5-31-2021

Finite element modeling of underwater acoustic environments and domain decomposition methods

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ABSTRACT

FINITE ELEMENT MODELING OF UNDERWATER ACOUSTIC ENVIRONMENTS AND DOMAIN DECOMPOSITION METHODS

by
General Ozochiawaeze

Underwater acoustic scattering problems have several important applications ranging from sonar imaging in target detection to providing information for sediment classification and geoacoustic inversion. This work presents numerical methods for time-harmonic acoustic scattering problems, specifically, finite element methods for the Helmholtz equation. Furthermore, an iterative domain decomposition formulation is introduced for acoustic scattering problems where the physical domain consists of multiple layers of different materials.

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by
General Ozochiawaeze

A Thesis
Submitted to the Faculty of
New Jersey Institute of Technology
in Partial Fulfillment of the Requirements for the Degree of
Master of Science in Applied Mathematics

Department of Mathematical Sciences

May 2021

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APPROVAL PAGE

FINITE ELEMENT MODELING OF UNDERWATER ACOUSTIC ENVIRONMENTS AND DOMAIN DECOMPOSITION METHODS

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Keep working, keep striving, never give up– fall down 7 times, get up 8.

Denzel Washington

ACKNOWLEDGMENT

Firstly, I like to express my sincere gratitude to my thesis advisor Dr. Christina Frederick. Without your assistance, patience, and dedicated involvement in every step throughout the process, this paper would have never been accomplished. You have been a blessing, and I thank you for also helping to fund this research.

I would also like to show my gratitude to my committee members, Dr. Brittany Hamfeldt and Dr. Yassine Boubendir, for taking the time to help me with both their valuable comments and corrections.

I would like to thank the following people who made my graduate studies here at NJIT both fun and enlightening, especially during such challenging times: Jose Pabon, Austin Juhl, Fatou Ouro, Jake Brusca, Sam Evans, Moshe Silverman, John Wu, and Nicholas Dubicki, and Atul Anurang. Special thanks to Jose and Austin for helping me get more comfortable with LaTeX and coding as well as continual social support.

Finally, I would like to express my very profound gratitude my parents and siblings for providing me with love and encouragement. I would not have gotten as far as I have without them. Thank you for everything.

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CHAPTER 1

INTRODUCTION

1.1 Helmholtz Equation and Acoustic Waves

Acquiring and interpreting sonar imagery of the seafloor is useful to a wide range of oceanographic applications. Of special interest is in the forward modeling of acoustic scattering waves in a rough, two-layered seafloor. To improve methods of underwater communication, we need to model, and thus control, sound propagation arising in the seafloor. In this chapter, we first give an overview of the mathematical background for acoustic scattering problems. Then we discuss the theoretical preliminary background required for the finite element method approach to solving boundary value problems of the Helmholtz equation.

Consider the propagation of acoustic waves in an inhomogeneous medium consisting of a material having distinct properties contained inside the volume $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ (Meury, 2007). Seafloor acoustic propagation is examined analytically and numerically by the Helmholtz equation:

$$\Delta u + \frac{\omega^2}{c^2} u = 0, \tag{1.1}$$

which describes all time-harmonic solutions of the wave equation. A time-harmonic function is a scalar field whose time dependence is a sinusoidal, in the form

$$u(\mathbf{x}, t) = \Re\{u(\mathbf{x}) \exp(-i\omega t)\} = \Re\{u(\mathbf{x})\} \cos \omega t + \Im\{u(\mathbf{x})\} \sin \omega t, \tag{1.2}$$

where $\omega > 0$ denotes the angular frequency and a complex-valued u depends on the position in space $\mathbf{x}$ but not time variable t. In other words, the equation explains the flux of particles as they propagate through some medium. Here u represents the acoustic wave pressure, $k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$ represents the wavenumber, where λ refers to the wavelength associated with a specific sound frequency. The wavenumber is also equal to the ratio of angular frequency ω and sound speed c.

We can derive the Helmholtz from the time-dependent wave equation, namely:

$$v_{tt} = (c(x))^2 \Delta v + F(t, x) \tag{1.3}$$

where c(x) is local wave speed and $F(t, x)$ is a source that injects waves into a solution. Now suppose we look for solutions that generate plane waves with a singular angular time frequency ω, i.e.,

$$v(t, x) = u(x) \exp(-i\omega t), \tag{1.4}$$

$$F(t, x) = g(x) \exp(-i\omega t) \tag{1.5}$$

Substituting (1.4) and (1.5) into (1.3) yields:

$$v_{tt} = u(x)[(-i\omega) \exp(-i\omega t)_t = u(x)(\omega)^2 \exp(-i\omega t) = \omega^2 u(x) \exp(-i\omega t).$$

Hence, $\Delta v = \Delta u \cdot \exp(-i\omega t)$. Finally, we get after dividing both sides by $\exp(-i\omega t)$ the following result:

$$v_{tt} + c(x)^2 \Delta v = u(x)\omega^2 + c(x)^2 \Delta u = g(x).$$

In conclusion, we get the Helmholtz equation: $\Delta u(c(x))^2 + \omega^2 u = g(x)$.

If we assume $c(x) > 0$, then $\Delta u(x) + \frac{\omega^2}{c(x)^2} u(x) = g(x)$. We can denote $\frac{1}{c(x)}$ by $n(x)$, the index of refraction. So altogether we derived the inhomogeneous Helmholtz equation from the wave equation:

$$\Delta u(x) + n(x)^2 \omega^2 u(x) = g(x), \quad x \in \mathbb{R}^d \tag{1.6}$$

The time-dependent wave equation reduces to the time-harmonic Helmholtz equation (Runberg, 2012-04).

1.1.1 Helmholtz Equation and the Fourier Transform

One important implication of Fourier Analysis is that any square-integrable time-dependent field U can be written as a continuous linear combination of time-harmonic fields $\exp(i\omega t)\hat{U}(\mathbf{x}, \omega)$ with varying frequencies $\omega \in \mathbb{R}$, where $\hat{U}$ is the Fourier transform of the time-dependent field U (Moiola, n.d.):

$$U(\mathbf{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(i\omega t)\hat{U}(\mathbf{x}, \omega)\, d\omega \tag{1.7}$$

$$\hat{U}(\mathbf{x}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-i\omega t) U(\mathbf{x}, t)\, dt. \tag{1.8}$$

If U is a solution to (1.3), i.e., the wave equation with wave speed c, then its Fourier transform $\hat{U}$ evaluated at a given frequency, i.e., $u(\mathbf{x}, \omega) = \hat{U}(\mathbf{x}, \omega)$, is a solution to the Helmholtz equation with wavenumber $k = \frac{\omega}{c}$ (Moiola, n.d.). Thus, when studying u and the Helmholtz equation, we are working in the "frequency domain" as opposed to the "time domain" for the wave equation.

1.2 Canonical Solutions to the Helmholtz Equation

The solutions to the Helmholtz equations are generally complex and usually cannot be written explicitly. There are exceptions, however, which we present here.

1.2.1 Plane Wave Solutions

We take the index of refraction $n(x) \equiv n$ to be constant. Then we obtain the general solution

$$u(x) = Ae^{i\omega n\hat{k}\cdot x} \tag{1.9}$$

Here A is the amplitude and $\hat{k}$ indicates the direction of propagation where $|\hat{k}| = 1$. (1.9) is a *plane wave* solution. Furthermore, in the time-dependent setting we also obtain a plane wave:

$$v(t,x) = u(x)e^{-i\omega t} = Ae^{i\omega n\hat{k}\cdot x} \cdot e^{-i\omega t} = Ae^{i\omega n(\hat{k}\cdot x - ct)} \tag{1.10}$$

where $c = \frac{1}{n}$, where n is constant.

