount of water flowing into a given volume be equal to the amount of water flowing out. Two assumptions are made. First, water is assumed to be incompressible. Both mass and volume are conserved for an incompressible fluid. Second, the volume contains no sources or sinks. That is, no water is allowed to be added or removed from a given volume by an outside force.

Consider the flow into and out of a unit volume of porous media as shown below in Illustration 3. The cube is defined by the length of its sides, dx, dy, and dz. Its volume is defined by $dV = dxdydz$. For steady-state conditions, the law of conservation of volume states that:

Volume In = Volume Out (13)

As seen in Illustration 3, q_x(out) is different from q_x(in) by an amount equal to $(\partial q_x/\partial x)dx$. Therefore, the net change in the discharge rate in the x-directon is equal to $(\partial q_x/\partial x)dx(dydz)$ or simply $(\partial q_x/\partial x)dV$. Similarly, in the y and z-directions, the net change in the discharge rate is equal to $(\partial q_y/\partial y)dV$ and $(\partial q_z/\partial z)dV$ repectively. From the law of conservation of volume:

$$(\partial q_x/\partial x)dV + (\partial q_y/\partial y)dV + (\partial q_z/\partial z)dV = 0 \quad (14)$$

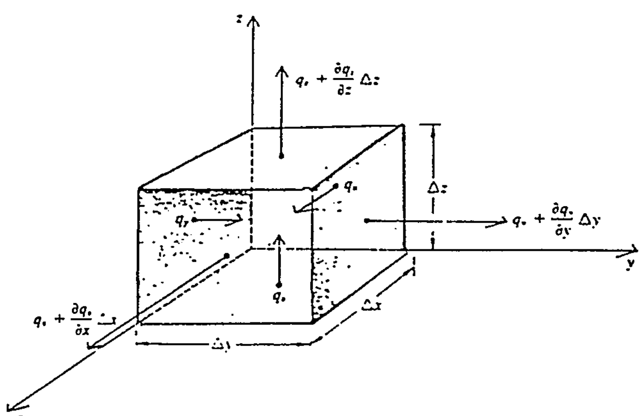


Illustration 3: Analysis of flow into and out of a unit volume of porous media

By dividing through by the quantity dV, the continuity equation for steady-state conditions is attained:

$$(\partial q_x/\partial x) + (\partial q_y/\partial y) + (\partial q_z/\partial z) = 0 \qquad (15)$$

LAPLACE'S EQUATION:

Laplace's equation combines Darcy's law and continuity equation into a single partial differential equation. Substitution of equations (12) into equation (15) results in:

$$\frac{\partial}{\partial x}(-K_x)\frac{\partial h}{\partial x} + \frac{\partial}{\partial y}(-K_y)\frac{\partial h}{\partial y} + \frac{\partial}{\partial z}(-K_z)\frac{\partial h}{\partial z} = 0 \qquad (16)$$

For an isotropic, homogeneous medium, the hydraulic conductivity (K) is equal in every direction. As a result, Laplace's equation for flow through an isotropic, homogeneous porous medium under steady-state conditions reduces to:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0 \qquad (17)$$

THEORY BEHIND THE FINITE ELEMENT METHOD AS USED IN THE PROGRAM PACKAGE 2DFLOW/2DPLOT

In the computer program 2DFLOW, the flow domain is divided into a mesh of quadrilateral elements and equation (16) is solved in two dimensions. Each quadrilateral element is defined by four nodes, one at each corner. These nodes are the points within the problem domain, at which the hydraulic head potentials are computed. The hydraulic head potential within each element is defined in terms of the nodal values by basis functions.

The use of basis functions to define the hydraulic head throughout the problem domain distinguishes the finite element method from the finite difference method. In the finite difference method, the hydraulic head is defined only at the nodal points themselves. In the finite element method, the hydraulic head is defined throughout the problem domain. This permits application of Galerkin's weighted residual method.

Galerkin's weighted residual principle is expressed directly in terms of the governing Laplace partial differential equation. The residual at each point in the problem domain is a measure of the degree to which the hydraulic head potential does not satisfy the governing equations. If a particular weighted average of the residual is forced to vanish, the nodal hydraulic head potentials are obtained from the solution of a system of algebraic equations.

The first step in applying Galerkin's method to Laplace's equation is to define an approximate solution, h(x,y). This is expressed as a series summation, in which each term is a product of a nodal hydraulic head potential h_i and a nodal basis function $\phi_i(x,y)$:

$$h(x,y) = \sum_{i=1}^{N} h_i \, \phi_i(x,y) \qquad (18)$$

where h_i is the hydraulic head potential at node i and $\phi_i(x,y)$ is the shape factor. $\phi_i(x,y) = 1$ for node i and zero for all the other nodes. The subscript i indicates the nodal number and N indicates the total number of nodes in the problem domain.

The residual due to the approximate solution h(x,y) may be written as:

$$r = r(x,y) + Q(s) \qquad (19)$$

The quantities r(x,y) and Q(s) are defined by the following equations:

$$r(x,y) = -\left[\frac{\partial}{\partial x}\left(K\frac{\partial h}{\partial x}\right) - \frac{\partial}{\partial y}\left(K\frac{\partial h}{\partial y}\right)\right] \qquad (20)$$

$$Q(s) = -\left(-q_{nk+} + q_{nk-} + Q_k\right) \qquad (21)$$

where q_{nk+} = mass flux going into line "k" from the right of "k"

q_{nk-} = mass flux coming out of line "k" from the left of "k"

Q_k = mass flux along "k"

Now using the Galerkin formulation with the weighing factor, which is equal to the basis function, defined as

$\phi i(x,y)$ = 1 for the node i

0 for all the other nodes

the weighted residual at node i may be written as:

$$R_i = \iint_A r\,\phi_i\,dA \qquad (22)$$

Substituting equations (19), (20), and (21) into equation (22) results in:

$$R_i = \sum_{j=1}^{NE} \iint_{A_j} -\left[\frac{\partial}{\partial x}\left(K\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial h}{\partial y}\right)\right]\phi_i\,dA_j$$

$$+ \sum_{k=1}^{NL} -\int_0^{l_k} \left[-q_{nk+} - q_{nk-} + Q_k\right]\phi_i\,ds \qquad (23)$$

where NE = the total number of elements

NL = the total number of lines in the medium

Using Green's Theorem, equation (23) can be written as:

$$R_i = \sum_{j=1}^{NE} R_{ij} - \sum_{k=1}^{NL} \int_0^{l_k} Qk\_\phi_i \, ds_k \qquad (24)$$

where the quantity R_{ij} is defined by:

$$R_{ij} = \iint_{A_j} \left(K \frac{\partial h}{\partial x} \frac{\partial \phi_i}{\partial x} + K \frac{\partial h}{\partial Y} \frac{\partial \phi_i}{\partial Y} \right) dA_j \qquad (25)$$

Next, consider any given quadrilateral element that comprises the mesh that defines any given problem domain. Each of these quadrilateral elements, with coordinates (x_1,y_1), (x_2,y_2), (x_3,y_3), and (x_4,y_4) can be transformed into a first order isoparametric element with coordinates of (1,1), (1.-1), (-1,1) and (-1,1). See Illustration 4.

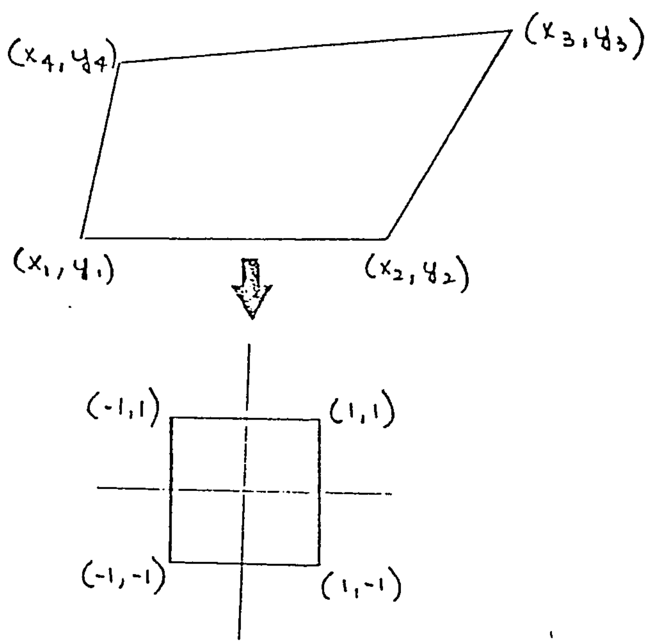


Illustration 4: Transformation from a quadrilateral element to an isoparametric element

For the resulting isoparametric element, we can define a shape function N_p where:

$$N_p = \frac{1}{4} (1+w_p w)(1+n_p n) \qquad (26)$$

where $p = 1$ for (x_1, y_1)

$p = 2$ for (x_2, y_2)

$p = 3$ for (x_3, y_3)

$p = 4$ for (x_4, y_4)

and $w_p = 1$ for $p = 1$ and 4

$w_p = -1$ for $p = 2$ and 3

$n_p = 1$ for $p = 1$ and 2

$n_p = -1$ for $p = 3$ and 4

Since the element is first order,

$$x = \sum_{p=1}^{4} N_p(w,n)\ X_p \qquad (27)$$

$$y = \sum_{p=1}^{4} N_p(w,n)\ Y_p \qquad (28)$$

and

$$h = \sum_{p=1}^{4} N_p\ u_p \qquad (29)$$

where u_p are the nodal hydraulic head potentials

Therefore, for element j,

$$\frac{\partial h}{\partial x} = \sum_{p=1}^{4} u_p \frac{\partial N_p}{\partial x} \qquad (30)$$

$$\frac{\partial h}{\partial y} = \sum_{p=1}^{4} u_p \frac{\partial N_p}{\partial y} \qquad (31)$$

and assuming $N_i = \phi_i$

$$\frac{\partial \phi_p}{\partial x} = \frac{\partial N_p}{\partial x} \qquad (32)$$

$$\frac{\partial \phi_p}{\partial y} = \frac{\partial N_p}{\partial y} \qquad (33)$$

also,

$$\frac{\partial N_p}{\partial w} = \frac{\partial N_p}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial N_p}{\partial y}\frac{\partial y}{\partial w} \qquad (34)$$

$$\frac{\partial N_p}{\partial n} = \frac{\partial N_p}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial N_p}{\partial y}\frac{\partial y}{\partial n} \qquad (35)$$

As a result,

$$\frac{\partial N_p}{\partial x} = \frac{\frac{\partial y}{\partial n}\frac{\partial N_p}{\partial w} - \frac{\partial y}{\partial w}\frac{\partial N_p}{\partial n}}{\frac{\partial x}{\partial w}\frac{\partial y}{\partial n} - \frac{\partial x}{\partial n}\frac{\partial y}{\partial w}} \qquad (36)$$

$$\frac{\partial N_p}{\partial y} = \frac{\frac{\partial y}{\partial n}\frac{\partial N_p}{\partial w} - \frac{\partial y}{\partial w}\frac{\partial N_p}{\partial n}}{\frac{\partial x}{\partial w}\frac{\partial y}{\partial n} - \frac{\partial x}{\partial n}\frac{\partial y}{\partial w}} \qquad (37)$$

Equations (36) and (37) can be rewritten in the following way:

$$\frac{\partial N_p}{\partial x} = F_p(w,n) = \frac{Ay - By}{C_1 - C_2} \qquad (38)$$

$$\frac{\partial N_p}{\partial y} = G_p(w,n) = \frac{Ax - Bx}{C_1 - C_2} \qquad (39)$$

where

$$Ay = w_p(1+n_p n) \sum_{q=1}^{4} Y_q n_q (1+w_q w) \qquad (40)$$

$$By = n_p(1+w_p w) \sum_{q=1}^{4} Y_q w_q (1+n_q n) \qquad (41)$$

$$C1 = \sum_{q=1}^{4} X_q w_q (1+n_q n) \sum_{r=1}^{4} Y_r n_r (1+w_r w) \qquad (42)$$

$$C2 = \sum_{q=1}^{4} Y_q w_q (1+n_q n) \sum_{r=1}^{4} X_r n_r (1+w_r w) \qquad (43)$$

$$Ax = w_p(1+n_p n) \sum_{q=1}^{4} X_q n_q (1+w_q w) \qquad (44)$$

$$Bx = n_p(1+w_p w) \sum_{q=1}^{4} X_q w_q (1+n_q n) \qquad (45)$$

Therefore, the values of $\partial h/\partial x$ and $\partial h/\partial y$ can be written as:

$$\frac{\partial h}{\partial x} = \sum_{p=1}^{4} u_p \frac{\partial N_p}{\partial x} = \sum_{p=1}^{4} u_p F_p \qquad (46)$$

$$\frac{\partial h}{\partial y} = \sum_{p=1}^{4} u_p \frac{\partial N_p}{\partial y} = \sum_{p=1}^{4} u_p G_p \qquad (47)$$

As a result, equation (25) can be written as:

$$R_{ij} = \int_{-1}^{1}\int_{-1}^{1} \left(K \sum_{p=1}^{4} u_p F_p F_i + K \sum_{p=1}^{4} u_p G_p G_i\right) J \, dwdn \qquad (48)$$

where $dA = dxdy = J\, dwdn$

and

$$Fi = \frac{\partial \phi_i}{\partial x} = \frac{\partial N_i}{\partial x} \qquad (49)$$

$$Gi = \frac{\partial \phi_i}{\partial y} = \frac{\partial N_i}{\partial y} \qquad (50)$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} \\ \frac{\partial x}{\partial n} & \frac{\partial y}{\partial n} \end{vmatrix} \qquad (51)$$

Furthermore, equation (48) can be rearranged to the following form:

$$R_{ij} = \sum_{p=1}^{4} u_p \left(\int_{-1}^{1}\int_{-1}^{1} KF_p F_i + KG_p G_i\right) J \, dwdn \qquad (52)$$

The integral in equation (52) can be evaluated using the

Gauss quadrature, as depicted in Illustration 5, in the following way:

$$\int_{-1}^{1}\int_{-1}^{1} H(w,n)\, dwdn = \sum_{o=1}^{4} \sum_{m=1}^{4} H(w_o,n_m)\, v_o v_m \qquad (53)$$

where $v_o = v_m = 1$ for $m = 1,2,3,4$ and $o = 1,2,3,4$

$w_1 = w_4 = \frac{1}{(3)^{0.5}}$ and $w_2 = w_3 = \frac{-1}{(3)^{0.5}}$

$n_1 = n_2 = \frac{1}{(3)^{0.5}}$ and $n_3 = n_4 = \frac{-1}{(3)^{0.5}}$

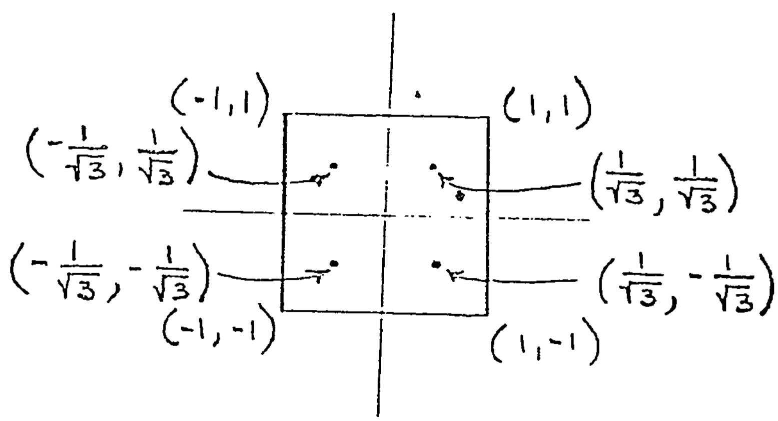


Illustration 5: Location of Gauss quadrature points

Consequently, if the value of the integral in equation (52) is set equal to the variable $[M_{ip}]_j$, $[M_{ip}]_j$ can be evaluated in the following manner:

$$[M_{ip}]_j = \sum_{o=1}^{4} \sum_{m=1}^{4} \{[KF_i(w_o,n_m) F_p(w_o,n_m)] + [KG_i(w_o,n_m) G_p(w_o,n_m)]\}\, J(w_o,n_m)\, v_o v_m \qquad (54)$$

As a result, for element j, equation (52) can be rewritten in the form:

$$(R_i)_j = [M_{ip}]_j \, [u_p]_j \qquad (55)$$

The result of equation (55) is then substituted into equation (24) to get the weighted residual at node i, and hence, defines the distribution of the hydraulic head potential at node i.