Radial Wave Solutions Consider the function

$$u = \frac{e^{ikr}}{r}, \tag{1.11}$$

where $r = |\mathbf{x} - \mathbf{x}_0|$ for some $\mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^d$. For this function to be a solution to (1.4), we need to consider some domain $\Omega \subset \mathbb{R}^3$ and there exists some neighborhood around the point $\mathbf{x}_0$ not included in Ω (Matheson, 2015). Then we can use the Laplace operator in spherical coordinates to check that the function is a solution to the Helmholtz equation (1.1) as follows:

$$\begin{aligned}\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big(\frac{e^{ikr}}{r}\Big)\Big)+k^2\frac{e^{ikr}}{r} &= \frac{1}{r^2}\frac{\partial}{\partial r}(e^{ikr}(ikr-1))+k^2\frac{e^{ikr}}{r}\\ &= -k^2\frac{e^{ikr}}{r}+k^2\frac{e^{ikr}}{r}=0,\end{aligned}$$

where $k=\frac{\omega}{c}$ is the wave number. Note that since the function is radial it can only satisfy boundary conditions that are also radial (Matheson, 2015).

In $\mathbb{R}^3$ the solution we obtain for (1.6) is a circular wave, a wave emanating from a point source. The solution is given by

$$u_c(x)=\frac{e^{i\omega n|\mathbf{x}|}}{4\pi|\mathbf{x}|}, \tag{1.12}$$

which is the Green's function for the Helmholtz equation in three dimensions. In two dimensions the corresponding Green's function is given by the first Hankel function.

In order to find a unique solution to the Helmholtz equation, one needs to specify boundary conditions at infinity. We typically employ the Sommerfield radiation condition and say a solution to this equation is radiating:

$$\lim_{|x|\to\infty}|x|^{n-\frac{1}{2}}\left(\frac{\partial}{\partial|x|}-ik\right)u(x)=0. \tag{1.13}$$

1.2.2 Circular Waves and Bessel Functions

We look for solutions to the Helmholtz equation that are separable in polar coordinates $(x_1,x_2)=(r\cos\theta,r\sin\theta)$. In two dimensional polar coordinates, the

Helmholtz equation can be rewritten as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta} + k^2 u = 0.$$

Employing separation of variables $u(r,\theta) = R(r)\Theta(\theta)$, the Helmholtz equation can be rewritten as

$$R''(r)\Theta + \frac{1}{r}R'(r)\Theta + \frac{1}{r^2}\Theta''(\theta)R + k^2 R\Theta = 0.$$

Multiplying both sides by $r^2/(R\Theta)$ yields the result

$$\left(\frac{r^2}{R}R''(r) + \frac{r}{R}R'(r) + k^2 r^2\right) + \left(\frac{1}{\Theta}\Theta''(\theta)\right) = 0. \tag{1.14}$$

The second, angular part of (1.14) has to be periodic of period 2π, so we take the circular harmonic $\Theta(\theta) = \exp(im\theta)$ for $m \in \mathbb{Z}$. Then $\Theta''(\theta) = -m^2\Theta(\theta)$, so in cancelling Θ from (1.14) and multiplying both sides by r^2, we obtain that R satisfies

$$r^2 R''(r) + rR'(r) + (r^2k^2 - m^2)R(r) = 0. \tag{1.15}$$

Setting $k = 1$, (1.15) is a second-order linear ODE called the *Bessel differential equation.* This ODE has the solution

$$R(r) = C_m J_m(kr) + D_m Y_m(kr), \tag{1.16}$$

where C_m, D_m are constants and $J_m(x), Y_m(x)$ are *Bessel functions of the first kind* and *Bessel functions of the second kind* respectively (Weisstein, 2015).

The *Hankel functions* are complex-valued linear combinations of the Bessel functions

$$H_m^{(1)}(r) \coloneqq J_m(r) + iY_m(r) \tag{1.17}$$

$$H_m^{(2)}(r) \coloneqq J_m(r) - iY_m(r). \tag{1.18}$$

Thus, we deduce that for any $m \in \mathbb{Z}$, the two fields

$$J_m(kr)\exp(im\theta)$$

$$Y_m(kr)\exp(im\theta),$$

and their linear combinations are the solutions of the Helmholtz equations separable in polar coordinates; they are called *circular waves* or *Fourier-Bessel* functions (Moiola, n.d.). Taking complex-valued linear combinations of the Fourier-Bessel functions yields special circular waves called the *Fourier-Hankel* functions, namely

$$H_m^{(1)}(kr)\exp(im\theta) \coloneqq J_m(kr)\exp(im\theta) + iY_m(kr)\exp(im\theta) \tag{1.19}$$

$$H_m^{(2)}(kr)\exp(im\theta) \coloneqq J_m(kr)\exp(im\theta) - iY_m(kr)\exp(im\theta). \tag{1.20}$$

These circular waves are of prime importance for exterior problems posed in unbounded domains in $\mathbb{R}^2$ (Moiola, n.d.).

1.3 Domain Problems

Solutions to the Helmholtz equations exhibit some general properties that differ depending on the type of domain problem.

Assuming time-harmonic waves, i.e., waves of the form:

$$v(t,x) = u(x)e^{i\omega t}, \tag{1.21}$$

the wave equation can then be reduced to the Helmholtz equation:

$$\Delta u(x) + \frac{\omega^2}{c(x)^2}u(x) = 0,\ x \in \Omega, \tag{1.22}$$

with boundary conditions, which can be Dirichlet, i.e.,

$$u(x) = 0, \quad x \in \partial\Omega \tag{1.23}$$

or Neumann, i.e.,

$$\frac{\partial u(x)}{\partial n} = 0, \quad x \in \partial\Omega \tag{1.24}$$

Here the domain Ω is bounded and what we have is an interior problem. The interior problem formulation is well-posed for almost all values of ω. However, the problem is ill-posed for a discrete set of ω, which corresponds to the eigenvalues of operator $S = -\frac{1}{n^2}\Delta$. That is, $Su = \omega^2 u$, so ω denotes eigenvalues corresponding to eigenfunction

u. Here we are treating Helmholtz equation as a solution operator:

$$\begin{aligned} T\colon u &\to \Delta u + n^2\omega^2 \\ \mathbf{x} &\to \quad \mathbf{x} \end{aligned} \tag{1.25}$$

Then T is singular and there is either an infinite set of solutions or no solution (Runberg, 2012-04).

In exterior problems, the Helmholtz equation is set in an unbounded domain. The exterior problems of present concern are scattering problems, which refer to the propagation of waves colliding with some object. That is, we are considering the problem of a wave hitting an impenetrable obstacle. More precisely, we let Ω denote some object, or scatterer, illuminated by an incident wave u_{inc}. Specifically, let $u_{inc}(x) = \exp\Big(ikx \cdot \hat{\theta}\Big)$ be a plane wave with $|\hat{\theta}| = 1$ that is propagating rightward and either upward or downward in the plane. The incident wave is also called the "incoming field", "incoming wave", or "incident field". Then the scattered field u_{scat} is the wavefield generated by u_{inc} colliding with bounded domain Ω. u_{scat} can also be thought of as the "reflected wave".

More formally, if we let u_{tot} denote the sum of the known incident wave u_{inc} and the unknown scattered wave u_{scat}, and assume $u_{tot} = 0$ on the boundary of the object $\partial\Omega$, then the scattering problem is to find the scattered field u_{scat} that satisfies:

$$\Delta u_{scat}(x) + \omega^2 u_{scat}(x) = 0,\ x \notin \overline{\Omega} \tag{1.26}$$

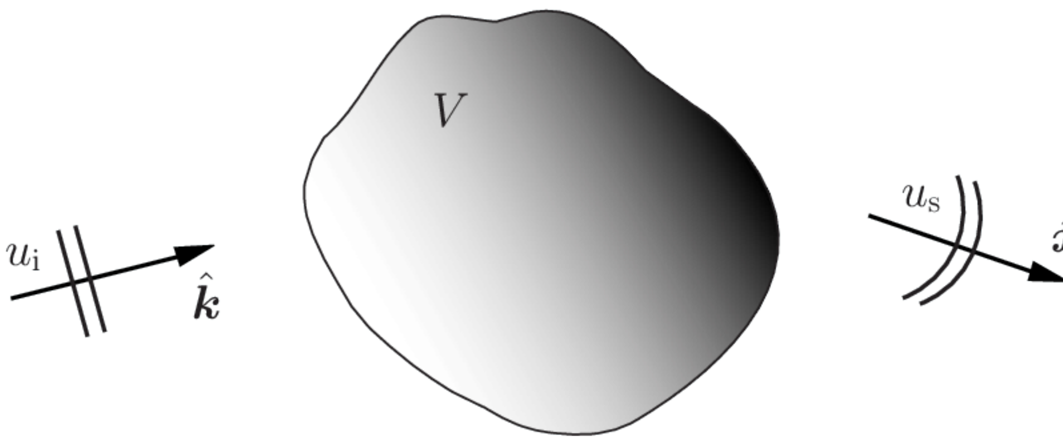

Figure 1.1 Illustration of a direct scattering problem: The scatterer V is subject to the incident plane wave u_i in the $\hat{\mathbf{k}}$-direction. Scattering wave u_s is detected in the $\hat{\mathbf{x}}$-direction, adopted from (Sohl et al., 2008).

and one of either

$$u_{scat}(x) = -u_{inc}(x),\ x \in \partial\Omega \tag{1.27}$$

which denotes inhomogeneous Dirichlet boundary condition or

$$\frac{\partial u_{scat}(x)}{\partial n} = -\frac{\partial u_{inc}(x)}{\partial n},\ x \in \partial\Omega \tag{1.28}$$

which denotes the inhomogeneous Neumann boundary condition.

The scattering problem is well-posed if additional boundary conditions are given at infinity, namely, the Sommerfield radiation conditions (8) are satisfied (guarantees the scattered wave is outgoing).

In many cases when the inhomogeneous Helmholtz equation models a physical situation with waves inside a bounded domain (interior problem), there is often some damping in the material which defines solution at resonant frequencies. We thus add a damping term to the equation, i.e.,

$$\nabla^2 u(x) + n(x)^2\omega^2 u(x) + i\omega^2\alpha u(x) = 0,\ x \in \Omega, \tag{1.29}$$

where $\alpha > 0$ denotes the damping coefficient. This formulation is well-posed for all frequencies ω, with the damping term making it so that the waves eventually die off when traveling long distances and energy dissipating. (Runberg, 2012-04)

1.4 Fundamental Solution of the Helmholtz Equation

In acoustic scattering problems, the incident wave (or incident field) is generated by a point source, i.e., for some $\mathbf{z} \in \mathbb{R}^d \setminus \Omega$,

$$u_{inc}(\mathbf{x}) = G_k(\mathbf{x}, \mathbf{z}), \quad \mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{z}\}, \tag{1.30}$$

where G_k is the fundamental solution or Green's function of the Helmholtz equation, given in the two-dimensional or three-dimensional cases by

$$G_k(\mathbf{x}, \mathbf{y}) := \begin{cases} \frac{i}{4} H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|), & d = 2 \\ \frac{\exp(ik|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}, & d = 3, \end{cases} \tag{1.31}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \mathbf{x} \neq \mathbf{y}$, where $H_\nu^{(1)}$ denotes the Hankel function of the first kind of order ν. These Green's functions correspond to the solution of the Helmholtz equation with a Dirac δ-function source at $\mathbf{y}$ (Atle, 2006). When representing a source far from the scatterer, the incident field is a plane wave, i.e., for some $\hat{\theta} \in \mathbb{R}^d$ with $|\hat{\theta}| = 1$,

$$u_{inc}(\mathbf{x}) = \exp\Big(ik\mathbf{x} \cdot \hat{\theta}\Big), \quad \mathbf{x} \in \mathbb{R}^d. \tag{1.32}$$

1.5 Helmholtz Problem and Boundary Conditions

The Helmholtz equation is an elliptic PDE and to obtain a well-posed problem we need suitable boundary conditions. In this section, we summarize the different boundary conditions mentioned earlier along with their physical applications.

The Helmholtz problem is commonly considered in an unbounded exterior domain with scatterers at the boundary and the Sommerfield radiation condition at infinity. However, for numerical experimental purposes, we often formulate this problem on a bounded domain instead of an unbounded domain.

1.5.1 Sommerfield Radiation Condition

We need a condition that represents the behavior of a wave at infinity to guarantee unique a solution to wave problems on unbounded domains. We addressed earlier that we impose the Sommerfield radiation condition to do accomplish this.

Assuming no waves are reflected at infinity as is typical for wave propagation in free space, let $u(\mathbf{r})$ be the solution to a homogeneous Helmholtz equation in an exterior domain $\Omega^+ = \mathbb{R}^d \setminus \bar{\Omega}$, where $\bar{\Omega}$ is the closure of domain. We assume that a wave source is placed at the origin and denote by R the radial distance from the origin to the observation point.

To absorb waves at infinity, we impose the Sommerfield radiation condition, which can also be written in polar coordinates as

$$\lim_{R \to \infty} R^{\frac{d-1}{2}} \left(\frac{\partial u}{\partial R} - iku\right) = 0. \tag{1.33}$$

1.5.2 Dirichlet Boundary Condition

We impose Dirichlet boundary conditions in bounded domains when the material of a surface has much lower resistance (or impedance) than the carrier medium. The Dirichlet boundary conditions are as follows:

$$u = 0 \text{ on } \partial\Omega. \tag{1.34}$$

1.5.3 Neumann Boundary Condition

Neumann boundary conditions are imposed for bounded domains when the surface material has much higher resistance (or acoustic impedance) than the carrier medium:

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega. \tag{1.35}$$

Here $\mathbf{n}$ denotes the outer normal vector on the boundary of the domain.

1.5.4 Robin Boundary Condition

The Robin boundary condition is a generalization of the Dirichlet and Neumann boundary conditions. Namely, this condition models the acoustic impedance of the boundary in general:

$$\frac{\partial u}{\partial \mathbf{n}} + i\beta u = 0 \quad \text{on } \partial\Omega. \tag{1.36}$$

Here i is the imaginary unit and β is a modifying coefficient that measures the admittance of the surface.

1.6 Green's Identities

In this section, we briefly recall Green's identities which will prove useful for the remainder of this thesis.

Let Ω be simply connected and a bounded region in $\mathbb{R}^2$ with a C^2 boundary $\partial\Omega$ and let $\mathbf{F}(\mathbf{x}) \in C^1(\overline{\Omega})^3$ be a vector-valued function or vector field. If $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$, then we can state the divergence theorem as follows:

Theorem 1.6.1 (Divergence Theorem)**.**

$$\int_\Omega \nabla \cdot \mathbf{F}(\mathbf{x})\, d\mathbf{x} = \int_{\partial\Omega} \mathbf{F}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, d\mathbf{x}.$$

For $u, v \in C^2(\overline{\Omega})$, set $\mathbf{F}(\mathbf{x}) = u(\mathbf{x})\nabla v(\mathbf{x})$ and substitute into the divergence theorem to obtain the following result:

Green's First Identity:

$$\int_\Omega (u\Delta v + \nabla u \cdot \nabla v)\, d\mathbf{x} = \int_{\partial\Omega} u\frac{\partial v}{\partial \mathbf{n}}\, ds(\mathbf{x}).$$

You can also take $\mathbf{F}(\mathbf{x}) = v(\mathbf{x})\nabla u(\mathbf{x})$ and switch u and v in the integrands:

$$\int_{\Omega} (v\Delta u + \nabla v \cdot \nabla u)\; d\mathbf{x} = \int_{\partial\Omega} v \frac{\partial u}{\partial \mathbf{n}}\; ds(\mathbf{x}).$$

Subtract the previous two equations to obtain:

Green's Second Identity:

$$\int_{\Omega} (u\Delta v - v\Delta u)\; d\mathbf{x} = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}} \right)\; ds(\mathbf{x}).$$

We will mainly make use of Green's First identity in deriving the weak formulation of acoustic scattering problems.

1.7 Finite Element Method for the Helmholtz Equation

Two common approaches to solving elliptic PDEs like the Helmholtz equation are finite difference methods and variational methods. The finite element method (FEM) falls in the latter category. The FEM is a method for boundary value problems that discretize the domain, which is divided into small regions or elements. Meshing describes the process of subdividing into non-overlapping elements.

One of the first steps in FEM is to identify the PDE associated with the physical phenomenon we are studying. The PDE (or differential form) is the strong form and the integral form we derive is the weak form.

The measurable, bounded domain, $\Omega \in \mathbb{R}^d$, is discretized with a standard regular mesh. We partition the domain Ω into a finite set of disjoint cells $\mathcal{T} = \{K\}$, where $K \subset \Omega$, such that

$$\bigcup_{K \in \mathcal{T}} K = \Omega. \tag{1.37}$$

These cells form the regular mesh, which is typically made of simple polygonal shapes, though other more sophisticated shapes are possible, such as a non-polygonal domain generated by curved cells.