Finally, the flow flux at the centroid of any given element in the mesh is determined by the following relationship:

$$\begin{bmatrix} qx \\ qy \end{bmatrix} = - \begin{bmatrix} Kx\ (x,y) & 0 \\ 0 & Ky\ (x,y) \end{bmatrix} \begin{bmatrix} \partial h/\partial x \\ \partial h/\partial y \end{bmatrix} \qquad (56)$$

where the hydraulic conductivity is approximated as a constant within each element.

USER MANUAL TO THE SOFTWARE PACKAGE

2DFLOW/2DPLOT

USER MANUAL TO 2DFLOW/2DPLOT FOR STEADY STATE FLOW THROUGH SATURATED POROUS MEDIA

The computer program 2DFLOW describes two dimensional steady state flow through saturated porous media. The program 2DPLOT generates separate plots of the equipotential contours and flux vectors. The flux vectors show the relative magnitude and direction of the flow in any given element of the mesh that describes the problem domain.

In 2DFLOW, the input data is entered in formatted form into an input file. Thê structured format is necessary for the program generates the mesh, if the user specifies the boundary nodes. The input file can be created using any word processor available to the user. The units used to describe the material properties must be consistent with the units used to describe the geometry of the problem. The output is expressed in terms of these same units.

The following pages describe in detail, the form in which the input data must be entered into the input file.

INPUT DATA

The input data is entered into the input file in the following way:

I. TITLE BLOCK - Row 1, columns 1 - 72.

Enter any information that is deemed necessary to identify the problem, such as a title.

II. MATERIAL BLOCK - For each material type used, enter the following information:

Columns	2 - 5	MN	- Material number
	6 - 20	K_x	- Hydraulic conductivity in x-dir.
	21 - 35	K_y	- Hydraulic conductivity in y-dir.
	36 - 50	So	- Source (recharge)
	51 - 65	Si	- Sink (discharge)

The number 1 is entered in column 1, followed by:

III. <u>NODE POINT ARRAY BLOCK</u> - Use as many rows as necessary to specify the location of all the nodes in the mesh that describes the problem domain. Specify all the nodes located on the periphery of the problem domain, as well as along interior boundaries, such as sheet piles and the interface between two materials having a different hydraulic conductivity.

Columns	2 - 5	N -	Node point number
	6 - 15	X -	x-coordinate
	16 - 25	Y -	y-coordinate
	26 - 30	INC -	Numbering increment
	31 - 40	D -	Spacing ratio between any 2 successive pairs of points D = 1 - equal spacing between nodes D > 1 - spacing increases between nodes D < 1 - spacing decreases between nodes
	41 - 50	xc	Coordinates of a point on a circular arc located at some
	51 - 60	yc	intermediate point between the endpoints of the arc

The number 2 is entered in column 1, followed by:

IV. <u>ELEMENT ARRAY BLOCK</u> - Use as many rows as necessary to define all the the elements in the mesh that descibes the problem domain.

Columns 2 - 5 6 - 10 11 - 15 16 - 20		Reading counerclockwise around the element, the numbers of the 4 node points which describe the quadrilateral or triangular element. In the case of a triangle, the first node point is repeated as the fourth node point.
21 - 25	MN -	Material number (correspond- to the material description in section II)
26 - 30	NMIS -	Number of additional elements in the layer
31 - 35	INC -	Difference between corresponding node numbers of successive elements within a layer
36 - 40	NMISP -	Number of additional layers
41 - 45	INCP -	Difference between corresponding node numbers of successive layers

Note: The quantities NMIS, INC, NMISP, and INCP are associated wth the data generation option.

The number 3 is entered in column 1, followed by:

V. NODE POINT BOUNDARY SPECIFICATION ARRAY BLOCK - Use as many rows as necessary to specify the hydraulic head at the node points previously defined in the NODE POINT ARRAY BLOCK. Also specify any non-zero concentrated recharge or discharge sources.

Columns 2 -	N -	Node point number (initial node point in sequence)
10	IF -	Type of boundary IF = 0 - no flow boundary IF = 1 - flow occurs at boundary
11 - 20	V -	Value of the hydraulic head at the specified boundary
21 - 25	N^1 -	Final node point in sequence
26 - 30	INC -	Numbering increment in the sequence between N and N^1
31 - 40	Q_n	Value of hydraulic head at node points N and N1 respectively.
41 - 50	Q_n^1	These quantities are specified only if a linear varying hydraulic head is to be specified.

The number 4 is entered in column 1 to end the input file.

Save the input file on a diskette, and proceed with the execution of the 2DFLOW/2DPLOT software package.

OUTPUT

The items appearing in the output data file are listed below:

1. The input data is printed.
2. For each node in the mesh describing the problem domain, the hydraulic head is printed.
3. For each element, the following items are printed:
   A. Element number - The element numbers are listed in the order in which they are generated by the ELEMENT ARRAY BLOCK input. Also listed are the node point numbers that describe each element.
   B. The x and y-coordinates of the center of each element.
   C. The quantity of flow (flux) at the center of each element.

ELEMENT DESCRIPTION:

The cross section of a given problem domain is assumed to lie in the x-y plane. The problem domain may have any arbitrary shape, as can be seen from Illustration 6 on the following page. The cross section is described by a series of quadrilateral or triangular elements that form a mesh that describes the problem domain. A simple mesh system is shown in llustration 7. Note that every node point is defined by a number.

Each quadrilateral element is described by the numbers of the four node points. Each node point is located at one of the four corners of the element. A triangular element is also described by four node points. The fourth node point is the same as the first node point. For example, in Illustration 7, the triangular element would be defined by the node numbers 34,33,35,34 or 33,35,34,33, etc.

The program checks the area of each element. If any of these values are non-positive, the following error message is printed: ERROR IN ELEMENT N. This error is normally a result of the following reasons:

1. The node points describing the element were entered in a clockwise manner instead of counterclock-

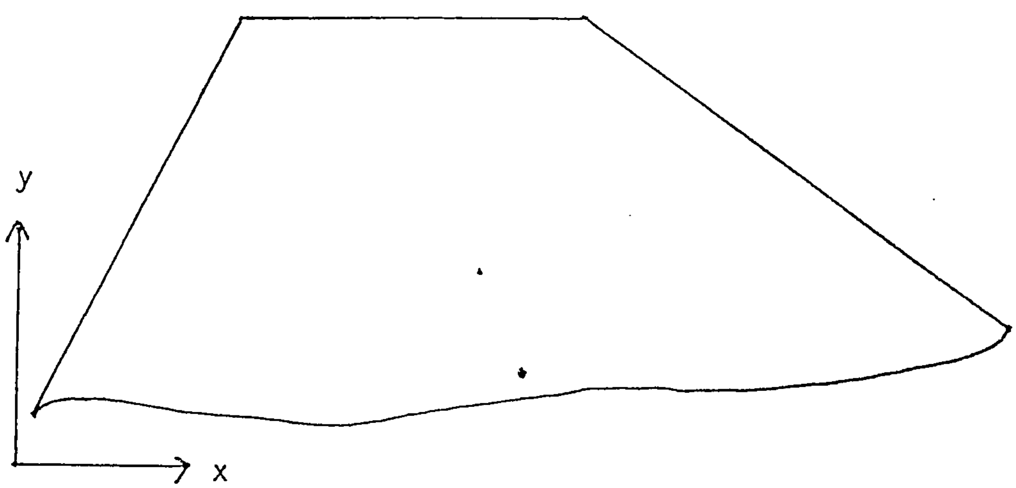


Illustration 6: Sketch of a typical problem domain

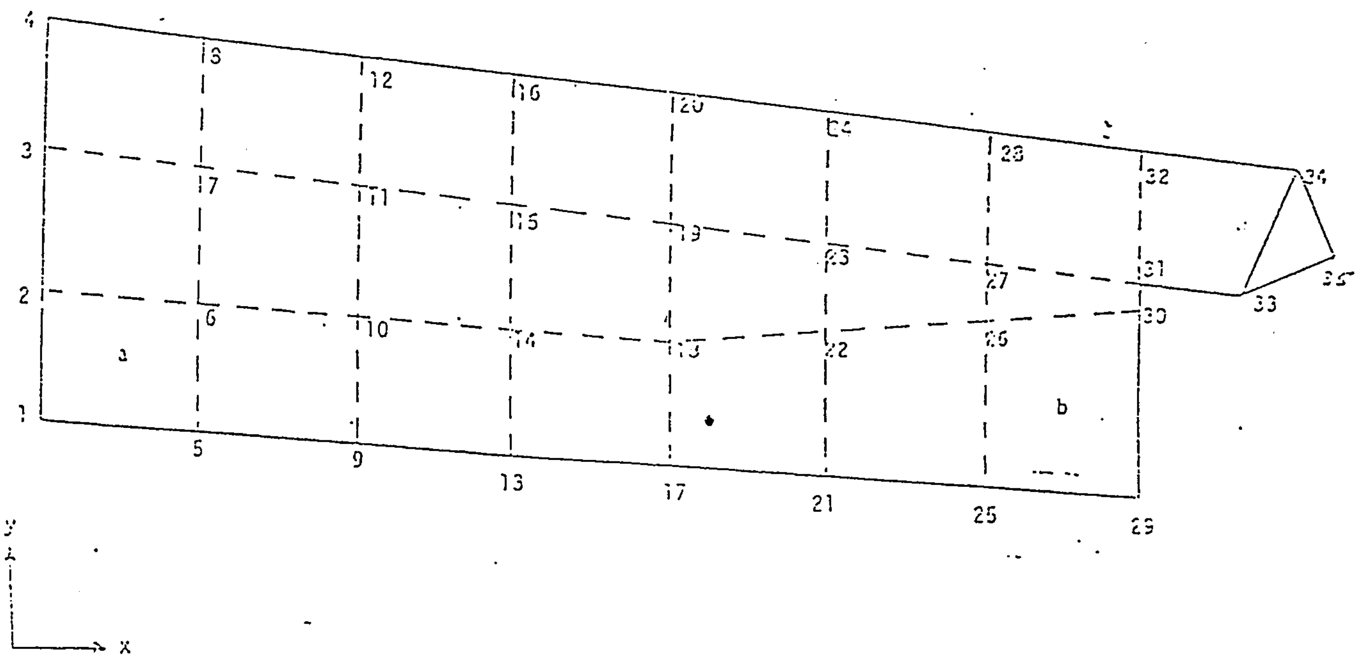


Illustration 7: A mesh of quadrilateral elements

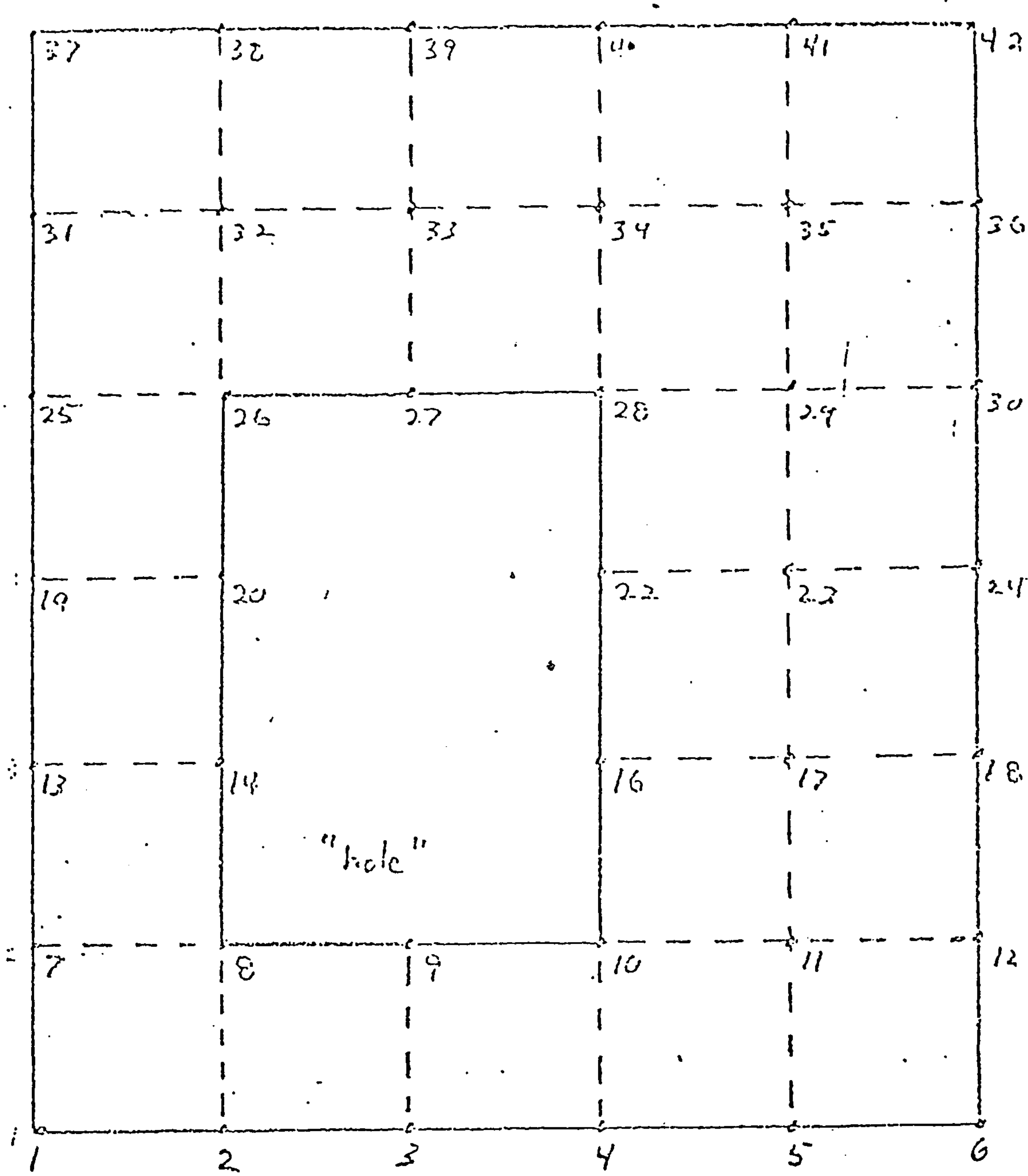


Illustration 8: Example of a mesh with missing node numbers

wise.

2. One of the node numbers describing the element was entered incorrectly.

3. The coordinates of one of the node points describing the element was entered incorrectly.

Not all numbers between 1 and the maximum node number have to correspond to actual node points in the mesh describing the problem domain. For example, the mesh configuration in Illustration 8 is allowed. In this mesh system, the node points 15 and 21 do not exist. This feature facilitates the use of various node point generation options available within the program.

NODE POINT GENERATION OPTIONS:

CIRCULAR AND STRAIGHT LINE COORDINATE GENERATION OPTION:

The CIRCULAR ARC COORDINATE GENERATION OPTION may be used whenever several sequential points lie along an circular arc. Similarly, the STRAIGHT LINE COORDINATE GENERATION OPTION may be used whenever several sequential points lie along a straight line. For such mesh systems, it is necessary only to enter data for the end node points (denoted as N and N') of the sequence and the values for INC and D. The value INC is the difference between any two successive node numbers in the sequence and the value D is the ratio of the lengths of successive segments between two pairs of node points.

If for a node point N, INC is not equal to 0, then intermediate node points are generated along a straight line ($x_C=y_C=0$) or a circular arc (x_C and y_C are not equal to zero) between node points N and N'. That is, node points N'+INC, N'+2xINC,...,N-INC are generated. Circular arcs are defined as passing through the end node points N and N', and some intermediate point (not necessarily one of the node points) whose coordinates are (x_C, y_C).

The end node points of a straight line or circular segment may be entered in any order. The segments shown in Illustrations 9 and 10 on the following page may be defined by specifying the end node points in the order 7 to 22 or 22 to 7. The spacing of the intermediate node points is controlled by the value of the spacing ratio D. D is equal to the ratio of the lengths of the successive segments between two pairs of node points. A value of D=1.0 generates equally spaced node points between the two end node points. The locations of the intermediate node points 12 and 17 in Illustrations 9 and 10 could be generated by either specifying node points 7 to 22 and setting D = 2.0 (D = 2.0/1.0 = 4.0/2.0), or 22 to 7 and setting D = 0.5 (D = 2.0/4.0 = 1.0/2.0). In both cases, the value of INC specified would be 5.

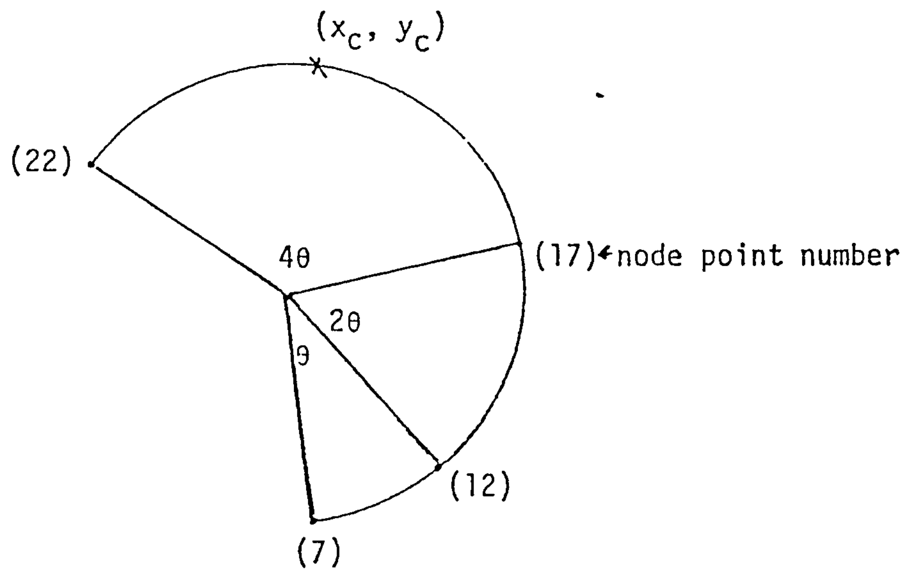


Illustration 9: Node points lying on a circular arc

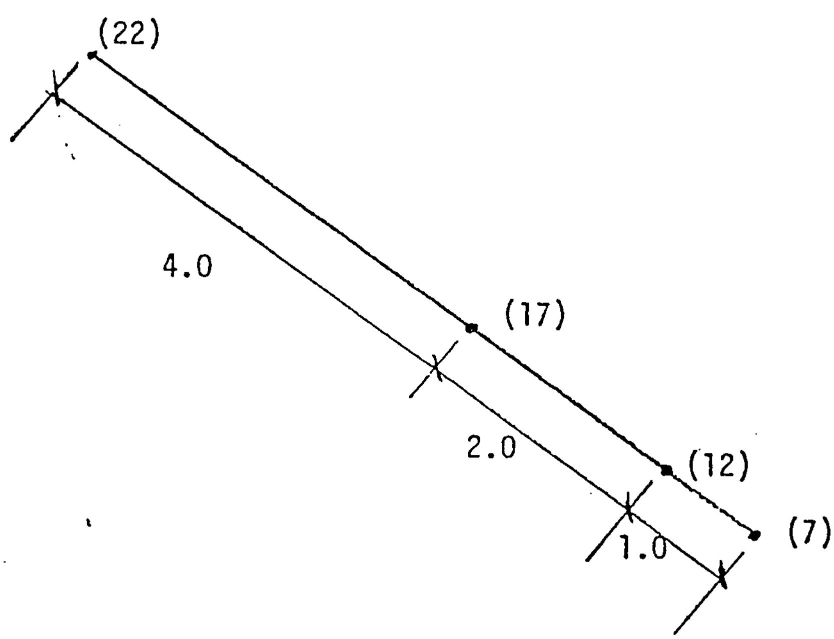


Illustration 10: Node points lying on a straight line

INTERIOR NODE POINT GENERATION OPTION:

The INTERIOR NODE POINT GENERATION OPTION locates all node points interior to the problem domain, whose coordinates have not been specifically defined by the user. The location of the interior node points are determined by the the Laplacian Isoparametric generation scheme. That is, the coordinates of the interior node points are selected so that they are equal to a weighted average of the coordinates of the neighboring nodes. All node points that define a boundary in the problem domain must be either directly specified by the user or generated by means of the line segment generation option.

ELEMENT DATA GENERATION OPTION:

If the mesh that defines the problem domain is divided into several layers of individual elements, and the value MN (material number) remains the same for all the elements, the node numbers for these elements can be specified by using the ELEMENT DATA GENERATION OPTION. As an example of this generation option, the elements shown in Grid 1 of Illustration 11 could all be described by entering, in counterclockwise order, the node numbers of the four node points that define the element located in the lower left hand corner and the following data:

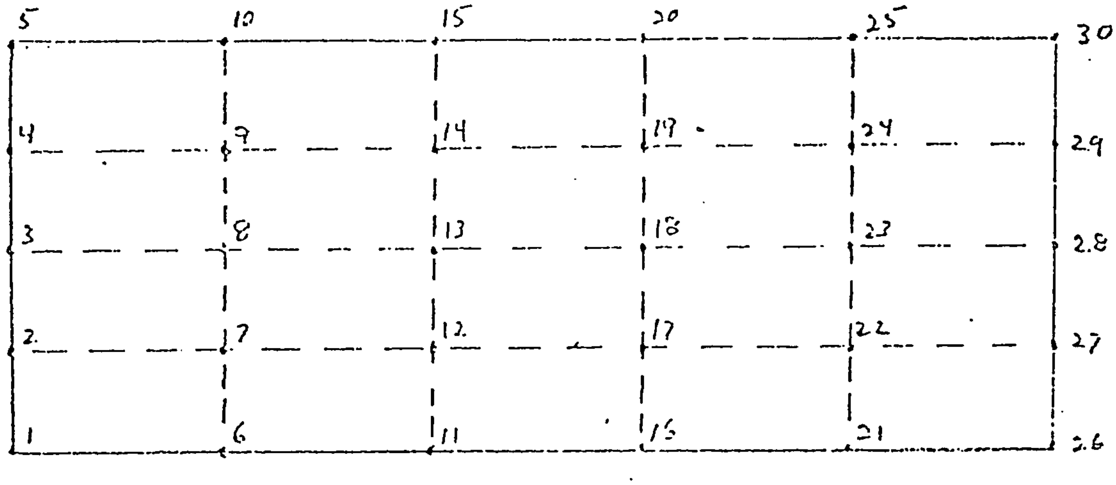
NMIS = 4
INC = 5
NMISP = 3
INCP = 1

where NMIS = Number of additional elements in the layer

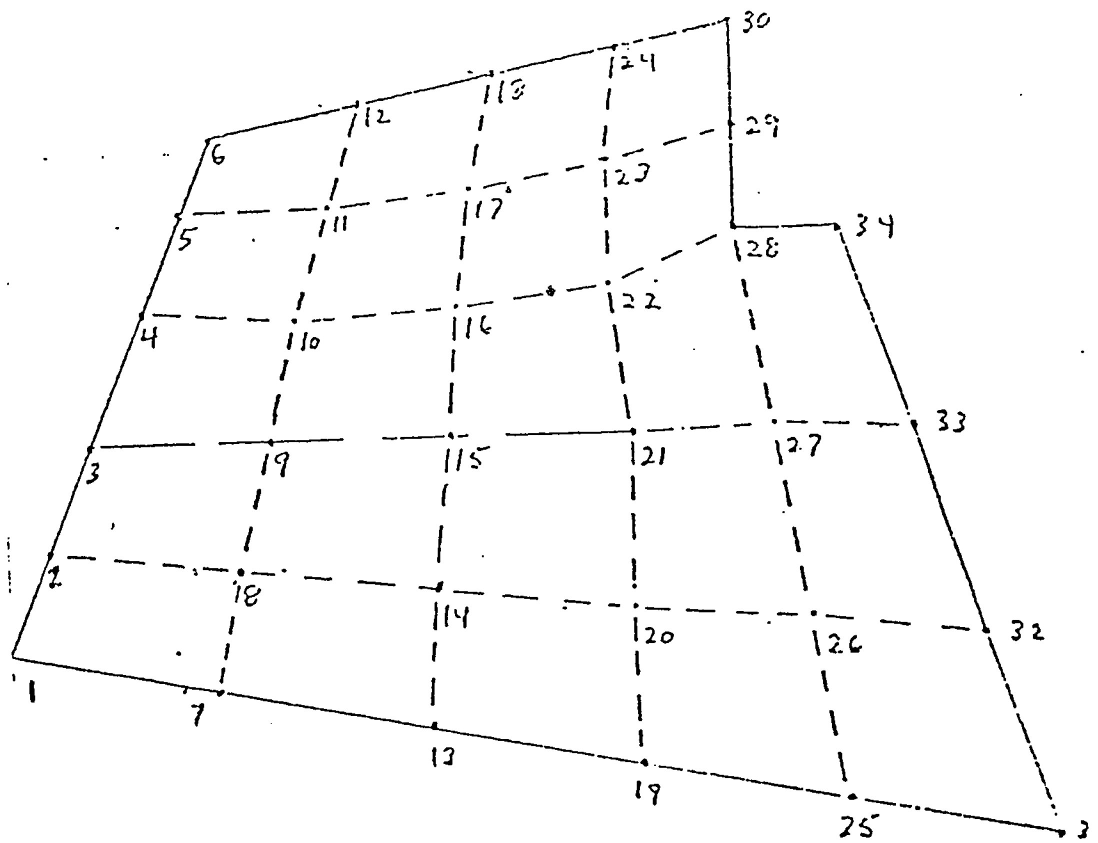
INC = Difference between corresponding node numbers of successive elements within a layer

NMISP = Number of addtional layers

INCP = Difference between corresponding node numbers of successive layers.



Grid 1



Grid 2

Illustration 11: Grids prepared with the aid of generation options

NODE POINT BOUNDARY SPECIFICATION:

The hydraulic head is specified at the node points previously defined in the NODE POINT ARRAY BLOCK. If several sequential node points have identical boundary specifications, they may be defined by specifying the initial node point (N) and the final node point (N^1) in the sequence, as well as the increment (INC) between node numbers of successive node points.

A linearly changing hydraulic head along a straight or curved boundary can be specfied in a similar fashion. The spaces for IF and V are left blank and the appropriate values are specified for Q_n and $Q_n{}^1$. For example, the boundary shown in Illustration 12 could be specified in the following way:

N = 11
N1 = 2
INC = -3
Q_n = 0.01
$Q_n{}^1$ = 0.005

where Q_n = Hydraulic head at node point N

$Q_n{}^1$ = Hydraulic head at node point N^1

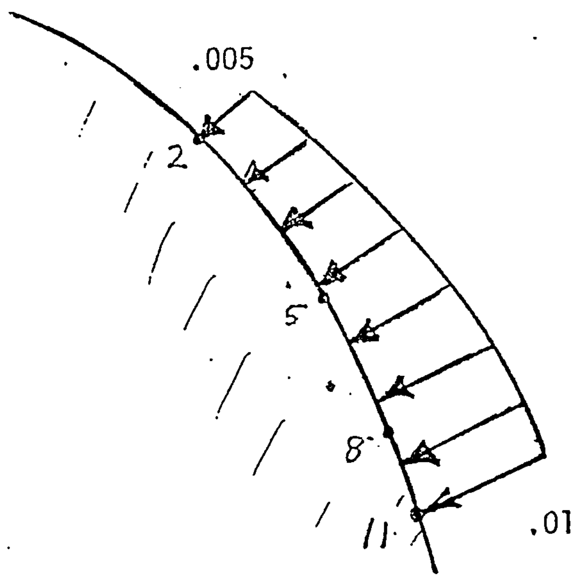


Illustration 12: Varied hydraulic head along a boundary

LIMITATIONS:

The limitations of the 2DFLOW/2DPLOT software package are the following:

1. The number of elements within a layer may not exeed 20.
2. The maximum number of node points that can be used is 255.
3. The maximum number of elements that can be used is 255.
4. The maximum number of different materials that can be used is 11.
5 Each element must be defined by 4 node points.