The original PDE (1.1) is referred to as the strong form, and the weak formulation $a(u, v) = L(v)$ is a re-formulation of the strong form. Here, H^1 is a function space known as a Sobolev space where all the functions are bounded and quadratic integrable.

1.7.1 Mathematical Formalism of Finite Element Method

In this section, we first collect some necessary results from functional analysis and the weak theory of elliptic PDEs to better formalize the FEM method. We start by formally defining the function spaces we need.

Function Spaces

Let Ω be an open subset of $\mathbb{R}^d$, $d \in \mathbb{N}$. We restrict our attention to real-valued functions, $f : \mathbb{R}^d \to \mathbb{R}$, on the given domain, Ω, that are Lebesgue measurable. We denote the Lebesgue integral of $f : \mathbb{R}^d \to \mathbb{R}$ with respect to the Lebesgue measure μ by

$$\int_\Omega f(x) d\mu. \tag{1.38}$$

Let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$. Then we define L^p spaces in the following manner:

$$L^p(\Omega) := \Big\{ f : \Omega \to \mathbb{R}^d : \|f\|_{L^p(\Omega)} < \infty \Big\}. \tag{1.39}$$

with

$$\|f\|_{L^p(\Omega)} := \|f\|_p = \Big[\int_\Omega |f(x)|^p d\mu \Big]^{\frac{1}{p}}, \quad 1 \leq p \leq \infty, \tag{1.40}$$

and

$$\|f\|_{L^\infty(\Omega)} := \|f\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|. \tag{1.41}$$

L^p spaces are function spaces that generalize the p-norm for finite-dimensional vector spaces and are Banach spaces of Lebesgue integrable functions. (A Banach space is a complete normed vector space, i.e., all Cauchy sequences of vectors are convergent to a vector in the space under the norm).

For $p = 2$, $L^p(\Omega)$ is a Hilbert space with inner product

$$(f, g) \coloneqq \langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)\, dx. \tag{1.42}$$

A Hilbert space $\mathcal{H}$ is a complex inner product space that is complete under the associated norm, and so is a strict subset of a Banach space.

We also define the following:

$$L_1^{loc}(\Omega) \coloneqq \Big\{ f : \Omega \to \mathbb{R} \text{ is measurable and locally integrable } \Big\}.$$

By "locally integrable", we mean that $f \in L^1(K)$ $(p = 1)$ for all compact subsets $K \subset \Omega$. $L_1^{loc}(\Omega)$ is the *space of all locally integrable functions*. We can now define the weak derivative:

We say a function $f \in L_1^{loc}(\Omega)$ (f is locally integrable) is *weakly differentiable* with respect to x_i if there is a $g \in L_1^{loc}(\Omega)$ (that is, if there exists a locally integrable g) such that

$$\int_\Omega g(x)\phi(x)dx = -\int_\Omega f(x)\partial_{x_i}\phi(x)dx \quad \forall \phi \in C^\infty(\Omega). \tag{1.43}$$

Then g is referred to as the *weak (partial) derivative* and can be written as $g = \partial_{x_i} f$.

We briefly introduce notation to define a Sobolev space. A *multi-index* is an n-tuple of nonnegative integers, usually denoted by α or β:

$$\alpha = (\alpha_1, ..., \alpha_n), \quad \beta = (\beta_1, ..., \beta_n).$$

If α is a multi-index, then we define

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$

which we call the *order* or *degree* of our multi-index α. We can then define higher-order weaker derivatives as:

$$\partial^\alpha(f) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}. \tag{1.44}$$

We can now say that a locally integrable function $g(x) = \partial^\alpha f(x)$ is a higher-order weak derivative of $f(x)$ if

$$\int_\Omega g(x)\phi(x)dx = (-1)^{|\alpha|} \int_\Omega f(x)\partial^\alpha \phi(x)dx \quad \forall \phi \in C^\infty(\Omega), \tag{1.45}$$

that is, for all infinitely differentiable functions ϕ with compact support in Ω. We can now define Sobolev spaces:

For $k \in \mathbb{N} \cup \{0\}$ and $1 \leq p \leq \infty$. The *Sobolev space of order* k is defined as

$$W^{k,p}(\Omega) := \Big\{ f \in L^p(\Omega) : \partial^\alpha(f) \in L^p(\Omega) \Big\}. \tag{1.46}$$

In other words, a Sobolev space is a function space (vector space of functions) equipped with a norm that is a combination of L^p-norms of the function together with its weak derivatives up to a given order. (Nair, 2007)

Remark:

(1) $W^{k,p}(\Omega)$ is a complete, normed function space and thus a Banach space with respect to the norm $\|\cdot\|_{k,p}$. In particular, $W^{k,p}(\Omega)$ is a subspace of the Banach space $L^p(\Omega)$.

(2) We introduce the standard norm in $W^{k,p}(\Omega)$:

$$\|u\|_{W^{k,p}(\Omega)} = \left(\int_\Omega \sum_{|\alpha| \leq k} |\partial^\alpha u|^p dx \right)^{\frac{1}{p}}. \tag{1.47}$$

(3) $W^{p,0}(\Omega) = L^p(\Omega)$.

(4) $H^k(\Omega) := W^{k,2}(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_k := \sum_{|\alpha| \leq k} \langle \partial^\alpha f, \partial^\alpha g \rangle, \quad f, g \in H^2(\Omega). \tag{1.48}$$

(5) The Sobolev space $W^{1,p}(\Omega)$ can be also introduced by the following definition:

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \middle| \exists g_1, g_2, \ldots, g_N \in L^p(\Omega) \text{ such that} \right. \tag{1.49}$$

$$\left. \int_\Omega u \frac{\partial \varphi}{\partial x_i} = - \int_\Omega g_i \varphi \quad \forall \varphi \in C_c^\infty(\Omega), \quad \forall i = 1, 2, \ldots, N \right\}. \tag{1.50}$$

We then set

$$H^1(\Omega) = W^{1,2}(\Omega). \tag{1.51}$$

In other words, the Sobolev space $H^1(\Omega)$ is the space of complex-valued $L^2(\Omega)$ functions, whose first distributional partial derivatives are in $L^2(\Omega)$. (By distributional derivative, we mean there is a function $w \in L^2(\Omega)$ such that $\int_\Omega v \frac{\partial \varphi}{\partial x_1} = -\int_\Omega w\varphi \quad \forall \varphi \in C_c^\infty(\Omega)$) (Nair, 2007).

That is,

$$H^1(\Omega) := \Big\{ v \in L^2(\Omega) \ : \ \partial_{x_i} v \in L^2(\Omega) \text{ exists for all } i = 1, \ldots, d \Big\}.$$

$$\langle u, v \rangle_{H^1(\Omega)} := \langle u, v \rangle_{L^2(\Omega)} + \sum_{i=1}^{d} \langle \partial_{x_i} u, \partial_{x_i} v \rangle_{L^2(\Omega)}.$$

$$\|v\|_{H^1(\Omega)} = \left(\|v\|^2_{L^2(\Omega)} + \sum_{i=1}^{d} \|\partial_{x_i} v\|^2_{L^2\Omega)} \right).$$

For $u \in W^{1,p}(\Omega)$ we define $\frac{\partial u}{\partial x_i} = g_i$, and write

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N} \right). \tag{1.52}$$

The space $W^{1,p}(\Omega)$ is then equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_p + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_p. \tag{1.53}$$

(6) Building on (5), we can alternatively define the *Sobolev spaces of order k* as

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega) \middle| \ \forall \alpha \text{ with } |\alpha| \le k, \ \exists g_\alpha \in L^p(\Omega) \text{ such that} \right. \tag{1.54}$$

$$\left. \int_\Omega u \, \partial_\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g_\alpha \varphi \quad \varphi \in C_c^\infty(\Omega) \right\}. \tag{1.55}$$

Note that $C_c^\infty(\Omega)$ refers to the set of complex-valued C^∞ functions defined on Ω whose support is compactly contained in Ω.