<u>NOTE</u>: If any of these restrictions are violated, an error message is printed. However, the user can specify higher values in the program code for items 1 and 4 above, and recompile the program if conditions necessitate such a change.

EXECUTION OF THE 2DFLOW/2DPLOT SOFTWARE PACKAGE

Once the input data file has been created, it is ready for processing using the 2DFLOW/2DPLOT program package. Using an IBM compatible, two disk drive computer system, 2DFLOW/2DPLOT can be executed in the following manner:

1. Place diskette containing the input data file in drive A.
2. Place diskette containing the 2DFLOW/2DPLOT program package in drive B.
3. Call up the processing program

   ```
   A> B:2DFLOW
   ```

4. Answer the following statements as they appear:

   ```
   Input filename :
   Output filename :
   Do you want to write an output plot file (Y/N)?
   Plot filename :
   ```

5. Processing of the data is finished when the following statement is encountered:

   ```
   Stop - Program Terminated
   ```

6. Invoke GWBASIC VERSION 3.20 or higher (included on program diskette). Type the following:

```
A> B:GWBASIC
```

7. Load the plotter program 2DPLOT at the OK prompt. Type the following:

```
LOAD "B:2DPLOT.BAS"
```

8. Run 2DPLOT by typing the following after the OK prompt:

```
RUN
```

9. Answer the following statements as they appear:

```
INPUT PLOT FILENAME :
INPUT DRIVE THAT CONTAINS PLOT FILE :
```

10. Prompts for mesh plots, contour plots, and flux vector plots appear one at a time. Answer each prompt accordingly.

11. Printouts of the mesh plot, contour plot, and flux vector plot can be attained by pressing simultaneously Shift-PrtSc on the keyboard. This should be done after one of the following statements appears on the screen:

DO YOU WISH TO REDRAW THE MESH (Y/N) ?

DO YOU WISH TO REDRAW THE CONTOURS (Y/N) ?

DO YOU WISH TO REDRAW THE FLUX VECTORS (Y/N) ?

ILLUSTRATED EXAMPLES

In this section, several sample computer generated flow nets are provided to demonstrate the capabilities of the 2DFLOW/2DPLOT program package. Also provided is a sample problem that illustrates in detail, how the input data file is created. Included in this example is a detailed illustrated sketch of the problem domain, a sample coding sheet, a printout of a typical input and output file, and a computer generated plot of the resulting flow net.

The sample problem depicts the flow of groundwater around a sheet-metal barrier wall.

SKETCH OF EXAMPLE #1

FLOW AROUND A SHEET-METAL BARRIER WALL

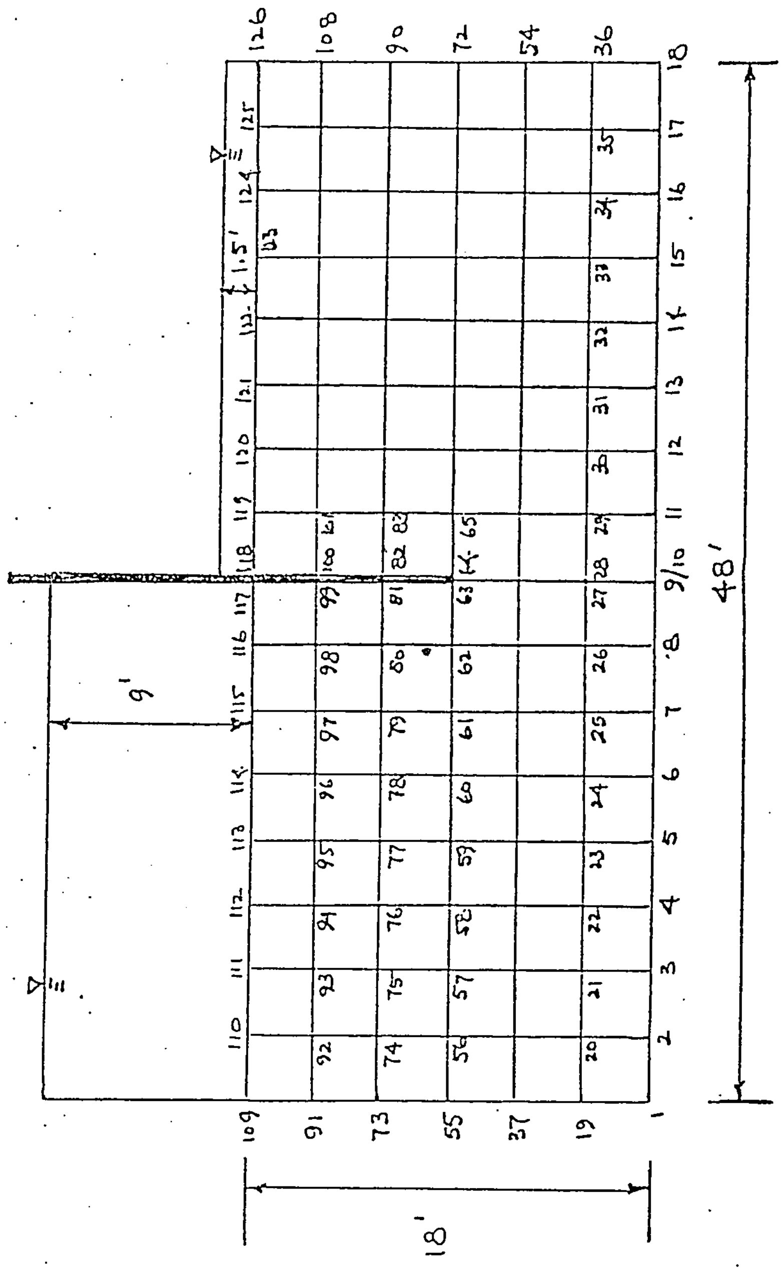


Illustration 13: Sketch of Example #1

SAMPLE CODING SHEET

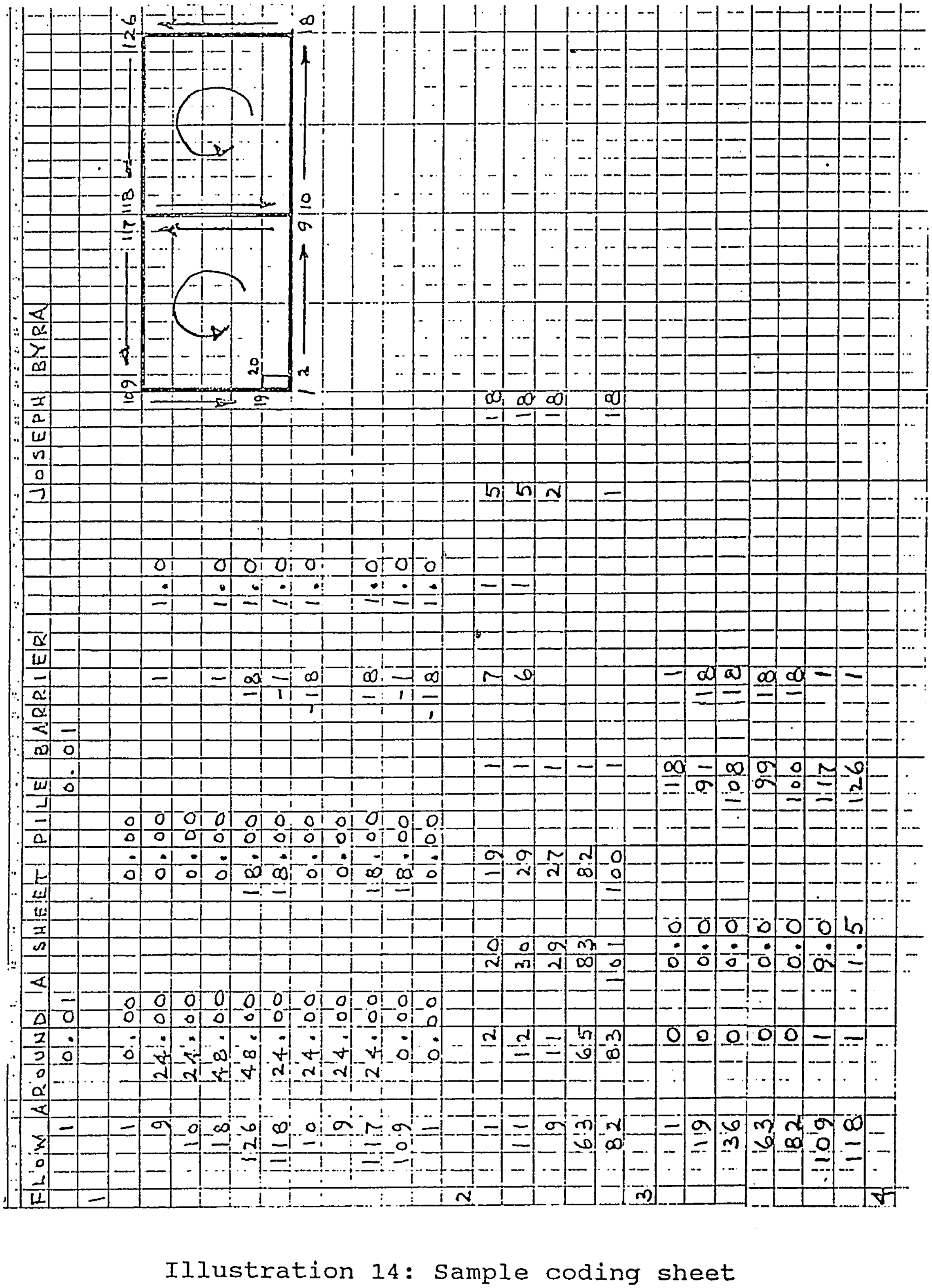


Illustration 14: Sample coding sheet

SAMPLE INPUT FILE

```
FLOW AROUND A SHEET PILE BARRIER        JOSEPH BYRA
    1   0.01              0.01
1
    1   0.00      0.00
    9  24.00      0.00        1    1.0
   10  24.00      0.00
   18  48.00      0.00        1    1.0
  126  48.00     18.00       18    1.0
  118  24.00     18.00       -1    1.0
   10  24.00      0.00      -18    1.0
    9  24.00      0.00
  117  24.00     18.00       18    1.0
  109   0.00     18.00       -1    1.0
    1   0.00      0.00      -18    1.0
2
    1    2   20   19    1    7    1    5   18
   11   12   30   29    1    6    1    5   18
    9   11   29   27    1              2   18
   63   65   83   82    1
   82   83  101  100    1              1   18
3
    1    0   0.0        18    1
   19    0   0.0        91   18
   36    0   0.0       108   18
   63    0   0.0        99   18
   82    0   0.0       100   18
  109    1   9.0       117    1
  118    1   1.5       126    1
4
```

SAMPLE OUTPUT FILE

PROPERTIES FOR MATERIAL 1 ********

K-X = 1.000E-02
K-Y = 1.000E-02
S0 = .000E+00
S1 = .000E+00

1

*****GEOMETRY*****

NODAL POINT	X-Y COORDINATES	
1	.00	.00
2	3.00	.00
3	6.00	.00
4	9.00	.00
5	12.00	.00
6	15.00	.00
7	18.00	.00
8	21.00	.00
9	24.00	.00
10	24.00	.00
11	27.00	.00
12	30.00	.00
13	33.00	.00
14	36.00	.00
15	39.00	.00
16	42.00	.00
17	45.00	.00
18	48.00	.00
19	.00	3.00
20	3.00	3.00
21	6.00	3.00
22	9.00	3.00
23	12.00	3.00
24	15.00	3.00
25	18.00	3.00
26	21.00	3.00
27	24.00	3.00
28	24.00	3.00
29	27.00	3.00
30	30.00	3.00
31	33.00	3.00
32	36.00	3.00
33	39.00	3.00
34	42.00	3.00

35	45.00	3.00
36	48.00	3.00
37	.00	6.00
38	3.00	6.00
39	6.00	6.00
40	9.00	6.00
41	12.00	6.00
42	15.00	6.00
43	18.00	6.00
44	21.00	6.00
45	24.00	6.00
46	24.00	6.00
47	27.00	6.00
48	30.00	6.00
49	33.00	6.00
50	36.00	6.00
51	39.00	6.00
52	42.00	6.00
53	45.00	6.00
54	48.00	6.00
55	.00	9.00
56	3.00	9.00
57	6.00	9.00
58	9.00	9.00
59	12.00	9.00
60	15.00	9.00
61	18.00	9.00
62	21.00	9.00
63	24.00	9.00
64	24.00	9.00
65	27.00	9.00
66	30.00	9.00
67	33.00	9.00
68	36.00	9.00
69	39.00	9.00
70	42.00	9.00
71	45.00	9.00
72	48.00	9.00
73	.00	12.00
74	3.00	12.00
75	6.00	12.00
76	9.00	12.00
77	12.00	12.00
78	15.00	12.00
79	18.00	12.00
80	21.00	12.00
81	24.00	12.00
82	24.00	12.00
83	27.00	12.00
84	30.00	12.00
85	33.00	12.00
86	36.00	12.00
87	39.00	12.00
88	42.00	12.00

89	45.00	12.00
90	48.00	12.00
91	.00	15.00
92	3.00	15.00
93	6.00	15.00
94	9.00	15.00
95	12.00	15.00
96	15.00	15.00
97	18.00	15.00
98	21.00	15.00
99	24.00	15.00
100	24.00	15.00
101	27.00	15.00
102	30.00	15.00
103	33.00	15.00
104	36.00	15.00
105	39.00	15.00
106	42.00	15.00
107	45.00	15.00
108	48.00	15.00
109	.00	18.00
110	3.00	18.00
111	6.00	18.00
112	9.00	18.00
113	12.00	18.00
114	15.00	18.00
115	18.00	18.00
116	21.00	18.00
117	24.00	18.00
118	24.00	18.00
119	27.00	18.00
120	30.00	18.00
121	33.00	18.00
122	36.00	18.00
123	39.00	18.00
124	42.00	18.00
125	45.00	18.00
126	48.00	18.00

1

*****ELEMENT INFORMATION*****

ELEMENT NUMBER	ELEMENT NODE POINTS				MATERIAL NUMBER
1	1	2	20	19	1
2	2	3	21	20	1
3	3	4	22	21	1
4	4	5	23	22	1
5	5	6	24	23	1
6	6	7	25	24	1
7	7	8	26	25	1

8	8	9	27	26	1
9	19	20	38	37	1
10	20	21	39	38	1
11	21	22	40	39	1
12	22	23	41	40	1
13	23	24	42	41	1
14	24	25	43	42	1
15	25	26	44	43	1
16	26	27	45	44	1
17	37	38	56	55	1
18	38	39	57	56	1
19	39	40	58	57	1
20	40	41	59	58	1
21	41	42	60	59	1
22	42	43	61	60	1
23	43	44	62	61	1
24	44	45	63	62	1
25	55	56	74	73	1
26	56	57	75	74	1
27	57	58	76	75	1
28	58	59	77	76	1
29	59	60	78	77	1
30	60	61	79	78	1
31	61	62	80	79	1
32	62	63	81	80	1
33	73	74	92	91	1
34	74	75	93	92	1
35	75	76	94	93	1
36	76	77	95	94	1
37	77	78	96	95	1
38	78	79	97	96	1
39	79	80	98	97	1
40	80	81	99	98	1
41	91	92	110	109	1
42	92	93	111	110	1
43	93	94	112	111	1
44	94	95	113	112	1
45	95	96	114	113	1
46	96	97	115	114	1
47	97	98	116	115	1
48	98	99	117	116	1
49	11	12	30	29	1
50	12	13	31	30	1
51	13	14	32	31	1
52	14	15	33	32	1
53	15	16	34	33	1
54	16	17	35	34	1
55	17	18	36	35	1
56	29	30	48	47	1
57	30	31	49	48	1
58	31	32	50	49	1
59	32	33	51	50	1
60	33	34	52	51	1
61	34	35	53	52	1

62	35	36	54	53	1
63	47	48	66	65	1
64	48	49	67	66	1
65	49	50	68	67	1
66	50	51	69	68	1
67	51	52	70	69	1
68	52	53	71	70	1
69	53	54	72	71	1
70	65	66	84	83	1
71	66	67	85	84	1
72	67	68	86	85	1
73	68	69	87	86	1
74	69	70	88	87	1
75	70	71	89	88	1
76	71	72	90	89	1
77	83	84	102	101	1
78	84	85	103	102	1
79	85	86	104	103	1
80	86	87	105	104	1
81	87	88	106	105	1
82	88	89	107	106	1
83	89	90	108	107	1
84	101	102	120	119	1
85	102	103	121	120	1
86	103	104	122	121	1
87	104	105	123	122	1
88	105	106	124	123	1
89	106	107	125	124	1
90	107	108	126	125	1
91	9	11	29	27	1
92	27	29	47	45	1
93	45	47	65	63	1
94	63	65	83	82	1
95	82	83	101	100	1
96	100	101	119	118	1

1

*****BOUNDARY CONDITIONS*****

BOUNDARY NODE		POTENTIAL
1	Q=	.00
2	Q=	.00
3	Q=	.00
4	Q=	.00
5	Q=	.00

6	Q=	.00
7	Q=	.00
8	Q=	.00
9	Q=	.00
10	Q=	.00
11	Q=	.00
12	Q=	.00
13	Q=	.00
14	Q=	.00
15	Q=	.00
16	Q=	.00
17	Q=	.00
18	Q=	.00
19	Q=	.00
37	Q=	.00
55	Q=	.00
73	Q=	.00
91	Q=	.00
36	Q=	.00
54	Q=	.00
72	Q=	.00
90	Q=	.00
108	Q=	.00
63	Q=	.00
81	Q=	.00
99	Q=	.00
82	Q=	.00

100	Q=	.00
109	T=	9.00
110	T=	9.00
111	T=	9.00
112	T=	9.00
113	T=	9.00
114	T=	9.00
115	T=	9.00
116	T=	9.00
117	T=	9.00
118	T=	1.50
119	T=	1.50
120	T=	1.50
121	T=	1.50
122	T=	1.50
123	T=	1.50
124	T=	1.50
125	T=	1.50
126	T=	1.50

1 NODE POTENTIALS

1	8.001
2	7.966
3	7.860
4	7.675
5	7.400
6	7.020
7	6.524
8	5.920
9	5.250
11	4.580
12	3.976
13	3.480
14	3.100

15	2.825
16	2.640
17	2.534
18	2.499
19	8.035
20	8.001
21	7.899
22	7.719
23	7.452
24	7.081
25	6.584
26	5.960
27	5.250
29	4.540
30	3.916
31	3.419
32	3.048
33	2.781
34	2.601
35	2.499
36	2.465
37	8.135
38	8.104
39	8.012
40	7.850
41	7.608
42	7.266
43	6.794
44	6.109
45	5.250
47	4.391
48	3.706
49	3.234
50	2.892
51	2.650
52	2.488
53	2.396
54	2.365
55	8.293
56	8.268
57	8.192
58	8.059
59	7.859
60	7.575
61	7.174
62	6.606
63	5.250
65	3.894
66	3.326
67	2.925
68	2.641
69	2.441
70	2.308
71	2.232

72	2.207
73	8.500
74	8.482
75	8.428
76	8.334
77	8.191
78	7.989
79	7.715
80	7.337
81	7.219
82	3.281
83	3.163
84	2.785
85	2.511
86	2.309
87	2.166
88	2.072
89	2.018
90	2.000
91	8.741
92	8.732
93	8.704
94	8.655
95	8.581
96	8.477
97	8.343
98	8.223
99	8.167
100	2.333
101	2.277
102	2.157
103	2.023
104	1.919
105	1.845
106	1.796
107	1.768
108	1.759
109	9.000
110	9.000
111	9.000
112	9.000
113	9.000
114	9.000
115	9.000
116	9.000
117	9.000
118	1.500
119	1.500
120	1.500
121	1.500
122	1.500
123	1.500
124	1.500
125	1.500

126 1.500

1ELEMENT FLUX VECTORS

	X	Y	Q0XQ-Y	
1	1.500	1.500	1.137E-04	-1.150E-04
2	4.500	1.500	3.487E-04	-1.226E-04
3	7.500	1.500	6.067E-04	-1.381E-04
4	10.500	1.500	9.034E-04	-1.610E-04
5	13.500	1.500	1.253E-03	-1.890E-04
6	16.500	1.500	1.655E-03	-2.025E-04
7	19.500	1.500	2.046E-03	-1.670E-04
8	22.500	1.500	2.301E-03	-6.637E-05
9	1.500	4.500	1.062E-04	-3.373E-04
10	4.500	4.500	3.259E-04	-3.600E-04
11	7.500	4.500	5.680E-04	-4.061E-04
12	10.500	4.500	8.490E-04	-4.764E-04
13	13.500	4.500	1.189E-03	-5.668E-04
14	16.500	4.500	1.614E-03	-6.581E-04
15	19.500	4.500	2.183E-03	-5.974E-04
16	22.500	4.500	2.615E-03	-2.473E-04
17	1.500	7.500	9.155E-05	-5.374E-04
18	4.500	7.500	2.812E-04	-5.741E-04
19	7.500	7.500	4.912E-04	-6.495E-04
20	10.500	7.500	7.376E-04	-7.673E-04
21	13.500	7.500	1.044E-03	-9.332E-04
22	16.500	7.500	1.454E-03	-1.147E-03
23	19.500	7.500	2.089E-03	-1.462E-03
24	22.500	7.500	3.691E-03	-8.290E-04
25	1.500	10.500	7.055E-05	-7.014E-04
26	4.500	10.500	2.169E-04	-7.501E-04
27	7.500	10.500	3.794E-04	-8.508E-04
28	10.500	10.500	5.714E-04	-1.011E-03
29	13.500	10.500	8.101E-04	-1.245E-03
30	16.500	10.500	1.124E-03	-1.594E-03
31	19.500	10.500	1.577E-03	-2.121E-03
32	22.500	10.500	2.457E-03	-4.499E-03
33	1.500	13.500	4.449E-05	-8.179E-04
34	4.500	13.500	1.369E-04	-8.753E-04
35	7.500	13.500	2.397E-04	-9.944E-04
36	10.500	13.500	3.608E-04	-1.185E-03
37	13.500	13.500	5.084E-04	-1.463E-03
38	16.500	13.500	6.810E-04	-1.859E-03
39	19.500	13.500	8.308E-04	-2.521E-03
40	22.500	13.500	2.898E-04	-3.057E-03
41	1.500	16.500	1.520E-05	-8.784E-04
42	4.500	16.500	4.678E-05	-9.404E-04
43	7.500	16.500	8.192E-05	-1.069E-03
44	10.500	16.500	1.232E-04	-1.274E-03
45	13.500	16.500	1.722E-04	-1.570E-03
46	16.500	16.500	2.245E-04	-1.966E-03
47	19.500	16.500	2.002E-04	-2.391E-03
48	22.500	16.500	9.231E-05	-2.684E-03
49	28.500	1.500	2.046E-03	1.670E-04
50	31.500	1.500	1.655E-03	2.025E-04

51	34.500	1.500	1.253E-03	1.890E-04
52	37.500	1.500	9.034E-04	1.610E-04
53	40.500	1.500	6.067E-04	1.381E-04
54	43.500	1.500	3.487E-04	1.226E-04
55	46.500	1.500	1.137E-04	1.150E-04
56	28.500	4.500	2.183E-03	5.974E-04
57	31.500	4.500	1.614E-03	6.581E-04
58	34.500	4.500	1.189E-03	5.668E-04
59	37.500	4.500	8.490E-04	4.764E-04
60	40.500	4.500	5.680E-04	4.061E-04
61	43.500	4.500	3.259E-04	3.600E-04
62	46.500	4.500	1.062E-04	3.373E-04
63	28.500	7.500	2.089E-03	1.462E-03
64	31.500	7.500	1.454E-03	1.147E-03
65	34.500	7.500	1.044E-03	9.332E-04
66	37.500	7.500	7.377E-04	7.673E-04
67	40.500	7.500	4.912E-04	6.495E-04
68	43.500	7.500	2.812E-04	5.741E-04
69	46.500	7.500	9.155E-05	5.374E-04
70	28.500	10.500	1.577E-03	2.121E-03
71	31.500	10.500	1.124E-03	1.594E-03
72	34.500	10.500	8.101E-04	1.245E-03
73	37.500	10.500	5.714E-04	1.011E-03
74	40.500	10.500	3.794E-04	8.508E-04
75	43.500	10.500	2.169E-04	7.501E-04
76	46.500	10.500	7.055E-05	7.014E-04
77	28.500	13.500	8.308E-04	2.521E-03
78	31.500	13.500	6.810E-04	1.859E-03
79	34.500	13.500	5.084E-04	1.463E-03
80	37.500	13.500	3.608E-04	1.185E-03
81	40.500	13.500	2.397E-04	9.944E-04
82	43.500	13.500	1.369E-04	8.753E-04
83	46.500	13.500	4.450E-05	8.179E-04
84	28.500	16.500	2.002E-04	2.391E-03
85	31.500	16.500	2.245E-04	1.966E-03
86	34.500	16.500	1.722E-04	1.570E-03
87	37.500	16.500	1.232E-04	1.274E-03
88	40.500	16.500	8.192E-05	1.069E-03
89	43.500	16.500	4.678E-05	9.403E-04
90	46.500	16.500	1.521E-05	8.784E-04
91	25.500	1.500	2.301E-03	6.638E-05
92	25.500	4.500	2.615E-03	2.473E-04
93	25.500	7.500	3.691E-03	8.290E-04
94	25.500	10.500	2.457E-03	4.499E-03
95	25.500	13.500	2.898E-04	3.057E-03
96	25.500	16.500	9.232E-05	2.683E-03