1.7.2 More Important Properties

$(H^k(\Omega), \langle\cdot,\cdot\rangle_{H^k})$ is a Hilbert space for every $k \in N_0$. Furthermore, we take $H^0(\Omega) = L^2(\Omega)$. We also note that

$$C^{\infty,k}(\Omega) := \left\{ v \in C^\infty(\Omega) \ : \ \int_\Omega |\partial^\alpha v(x)|^2 \, dx < \infty \text{ for all } \alpha \in N_0^d \text{ with } |\alpha|_1 \le k \right\}$$

is dense in $H^k(\Omega)$ with respect to $\|\cdot\|_{H^k(\Omega)}$, i.e., for every $u \in H^k(\Omega)$ and every $\epsilon > 0$ there is a $v_\epsilon \in C^{\infty,k}(\Omega)$ such that $\|v_\epsilon - u\|_{H^k(\Omega)} < \epsilon$ (Jahnke, n.d.). This leads to the following definition that will prove important when deriving the variational form of some elliptic problems:

The *Sobolev space* $H_0^k(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_{H^k(\Omega)}$, i.e.,

$$u \in H_0^k(\Omega) \iff \text{There are } v_n \in C_c^\infty \text{ such that } \lim_{n\to\infty} \|u - v_n\|_{H^k(\Omega)} = 0.$$

So $H_0^k(\Omega)$ is a closed subspace of $H^k(\Omega)$. Furthermore, if the boundary $\partial\Omega$ is a C^1 set, then $v \in C(\overline{\Omega}) \cap H_0^k(\Omega)$ implies that $v(x) = 0$ for all $x \in \partial\Omega$.

1.7.3 Weak Solution of Elliptic PDEs

Consider PDEs of the form

$$Lu = f,$$

where L is a linear differential operator of the form

$$Lu = -\sum_{j,k=1}^{n} \partial_j(a_{jk}(x)\partial_k u) + \sum_{j=1}^{n} b_j(x)\partial_j u + c(x)u \tag{1.56}$$

acting on functions $u : \Omega \to \mathbb{R}$ on a bounded open set $\Omega \subset \mathbb{R}^d$. Here $a_{jk}, b_j, c : \Omega \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ are functions given on Ω. The given coefficient functions satisfy

$$a_{jk}, b_j, c \in L^\infty(\Omega), \quad a_{jk} = a_{kj}.$$

We say the operator L is *elliptic* if the matrix (a_{jk}) is positive definite. *Ellipticity* can also be characterized as follows: the operator L is *elliptic* on Ω if there exists some constant $\theta > 0$ satisfying

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq \theta|\xi|^2 \tag{1.57}$$

for x almost everywhere in Ω and every $\xi \in \mathbb{R}^n$. The Laplacian operator $L = -\Delta$ is an example of an elliptic operator on any open set, with $\theta = 1$.

Assuming all functions and the domain are sufficiently smooth, we can multiply by a smooth *test function* (also known as a bump function) $v \in C_c^\infty(\Omega)$, integrate over $x \in \Omega$, and integrate by parts, noting that

$$\int_\Omega \partial_j(b_j u)v \; dx = -\int_\Omega b_j u \partial_j v \; dx,$$

which leads to the condition

$$\sum_{j,k=1}^{n}(a_{jk}\partial_j u, \partial_k v) + \sum_{j=1}^{n}(b_j\partial_j u, v) + (cu, v) - \sum_{j,k=1}^{n}(a_{jk}\partial_k u n_j, v)_{\partial\Omega} = (f, v), \tag{1.58}$$

where n is the outward unit normal on $\partial\Omega$ and

$$\begin{aligned}(f, g) &\coloneqq \int_{\Omega} f(x)g(x)\ dx \\ (f, g)_{\partial\Omega} &\coloneqq \int_{\partial\Omega} f(x)g(x)\ dx\ .\end{aligned}$$

This formulation requires $a_{jk}, b_j, c \in L^{\infty}(\Omega)$ and $f \in L^2(\Omega)$. We then search for the *weak solution* $u \in V$, where V is a suitably chosen function space satisfying (1.58) for all $v \in V$ including the boundary conditions. Our suitably chosen function space V will depend partly on our boundary conditions (Clason, 2017).

Dirichlet Conditions: Here $u = g$ on $\partial\Omega$ for a given $g \in L^2(\Omega)$. If $g = 0$, then we have a *homogeneous Dirichlet condition* and we take $V = H_0^1(\Omega)$, in which case the boundary integrals in (1.58) vanish since $v = 0$ on $\partial\Omega$. We can state the weak formulation as follows: We define a bilinear form

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

by

$$a(u,v) \coloneqq \sum_{j,k=1}^{n} (a_{jk}\partial_j u, \partial_k v) + \sum_{j=1}^{n} (b_j \partial_j u, v) + (cu, v) = (f, v),$$

for all $v \in H_0^1(\Omega)$.

Neumann Conditions: We require $\sum_{j,k=1}^{n} a_{jk}\partial_k u n_j = g$ on $\partial\Omega$ for a given $g \in L^2(\partial\Omega)$. Substitute this in boundary integral of (1.58) and take $V = H^1(\Omega)$. We then look for $u \in H^1(\Omega)$ satisfying

$$a(u,v) = (f,v) + (g,v)_{\partial\Omega}$$

for all $v \in H^1(\Omega)$ (Clason, 2017). There is also a weak formulation for Robin (or impedance) conditions which generalizes the Neumann conditions case.

Robin/Impedance Conditions: Set $\beta u + \sum_{j,k=1}^{n} a_{jk}\partial_k u n_j = g$ on $\partial\Omega$ for given $g \in L^2(\partial\Omega)$ and $\beta \in L^\infty(\partial\Omega)$. Substitute into the boundary integral and then the weak form will be to find a $u \in H^1(\Omega)$ satisfying

$$a_R(u,v) = a(u,v) + \beta(u,v)_{\partial\Omega} = (f,v) + (g,v)_{\partial\Omega}$$

for all $v \in H^1(\Omega)$ (Clason, 2017).

1.7.4 Lax-Milgram Theorem

To conclude, we state the following theorem necessary for guaranteeing the existence and uniqueness of a solution of the following general form of a linear variational problem:

for a given Hilbert space V, a bilinear form $a : \hat{V} \times V \to \mathbb{R}$ and a continuous linear function $L : \hat{V} \to \mathbb{R}$, find a $u \in V$ such that

$$a(u, v) = L(v) \quad \forall v \in V. \tag{1.59}$$

This theorem is a generalization of the Riesz representation theorem and is known as the Lax-Milgram theorem:

Theorem 1.7.1 (Riesz-Representation Theorem for Hilbert Spaces)**.** *Any continuous linear functional L on a Hilbert space $\mathcal{H}$ can be represented uniquely as*

$$L(v) = (u, v),$$

for some $u \in \mathcal{H}$. Additionally,

$$\|L\|_{\hat{\mathcal{H}}} = \|u\|_{\mathcal{H}},$$

where $\hat{\mathcal{H}}$ is the dual space of $\mathcal{H}$.

Thus, the Riesz representation theorem establishes a connection between the dual space of a Hilbert space and the Hilbert space itself, namely, there is a natural isometry L between $\mathcal{H}$ and $\hat{\mathcal{H}}$ (Brenner and Scott, 2007).

Theorem 1.7.2 (Lax-Milgram Theorem)**.** *Let a Hilbert space* V*, a bilinear form* $a : \hat{V} \times V \to \mathbb{R}$*, and a continuous linear function* $L : \hat{V} \to \mathbb{R}$ *be given satisfying the following conditions:*

- *There exists* $c_1 > 0$ *such that* $a(v, v) \geq c_1 \|v\|_V^2 \quad \forall v \in V$ *(coercivity),*

- *There exist* $c_1, c_2 > 0$ *such that* $a(u, v) \geq c_2 \|u\|_V \|v\|_V \quad \forall\, u, v \in V$ *(continuity).*

Then the linear variational problem stated has a unique solution, i.e., there exists $u \in V$ *to our problem satisfying*

$$\|u\|_V \leq \frac{1}{c_1} \|L\|_{\hat{V}}, \tag{1.60}$$

where $\hat{V}$ *is the dual space of* V*. (Nair, 2007).*

Proofs of both theorems can be found in Brenner and Scott (Brenner and Scott, 2007).