COMPUTER GENERATED PLOT OF THE FINISHED

FLOW NET FOR EXAMPLE #1

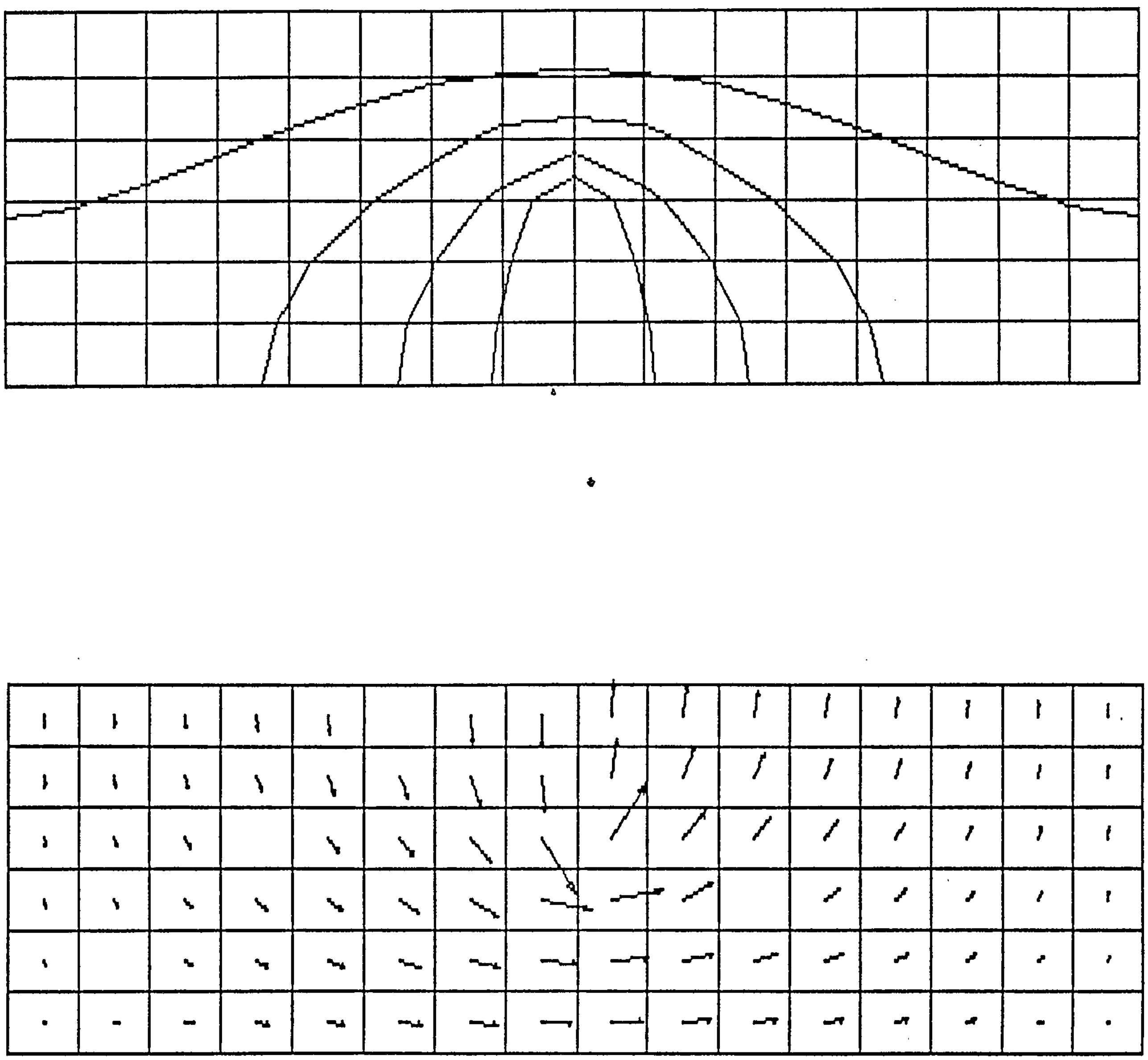


Illustration 15: Computer generated flow net
for Example #1

EXAMPLE #2

FLOW UNDERNEATH A CONCRETE DAM

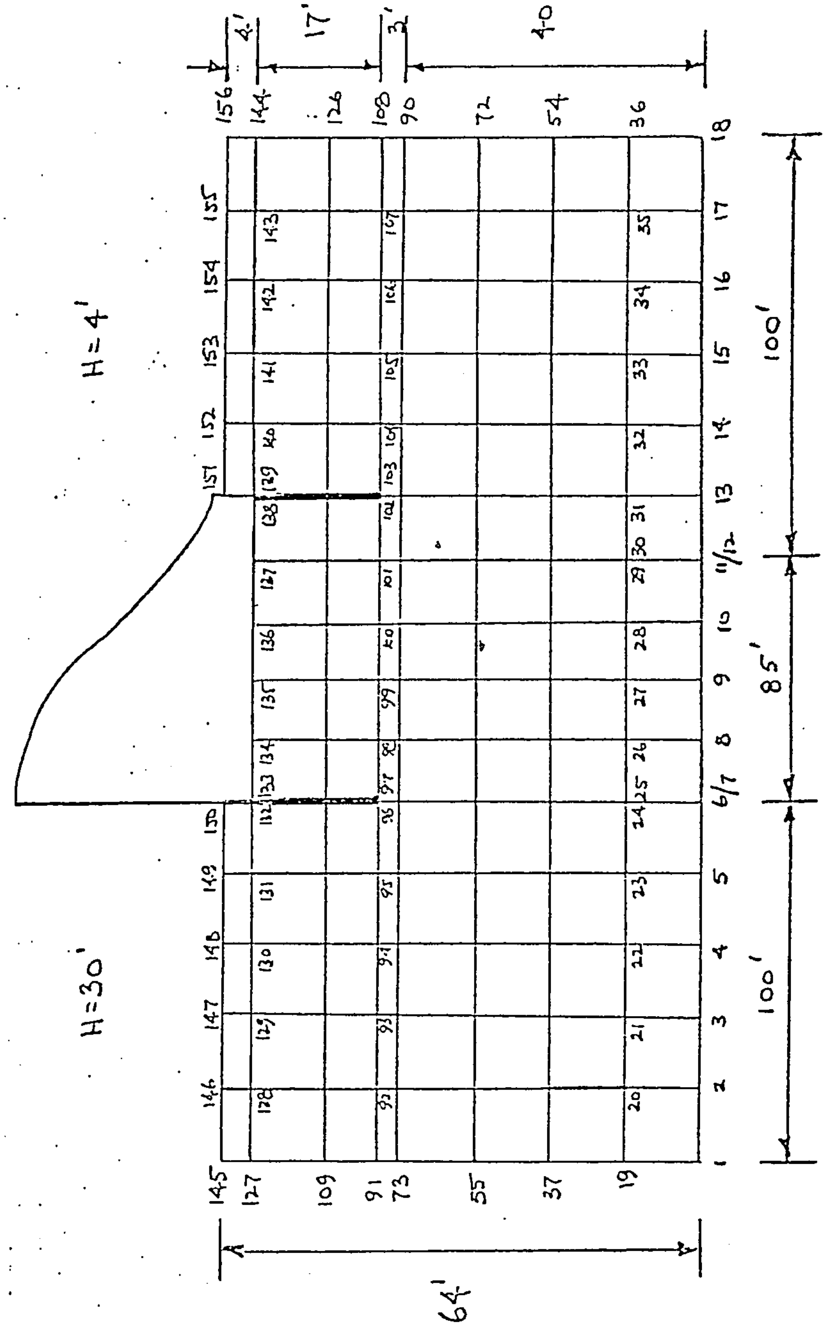


Illustration 16: Sketch of Example #2

```
FLOW UNDER A CONCRETE DAM                               JOSEPH BYRA
    1   0.001667         0.001667
1
    1   0.00          0.00
    6 100.00          0.00         1    1.0
    7 100.00          0.00
   12 185.00          0.00         1    1.0
   13 185.00          0.00
   18 285.00          0.00         1    1.0
   90 285.00         40.00        18    1.0
  108 285.00         43.00        18    1.0
  126 285.00         50.00        18    1.0
  144 285.00         60.00        18    1.0
  156 285.00         64.00        12    1.0
  151 185.00         64.00        -1    1.0
  139 185.00         60.00       -12    1.0
  121 185.00         50.00       -18    1.0
  103 185.00         43.00       -18    1.0
   85 185.00         40.00       -18    1.0
   13 185.00          0.00       -18    1.0
   12 185.00          0.00
   84 185.00         40.00        18    1.0
  102 185.00         43.00        18    1.0
  120 185.00         50.00        18    1.0
  138 185.00         60.00        18    1.0
  133 100.00         60.00        -1    1.0
  115 100.00         50.00       -18    1.0
   97 100.00         43.00       -18    1.0
   79 100.00         40.00       -18    1.0
    7 100.00          0.00       -18    1.0
    6 100.00          0.00
   78 100.00         40.00        18    1.0
   96 100.00         43.00        18    1.0
  114 100.00         50.00        18    1.0
  132 100.00         60.00        18    1.0
  150 100.00         64.00        18    1.0
  145   0.00         64.00        -1    1.0
  127   0.00         60.00       -18    1.0
  109   0.00         50.00       -18    1.0
   91   0.00         43.00       -18    1.0
   73   0.00         40.00       -18    1.0
    1   0.00          0.00       -18    1.0
2
    1      2     20     19      1      4      1      7     18
    8      9     27     26      1      3      1      6     18
    6      8     26     24      1                    4     18
   96     98    116    115      1
  115    116    134    133      1
   14     15     33     32      1      3      1      6     18
   12     14     32     30      1                    4     18
  102    104    122    121      1
  121    122    140    139      1
  139    140    152    151      1
  140    141    153    152      1      3      1
```

3

1	0	0.0	18	1
36	0	0.0	144	18
151	1	68.0	156	1
103	0	0.0	139	18
102	0	0.0	138	18
134	0	0.0	137	1
97	0	0.0	133	18
96	0	0.0	132	18
145	1	94.0	150	1
19	0	0.0	127	18

4

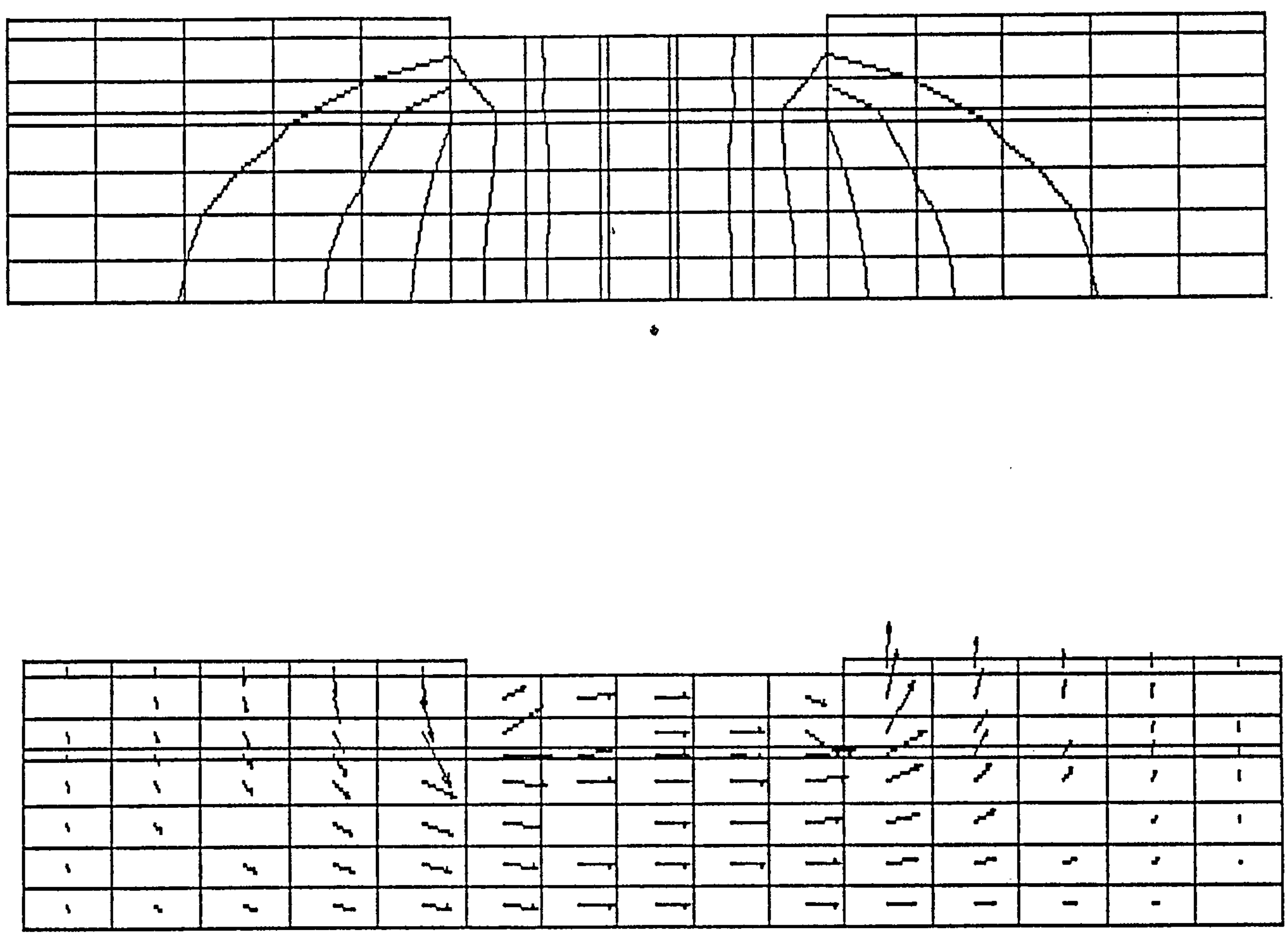


Illustration 17: Computer generated flow net
for Example #2

EXAMPLE #3

FLOW THROUGH AN EARTH DAM

In this example, the phreatic surface is described by Dupuit's formula

$$q = K\ (h_1^2 - h_2^2)\ /\ 2L$$

where q = flow per unit width of dam

K = hydraulic conductivity

h_1 = upstream hydraulic head potential

h_2 = downstream hydraulic head potential

L = length of the dam

The hydraulic head potential at any point on the phreatic surface can be defined by the following parabolic equation:

$$h^2 = h_1^2 - \{(h_1^2 - h_2^2)(x - x_1)/(x_2 - x_1)\}$$

where h = hydraulic head potential at any point on the phreatic surface

x = horizontal coordinate of a point located on the phreatic surface

x_1 = horizontal coordinate where $h = h_1$

x_2 = horizontal coordinate where $h = h_2$

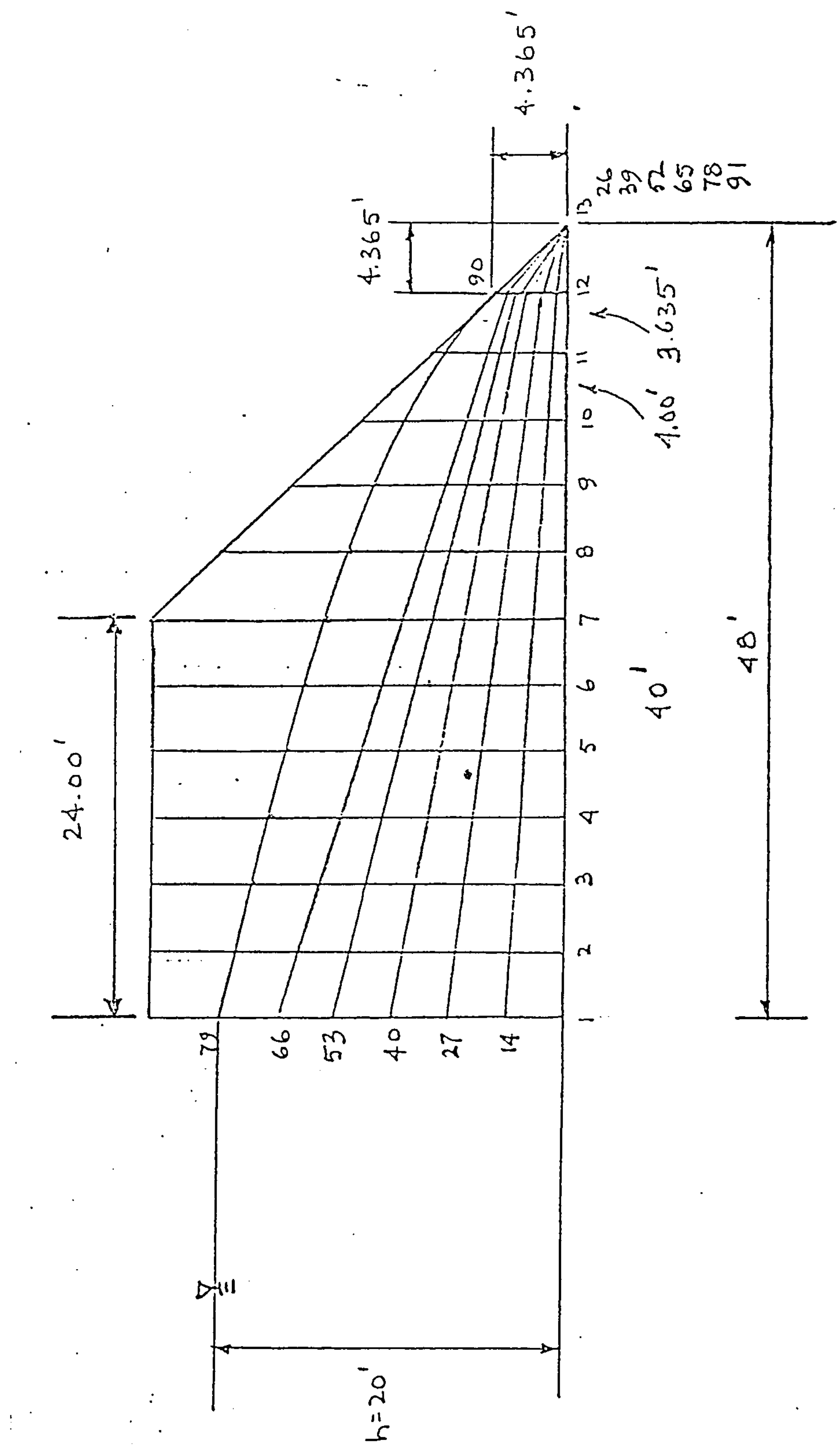
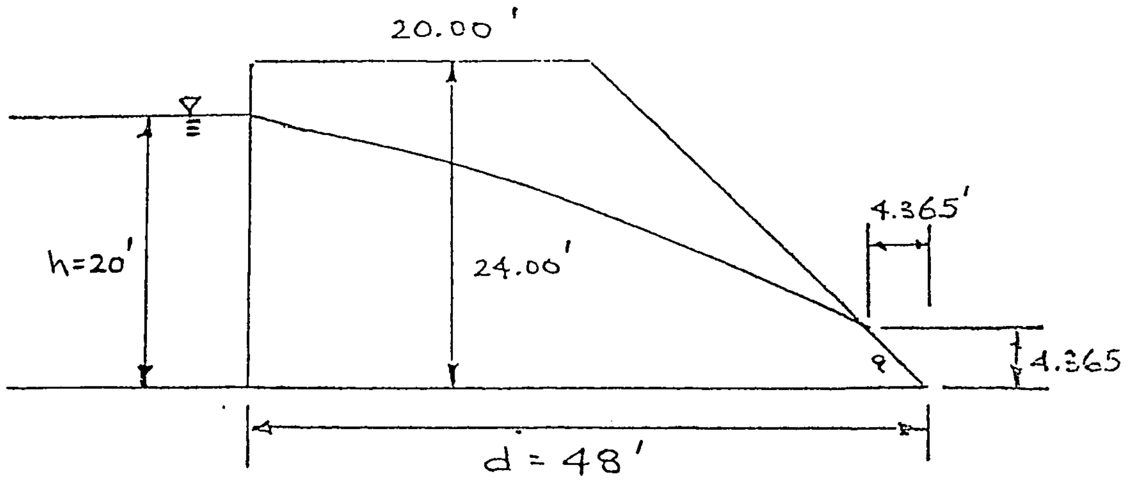


Illustration 18: Sketch of Example #3



$$a = \frac{d}{\cos\alpha} - \sqrt{\frac{d^2}{\cos^2\alpha} - \frac{h^2}{\sin^2\alpha}}$$

$$a = \frac{48}{\cos 45^\circ} - \sqrt{\frac{48^2}{\cos^2 45^\circ} - \frac{20^2}{\sin^2 45^\circ}} = 6.173'$$

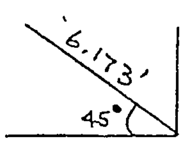


Illustration 19: Location of the phreatic surface for Example #3

$$h^2 = h_1^2 - (h_1^2 - h_2^2)\frac{(x - x_1)}{(x_2 - x_1)} =$$

$$h^2 = 20^2 - (20^2 - 4.365^2)\left(\frac{x - 0}{43.635 - 0}\right)$$

$$h^2 = 400 - 8.730x$$

x	h
0.00'	20.00'
4.00'	19.107'
8.00'	18.170'
12.00'	17.182'
16.00'	16.134'
20.00'	15.013'
24.00'	13.801'
28.00'	12.472'
32.00'	10.983'
36.00'	9.258'
40.00'	7.127'
43.635'	4.365'

Illustration 20: Calculations using Dupuit's formula

```
FLOW THROUGH AN EARTH DAM                      JOSEPH BYRA
    1   0.00000001     0.00000001
1
    1   0.00        0.00
   11  40.00        0.00         1    1.0
   12  43.635       0.00         1    1.0
   13  48.00        0.00         1    1.0
   26  48.00        0.00
   39  48.00        0.00
   52  48.00        0.00
   65  48.00        0.00
   78  48.00        0.00
   91  48.00        0.00
   90  43.635       4.365       -1    1.0
   89  40.00        7.127       -1    1.0
   88  36.00        9.258       -1    1.0
   87  32.00       10.983       -1    1.0
   86  28.00       12.472       -1    1.0
   85  24.00       13.801       -1    1.0
   84  20.00       15.013       -1    1.0
   83  16.00       16.134       -1    1.0
   82  12.00       17.182       -1    1.0
   81   8.00       18.170       -1    1.0
   80   4.00       19.107       -1    1.0
   79   0.00       20.00        -1    1.0
    1   0.00        0.00       -13    1.0
2
    1    2    15    14    1    11    1    5    13
3
    2    0    0.0        12    1
   13    1    0.0
   26    1    0.0
   39    1    0.0
   52    1    0.0
   65    1    0.0
   78    1    0.0
   91    1    0.0
   80    0    0.0        90    1
    1    1   20.0        79   13
4
```

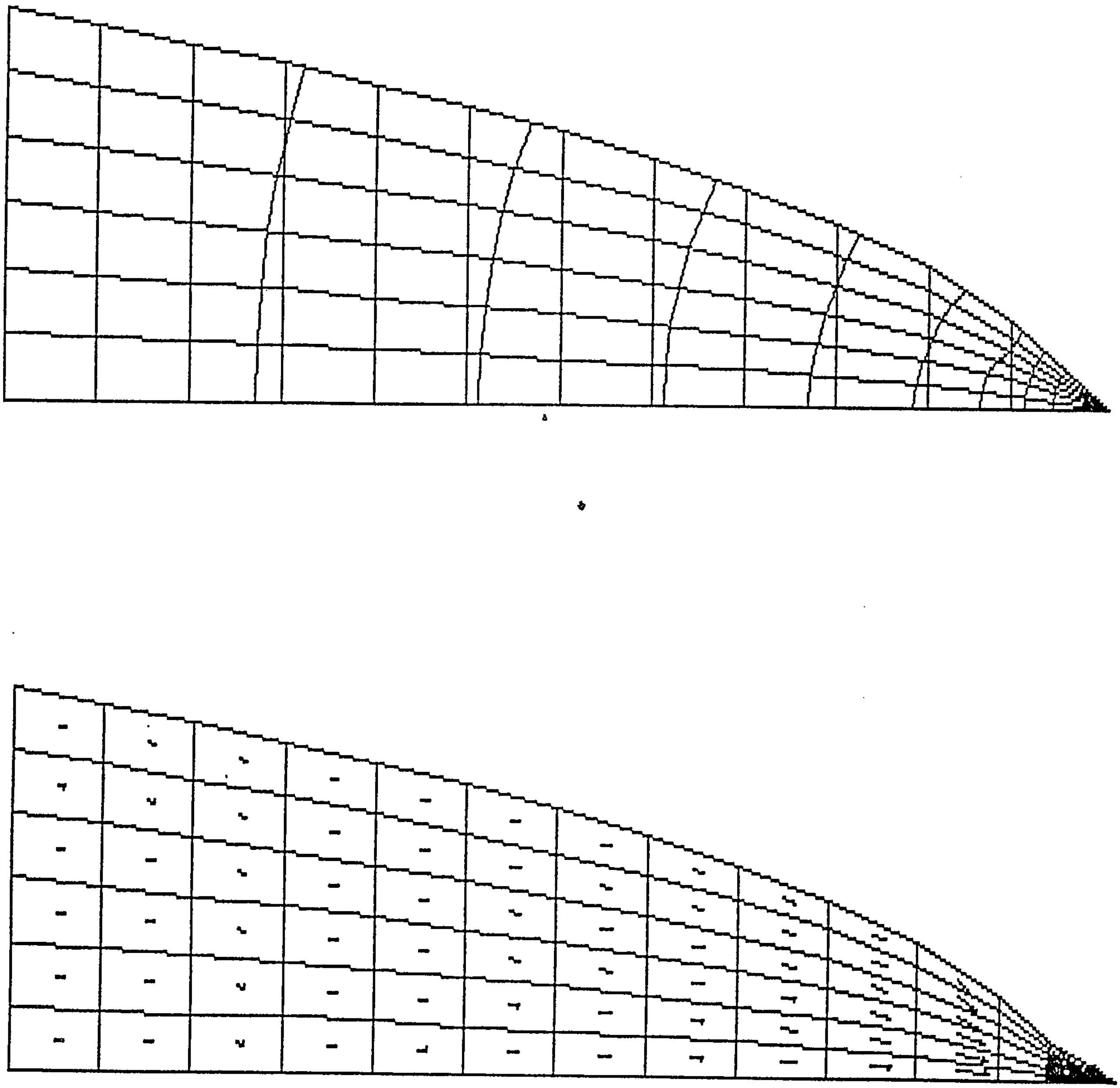


Illustration 21: Computer generated flow net
for Example #3

EXAMPLE #4

FLOW IN ANISOTROPIC MEDIA

This example illustrates the nature of groundwater flow in media that exhibits a different hydraulic conductivity in the x and y directions.

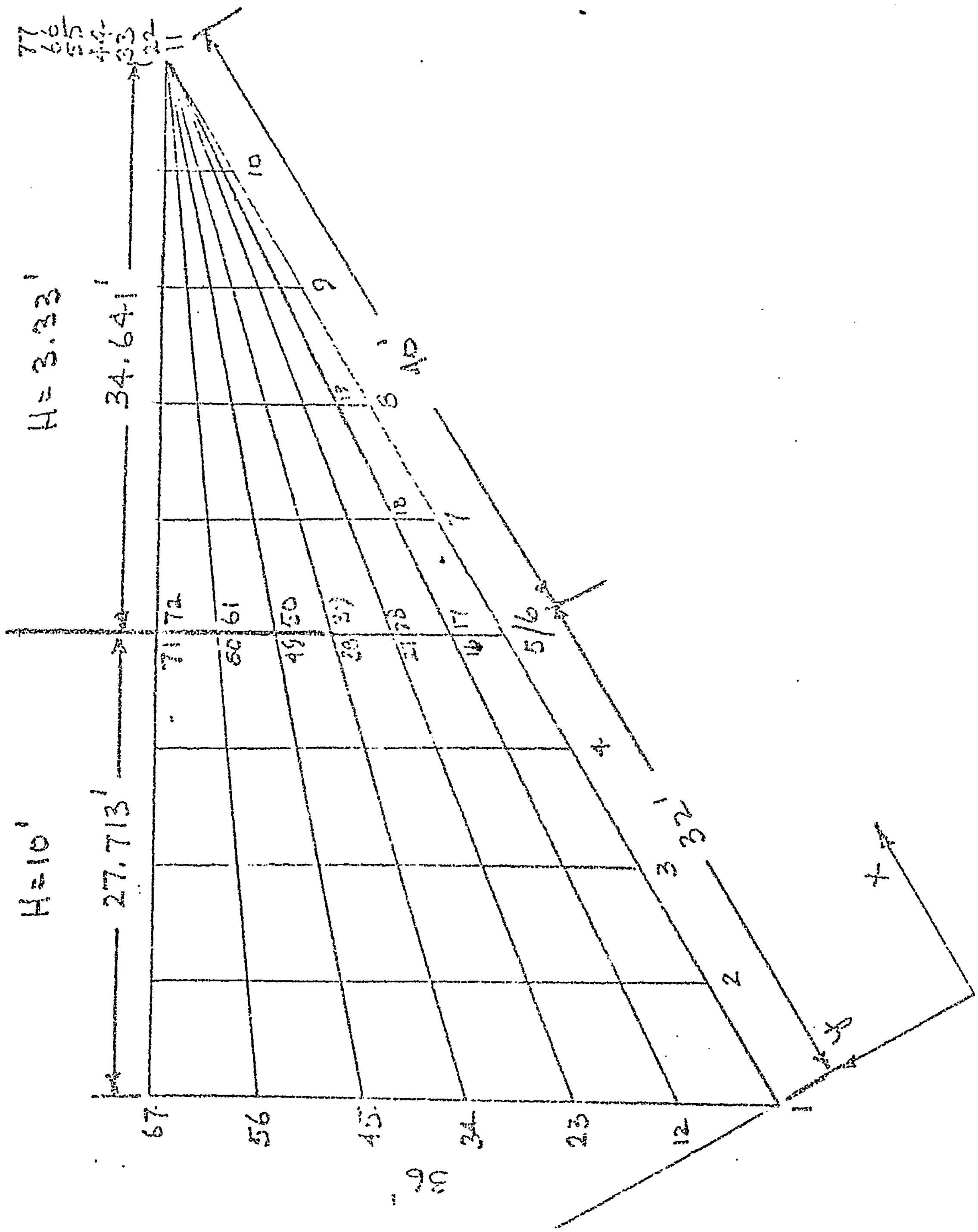


Illustration 22: Sketch of Example #4

```
FLOW IN ANISOTROPIC MEDIA                    JOSEPH BYRA
   1   0.09           0.01
1
   1   0.00       0.00
   5  32.00       0.00        1   1.0
   6  32.00       0.00
  11  72.00       0.00        1   1.0
  22  72.00       0.00
  33  72.00       0.00
  44  72.00       0.00
  55  72.00       0.00
  66  72.00       0.00
  77  72.00       0.00
  72  42.00      17.321      -1   1.0
   6  32.00       0.00      -11   1.0
   5  32.00       0.00
  71  42.00      17.321      11   1.0
  67  18.00      31.177      -1   1.0
   1   0.00       0.00      -11   1.0
2
   1     2    13    12     1     3     1     5    11
   7     8    19    18     1     3     1     5    11
   5     7    18    16     1                 2    11
  38    40    51    50     1
  50    51    62    61     1                 1    11
3
   1     0    0.0         11     1
  72     1    3.333       77     1
  67     1   10.0         71     1
  12     0    0.0         56    11
  38     0    0.0         60    11
  50     0    0.0         61    11
4
```

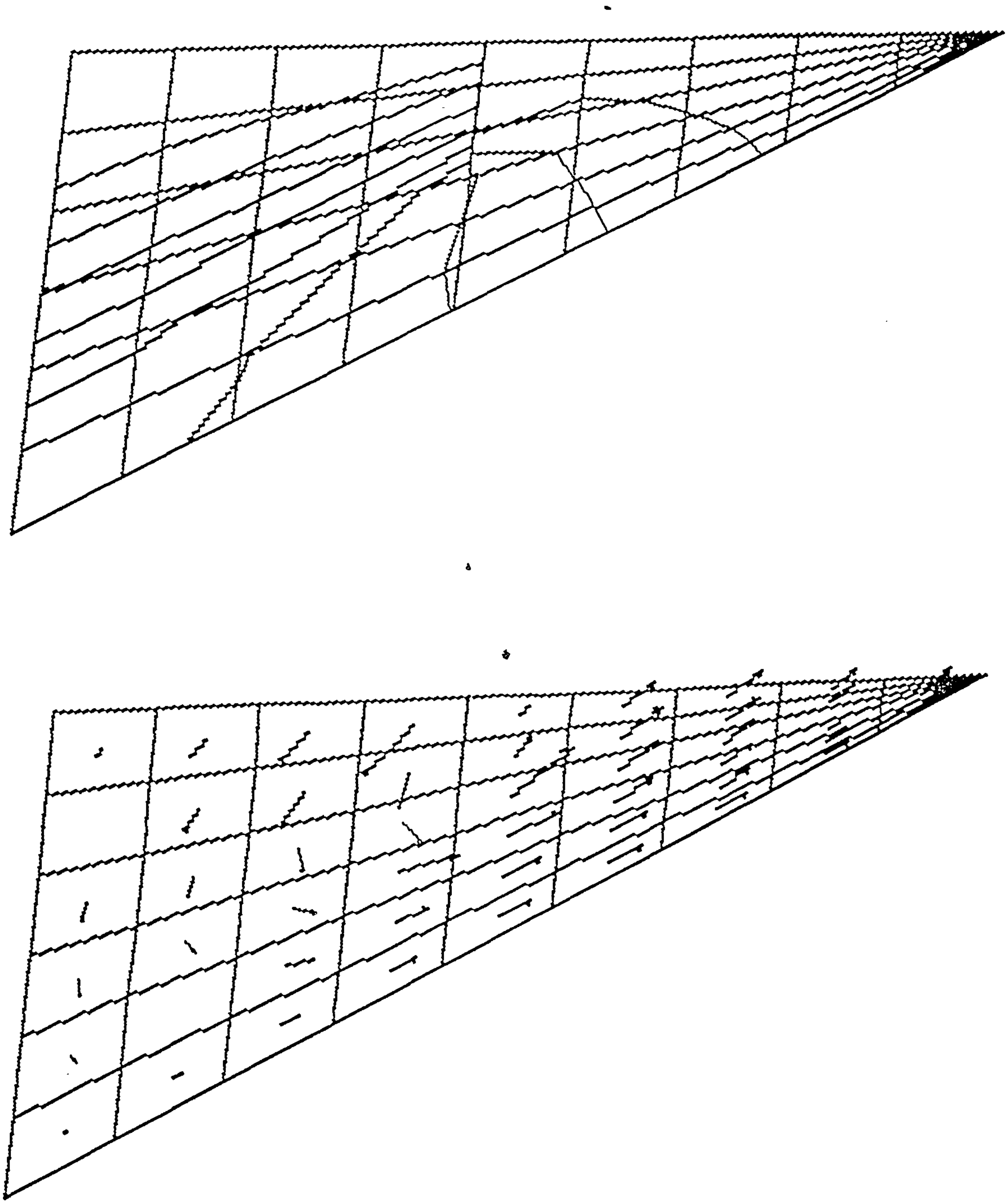


Illustration 23: Computer generated flow net
for Example #4

EXAMPLE #5

FLOW UNDERNEATH A COFFERDAM

This example illustrates the nature of groundwater flow through a multi-layer aquifer system. Typically, as in this example, each layer in a multi-layer aquifer system has a different hydraulic conductivity.