If the two properties continuity and coercivity hold, then there are two important consequences. The first is that the Lax-Milgram theorem implies that there exists a unique solution to the variational problem. The second consequence concerns the *weak Galerkin discretization* of the variational problem, namely, given V_h, a finite-dimensional subspace of V,

$$\text{we can find } u_h \in V_h \text{ such that } a(u_h, v_h) = L(v_h) \text{ for all } v_h \in V_h.$$

Hence, if continuity and coercivity hold, then the Lax-Milgram theorem implies that the Galerkin solution u_h exists and is unique (Moiola and Spence, 2014).

CHAPTER 2

OVERVIEW OF THE FINITE ELEMENT METHOD WITH NUMERICAL EXAMPLES

We again consider a general linear variational problem written in the following canonical form for elliptic PDEs: given a Hilbert space V, a bilinear form $a : \hat{V} \times V \to \mathbb{R}$ and a continuous linear function $L : \hat{V} \to \mathbb{R}$, we want to find $u \in V$ such that

$$a(u, v) = L(v) \quad \forall v \in \hat{V},$$

where $\hat{V}$ is the dual space of V. We know this problem has a unique solution if it satisfies the conditions stated in the Lax-Milgram theorem.

We then discretize the variational problem by restricting it to a pair of discrete test and trial spaces. That is, the function space V on which the variational formulation is defined is replaced by a finite-dimensional subspace $V_h \subset V$. The approximation u_h of the solution u is expressed as a linear combination of the finite number of *basis functions* $\phi_j(x)$ which are continuous, nonzero on only on small subdomains.

Hence, we approximate the solution to the linear elliptic boundary value problem with the weak Galerkin approximation: we want to find $u_h \in V_h \subset V$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall v \in V_h, \tag{2.1}$$

where $(\cdot, \cdot)$ is the $L^2(\Omega)$ inner product and $a(\cdot, \cdot)$ is a bilinear form related to the weak form of the PDE. The finite element method then becomes a systematic way to construct the subspace V_h and to derive a matrix equation from the approximate problem. The finite element space V_h constitutes a space spanned by a set of basis functions V_h=span$\{\phi_i\}$. Each element in V_h is related to a vector of coefficients

$\mathbf{x} \in \mathbb{R}^n$:

$$v = \sum_{i=1}^{n} (\mathbf{x})_i \phi_i. \tag{2.2}$$

We then obtain a matrix equation of the form:

$$A\mathbf{x} = \mathbf{b}. \tag{2.3}$$

where $A_{i,j} = a(\phi_h, \phi_i)$ and $\mathbf{b}_i = (f, \phi_i)$.

So in finite element analysis, we are taking a PDE (strong form), re-formulating it into its weak form, and reducing the boundary value problem to a matrix algebra one, as we shall see with the Helmholtz equation.

Finite Elements

A finite element, in the most abstract setting, is defined as a triple $(K, \mathcal{P}, \mathcal{N})$:

1. Let $K \subseteq \mathbb{R}^n$ denote a closed, bounded set with nonempty interior and a piecewise smooth boundary. We call K the *element domain*.

2. Let $\mathcal{P}_K \subset C(K)$ be a finite-dimensional space of continuous functions on K with $\dim \mathcal{P}_K = p_K$. We call $\mathcal{P}$ the space of *shape functions*.

3. $\mathcal{N}_K = \{N_1, N_2, \ldots, N_k\}$ is a basis for $\hat{\mathcal{P}}$ and denotes the set of *nodal variables*. That is, $\mathcal{N}_K$ is an indexed family of linear functionals on $\mathcal{P}_K$.

Then $(K, \mathcal{P}_K, \mathcal{N}_K)$ is called a *finite element*. Assume the nodal variables lie in the dual space of some larger function space, e.g., a Sobolev space (Brenner and Scott, 2007). Let Ω be the domain on which the problem is defined. Then a finite element method consists of defining n elements $\{(K_r, \mathcal{P}_{K_r}, \mathcal{N}_{K_r}\}_{r \in [1,n]}$ in such a way so that

(i) $\overline{\Omega} = \bigcup_{r=1}^{n} K_r$

(ii) $\dim(K_r \cap K_s) < \dim \Omega \quad \forall r, s \in [1, n],\ r \neq s$

(iii) $\mathcal{P}_h = \{u \in C(\Omega)\ :\ r \in [1, n],\ u\big|_{K_r} \in \mathcal{P}_{K_r}\},$

where $\overline{\Omega}$ denotes the closure of Ω (Matheson, 2015).

We define the *mesh* of the finite element method as the set $\mathcal{K} = \{K_r\}_{r \in [1,n]}$.

Note that for nodal elements, we can find a basis $\{\phi_1, \phi_2, \ldots, \phi_k\}$ of $\mathcal{P}$ dual to $\mathcal{N}$; i.e., $N_i(\phi_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta, defined as

$$\delta_{ij} \coloneqq \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{2.4}$$

This basis is the *nodal basis* of $\mathcal{P}$ (Brenner and Scott, 2007).

Example of Nodal Elements: Lagrange Elements

The Lagrange elements are a popular family of nodal elements where the function space $\mathcal{P}_K$ is the space of polynomials of degree $\leq k$ and a basis for $\mathcal{P}_K$ satisfying $N_i(\phi_j) = \delta_{ij}$ consists of the Lagrange polynomials, i.e.,

$$N_K^i(\phi) \coloneqq \prod_{0 < j \leq p} \frac{\phi - \phi_K^j}{\phi_K^i - \phi_K^j}, \quad i \neq j. \tag{2.5}$$

When the domains are simplexes, these elements are called P_k elements where k is the degree of the polynomial (Matheson, 2015). In this thesis, the simulations we produce incorporate piecewise linear and quadratic Lagrange finite elements, i.e., P_1 and P_2 finite elements respectively.