Flow lines deflect when they cross a boundary between soils of different hydraulic conductivities. The flow lines bend to conform to the following relationship:

$$\frac{\tan B}{\tan A} = \frac{K_1}{K_2}$$

where A = angle of reflection

B = angle of incidence

K1 and K2 are the hydraulic conductivities of the respective soils.

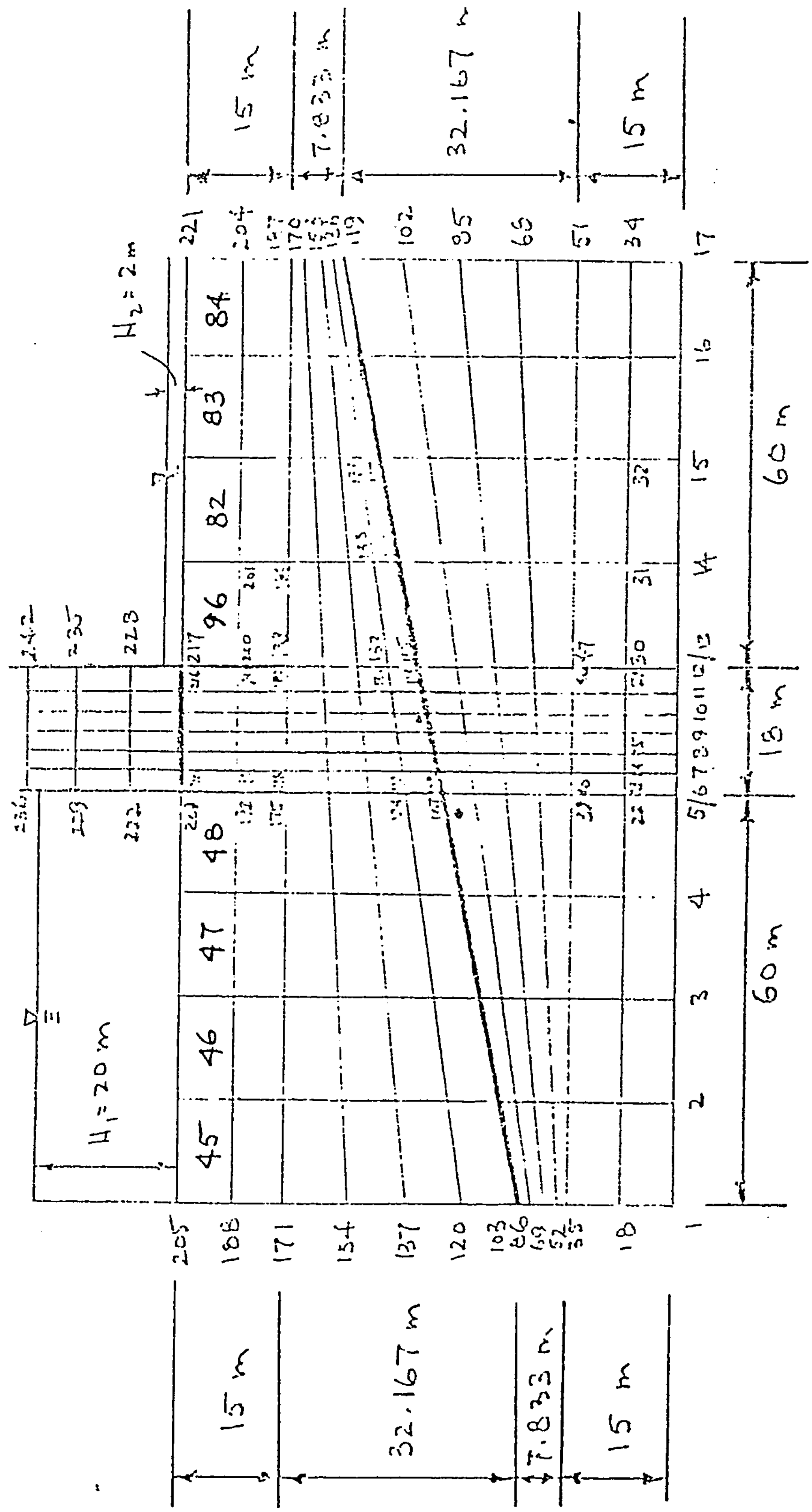


Illustration 24: Sketch of Example #5

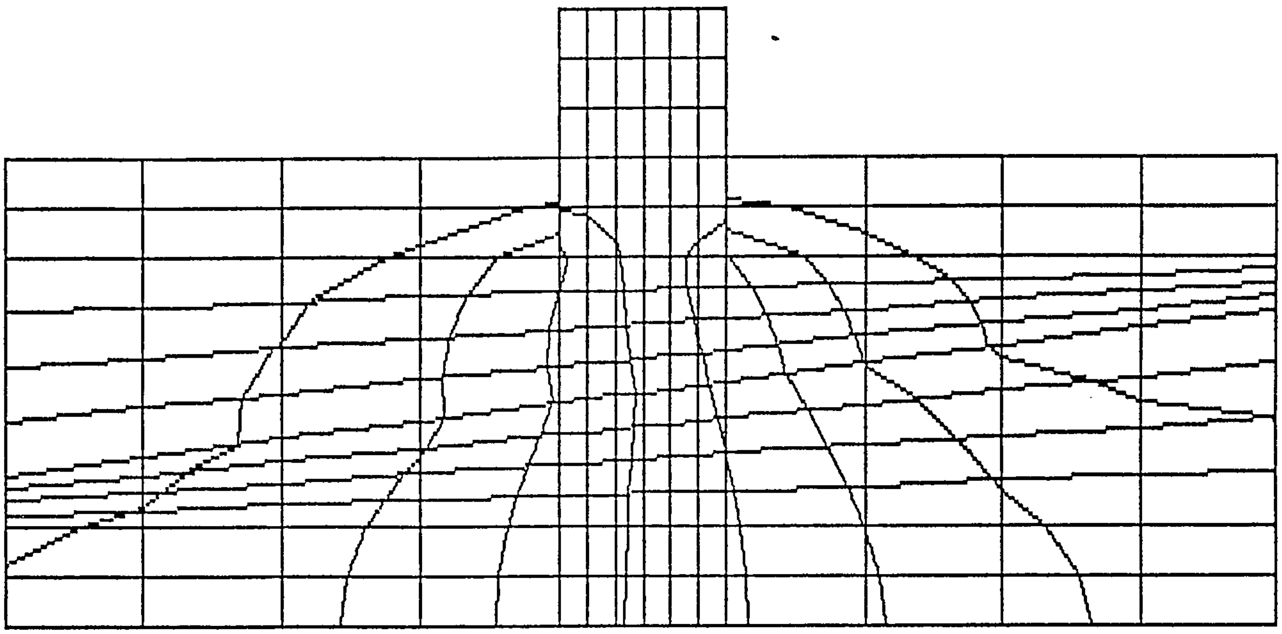
```
FLOW UNDERNEATH A COFFERDAM CONSTRUCTION          JOSEPH BYRA
    1 0.00015            0.00015
    2 0.00000015         0.00000015
    3 0.0000000015       0.0000000015
1
    1   0.00        0.00
    5  60.00        0.00        1    1.0
    6  60.00        0.00
   12  78.00        0.00        1    1.0
   13  78.00        0.00
   17 138.00        0.00        1    1.0
   51 138.00       15.00       17    1.0
  119 138.00       47.167      17    1.0
  187 138.00       55.00       17    1.0
  221 138.00       70.00       17    1.0
  217  78.00       70.00       -1    1.0
  183  78.00       55.00      -17    1.0
  115  78.00       36.587     -17    1.0
   47  78.00       15.00      -17    1.0
   13  78.00        0.00      -17    1.0
   12  78.00        0.00
   46  78.00       15.00       17    1.0
  114  78.00       36.587      17    1.0
  182  78.00       55.00       17    1.0
  216  78.00       70.00       17    1.0
  228  78.00       77.50       12    1.0
  242  78.00       92.50        7    1.0
  236  60.00       92.50       -1    1.0
  222  60.00       77.50       -7    1.0
  210  60.00       70.00      -12    1.0
  176  60.00       55.00      -17    1.0
  108  60.00       33.413     -17    1.0
   40  60.00       15.00      -17    1.0
    6  60.00        0.00      -17    1.0
    5  60.00        0.00
   39  60.00       15.00       17    1.0
  107  60.00       33.413      17    1.0
  175  60.00       55.00       17    1.0
  209  60.00       70.00       17    1.0
  205   0.00       70.00       -1    1.0
  171   0.00       55.00      -17    1.0
  103   0.00       22.833     -17    1.0
   35   0.00       15.00      -17    1.0
    1   0.00        0.00      -17    1.0
2
    1    2   19   18     3    3    1    5   17
  103  104  121  120     2    3    1    5   17
   14   15   32   31     3    2    1    5   17
  116  117  134  133     2    2    1    5   17
   12   14   31   29     3              5   17
  114  116  133  131     2              3   17
  182  184  201  200     2
  200  201  218  217     2
    5    7   24   22     3              5   17
```

107	109	126	124	2			3	17
175	177	194	193	2				
193	194	211	210	2				
7	8	25	24	3	4	1	5	17
109	110	127	126	2	4	1	5	17
210	211	223	222	1	5	1		
222	223	230	229	1	5	1	1	7

3

1	0	0.0	17	1
34	0	0.0	204	17
217	1	2.0	221	1
183	0	0.0	200	17
182	0	0.0	216	17
228	0	0.0	242	7
237	0	0.0	241	1
222	0	0.0	236	7
176	0	0.0	210	17
175	0	0.0	192	17
205	1	20.0	209	1
18	0	0.0	188	17

4



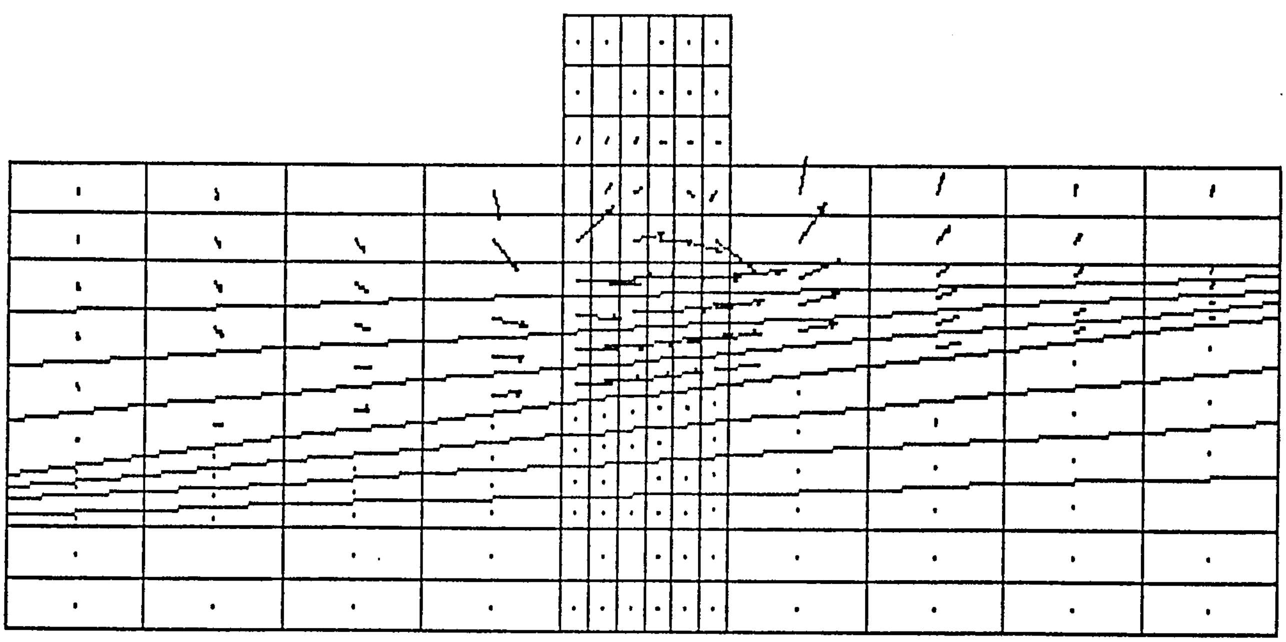


Illustration 25: Computer generated flow net
for Example #5

CONCLUSIONS

The program package 2DFLOW/2DPLOT calculates the hydraulic head potential at each node and the flow velocity at the center of each element. Laplace's equation is solved using the finite element method. As a result, 2DFLOW/2DPLOT does not have the inherent limitations that most of the other programs have, that utilize the finite difference method.

2DFLOW/2DPLOT has the capability to generate computer drawn flow nets for any saturated problem domain, regardless of its shape, geometry or boundary conditions. It can model any steady-state flow condition, such as constant hydraulic head potential, as well as varied hydraulic head potential along any given boundary.

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