2.1 Weak Form of Helmholtz Equation

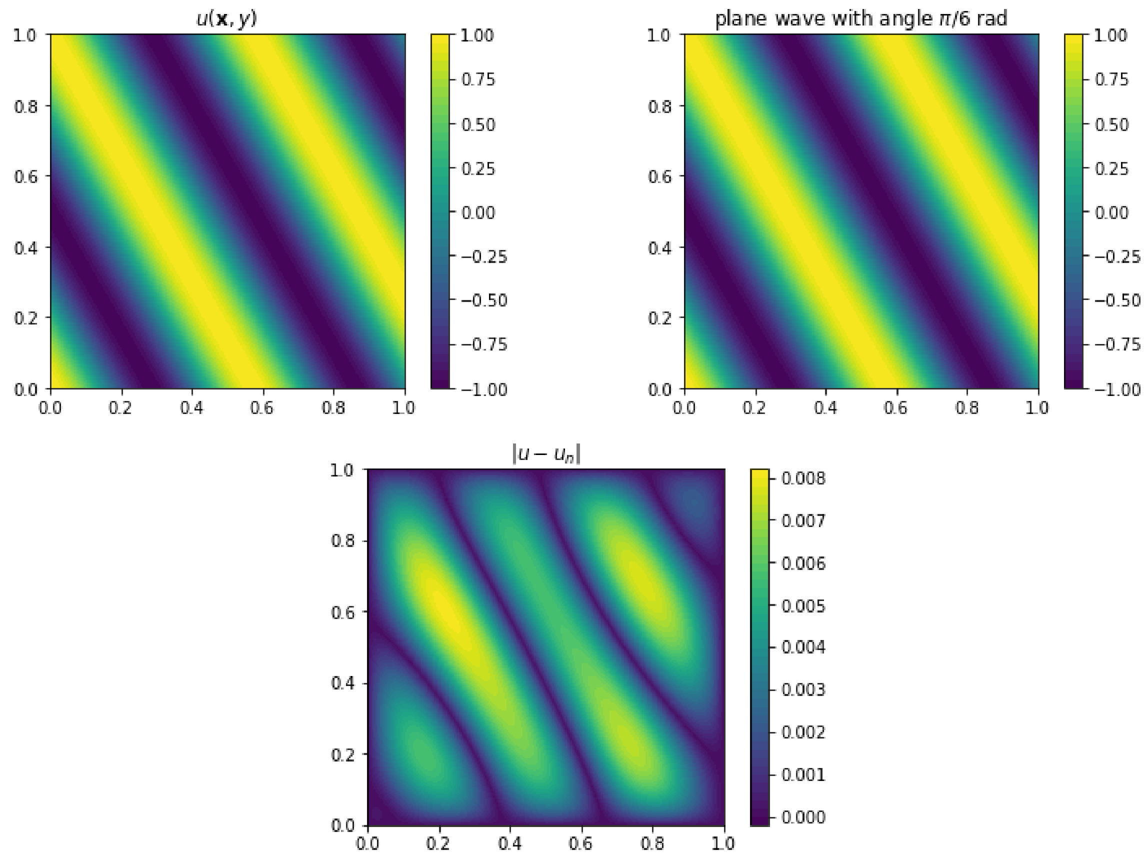

Figure 2.1 **Left**: Plot of the FEM solution to $u = \cos(2\pi\omega(k_1 x + k_2 y))$ of the Helmholtz boundary value problem (2.1), with $\omega = 2.0$, $k_1 = \cos\left(\frac{\pi}{6}\right)$, and $k_2 = \sin\left(\frac{\pi}{6}\right)$. Here we have a unit square 32-by-32 mesh. **Right**: Plot of the true solution, which is a plane wave rotated by $\frac{\pi}{6}$ rad. The third plot is the plot of the absolute error between the FEM solution and the exact solution. These plots were implemented using the FEnICS computing platform.

We will see later in this thesis that the FEM discretization of the governing time-harmonic acoustic Helmholtz equation in underwater domains enables accurate modeling of the seafloor environment. In this chapter, FEM discretization is used to simulate plane waves by solving the Dirichlet problem in 2D:

$$\begin{cases} \Delta u + k^2(x)u = f, & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.6}$$

where Ω is an open bounded domain in $\mathbb{R}^2$, $k(x) = \frac{\omega}{c(x)}$ is the inhomogeneous wave number, $c = c(x)$ is the sound speed, and $f \in L^2(\Omega)$ is the source.

We use a finite element method to approximate solution to (1.17) with Dirichlet boundary conditions. Here

$$V = H_0^1(\Omega) = \{v \in H^1(\Omega) \,:\, v|_{\partial\Omega} = 0\},$$

$$V_h = \{\text{piecewise linear polynomials that are 0 on } \partial\Omega\}.$$

The standard weak formulation is given by:

$$\text{Find } u \in V, \text{ such that } a(u, v) = L(v), \quad \forall v \in V,$$

where

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dV - \int_\Omega k^2 uv \, dx, \tag{2.7}$$

$$L(v) := \int_\Omega fv \, dx. \tag{2.8}$$

For the Helmholtz equation, we have the k-dependent inner product and norm

$$(u, v)_{1,k,\Omega} := \int_\Omega \left(\nabla u \cdot \nabla v + k^2 uv\right) \, dx,$$

$$\|v\|^2_{1,k,\Omega} := \|\nabla v\|_{L^2(\Omega)} + k^2 \|v\|^2_{L^2(\Omega)},$$

one the space $H^1(\Omega)$.

For the Dirichlet problem of the Helmholtz equation,continuity of $a(u,v)$ follows from the Cauchy-Schwarz inequality, namely

$$\begin{aligned}|a(u,v)| &\leq \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + k^2\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\leq (1+k^2)\|u\|_{1,k,\Omega}\|v\|_{1,k,\Omega}.\end{aligned}$$

However, the BVP does not have a unique solution if the wavenumber $k^2 = \lambda_j$ for λ_j an eigenvalue of the negative Laplacian in Ω with zero boundary conditions. That is, $a(u,v)$ cannot be bounded below by $\|v\|^2_{1,k,\Omega}$ for all $k > 0$. If $k^2 = \lambda_j$, then $a(u,v) = 0$ for u_j, the corresponding eigenfunction of the eigenvalue λ_j. Furthermore, if $k^2 > \lambda_1$ (the largest Laplace-Dirichlet eigenvalue), then the bilinear form a takes both positive and negative real values (Moiola and Spence, 2014). Thus, the bilinear form a is not coercive. Although $a(\cdot,\cdot)$ is not coercive, the Fredholm Alternative implies that if k^2 is not an eigenvalue of the negative Laplacian, then a solution to the variational problem exists and is unique (Moiola and Spence, 2014). In summary, the Helmholtz equation in Ω with Dirichlet boundary conditions is not well-posed for every k, especially large k, i.e., this makes high-frequency problems harder to solve than low-frequency problems.

Figure 2.4 and Tables 2.1 and 2.2 were produced from considering the Neumann problem on a unit square, Ω, with boundary Γ:

$$\begin{cases} \Delta u + k^2 u = f & x \in \Omega \\ \nabla u \cdot \mathbf{n} = 0 & x \in \Gamma, \end{cases} \tag{2.9}$$

for some known function f. Then the weak form is similar to the weak form for the Dirichlet problem, namely find a $u \in W$ such that $a(u,v) = L(v)$ for all $v \in W$,

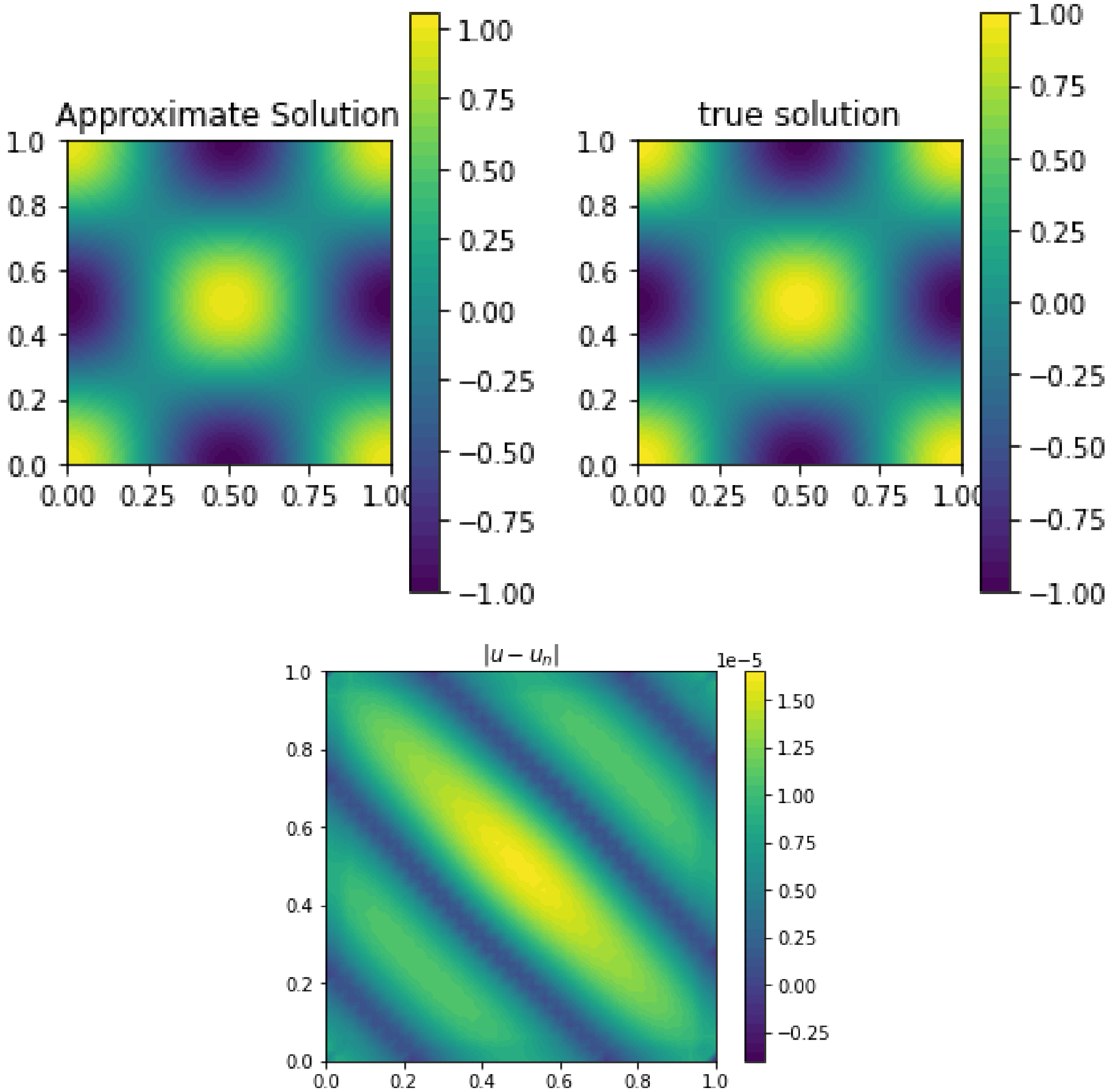

Figure 2.2 Plot of the true solution $u(x, y) = \cos(2\pi x)\cos(2\pi y)$ and the approximate FEM solution to the Neumann problem. The third plot is the absolute error between the true solution and the approximate FEM solution.

where

$$\begin{cases} a(u, v) \coloneqq \displaystyle\int_\Omega \left(\nabla u \cdot \nabla v - k^2 uv\right)\, d\mathbf{x}, \\ L(v) \coloneqq \displaystyle\int_\Omega fv \; d\mathbf{x}, \\ W \coloneqq H^1(\Omega) = \{v \in L^2(\Omega) \,:\, \partial_{x_i} v \in L^2(\Omega),\, 1 \leq i \leq 2\}. \end{cases}$$

The Dirichlet problem, unlike the Neumann problem, has zero boundary conditions on the unit square.

2.1.1 Convergence of Finite Element Method

Let s denote the degree of the polynomial of the finite element space, and assume that the solution $u \in H^{s+1}(\Omega)$. Let u_h denote the weak Galerkin approximation of u, i.e., the finite element approximation. Then we have

$$\|u - u_h\|_{L^2} \le Ch^{s+1}, \tag{2.10}$$

$$\|u - u_h\|_{L^\infty} \le Ch^{s+2}, \tag{2.11}$$

for some constant C (Villa, 2015). Tables 2.1 and 2.2 below show that the numerical results are consistent with the finite element convergence theory. In particular, for piecewise linear finite element $P1$ we observe second order convergence in the L^2-norm and third order convergence in the L^∞-norm. For piecewise quadratic finite element $P2$ we observe third order convergence in the L^2-norm and fourth order convergence in the L^∞-norm.

Table 2.1 Error At The Vertex Values Of Mesh With **N** Degrees Of Freedom With P_1 Elements

N	$\frac{\|u-u_n\|_{L^2}}{\|u\|_{L^2}}$	$\frac{\|u-u_n\|_\infty}{\|u_n\|_\infty}$
32	0.005409	0.00416
64	0.001358	0.00103
128	0.000339	0.000259
512	0.000021253	0.000001623
1024	5.313×10^{-6}	4.058×10^{-6}

Table 2.2 Error At The Vertex Values Of Mesh With **N** Degrees Of Freedom With P_2 Elements

N	$\frac{\|u-u_n\|_{L^2}}{\|u\|_{L^2}}$:	$\frac{\|u-u_n\|_{\infty}}{\|u_n\|_{\infty}}$:
32	6.93×10^{-5}	1.28×10^{-5}
64	8.62×10^{-6}	8.36×10^{-7}
128	1.07×10^{-6}	5.29×10^{-8}
512	1.68×10^{-8}	2.08×10^{-10}
1024	4.41×10^{-9}	3.73×10^{-11}

2.2 Point Source Problem: Helmholtz Equation With Damping Term

We derive the weak formulation of the Helmholtz equation with the damping term. Namely, we consider the Helmholtz problem with a point source in the domain and zero boundary conditions on the square:

$$\Delta u(x) + \omega^2 u(x) + i\omega\alpha u(x) = f, \tag{2.12}$$

where f is the point source in domain. Here u is a complex-valued function $u = u_r + iu_i$ where $\text{Re}(u) = u_r$ and $\text{Im}(u) = u_i$. Substituting complex-valued u into (2.5), we obtain

$$\Delta(u_r + iu_i) + \omega^2(u_r + iu_i) + i\alpha\omega(u_r + iu_i) = f,$$

and simplifying further, we then get

$$\Delta(u_r + iu_i) + \omega^2(u_r + iu_i) + i\alpha\omega u_r - \alpha\omega u_i = f.$$

We can split the terms into real and imaginary parts:

$$\Delta u_r + \omega^2 u_r - \alpha\omega u_i + i[\Delta u_i + \omega^2 u_i + \alpha\omega u_r] = f,$$

to obtain a coupled system:

$$\begin{cases} \Delta u_r + \omega^2 u_r - \alpha\omega u_i = f, \\ \Delta u_i + \omega^2 u_i + \alpha\omega u_r = 0. \end{cases} \tag{2.13}$$

We obtain the weak form by multiplying each equation by a separate test function $v_r \in H_0^1(\Omega)$ and $v_i \in H_0^1(\Omega)$ respectively and integrating both sides of each equation. We then integrate by parts and use of Green's first identity.

First, we focus on the real part of our coupled system using Green's first identity:

$$\int_\Omega \Delta u_r v_r \, d\mathbf{x} + \int_\Omega \omega^2 u_r v_r \, d\mathbf{x} - \int_\Omega \alpha\omega u_i v_r \, d\mathbf{x} = \int_\Omega f v_r \, d\mathbf{x}$$
$$\left(\int_\Omega \nabla(u_r) \cdot \nabla(v_r) \, d\mathbf{x} - \int_{\partial\Omega} u_r \frac{\partial v_r}{\partial \mathbf{n}} \, ds \right) + \int_\Omega \omega^2 u_r v_r \, d\mathbf{x} - \int_\Omega \alpha\omega u_i v_r \, d\mathbf{x} = \int_\Omega f v_r \, d\mathbf{x}.$$

We do the same for the imaginary part of the coupled system.

$$\int_\Omega \Delta u_i v_i \, d\mathbf{x} + \int_\Omega \omega^2 u_i v_i \, d\mathbf{x} - \int_\Omega \alpha\omega u_r v_i \, d\mathbf{x} = \int_\Omega 0 d\mathbf{x},$$
$$\left(\int_\Omega \nabla(u_i) \cdot \nabla(v_i) \, d\mathbf{x} - \int_{\partial\Omega} u_i \frac{\partial v_i}{\partial \mathbf{n}} \, ds \right) + \int_\Omega \omega^2 u_i v_i \, d\mathbf{x} - \int_\Omega \alpha\omega u_r v_i \, d\mathbf{x} = 0$$

(again applying Green's first identity). Note that the surface integrals will reduce to 0 since v_r, v_i are 0 in $\partial\Omega$, the boundary of our domain. In summary, we obtain:

$$\begin{cases} \int_\Omega (\nabla u_r \cdot \nabla v_r) + \omega^2 u_r v_r - \alpha\omega u_i v_r \, d\mathbf{x} = \int_\Omega f v_r \, d\mathbf{x}, \\ \int_\Omega (\nabla u_i \cdot \nabla v_i) + \omega^2 u_i v_i + \alpha\omega u_r v_i \, d\mathbf{x} = 0. \end{cases} \tag{2.14}$$

Thus, the weak form of the point source problem is

$$
\begin{cases}
a(u,v) := \displaystyle\int_\Omega (\nabla u_r \cdot \nabla v_r) + \omega^2 u_r v_r - \alpha\omega u_i v_r + (\nabla u_i \cdot \nabla v_i) + \omega^2 u_i v_i + \\
\alpha\omega u_r v_i \, d\mathbf{x}, \\
L(u,v) := \displaystyle\int_\Omega f v_r \, d\mathbf{x}, \\
V := H_0^1(\Omega) \times H_0^1(\Omega) = \{(v_r, v_i) \in H^1(\Omega) \, : \, v_r|_{\partial\Omega} = v_i|_{\partial\Omega} = 0\},
\end{cases}
$$

with $u = u_r + iu_i$ and $v = v_r + iv_i \in \mathbb{C}$.

The model problem we solve is the Helmholtz equation with a point source in the domain and zero boundary conditions on the square, illustrated in Figure 2.5.

2.3 Forward Modeling of Underwater Acoustic Scattering

Time-harmonic acoustic waves in an ocean are modeled by the Helmholtz equation inside a layer with suitable boundary conditions. When waves are intercepted by a physical boundary, reflection and scattering occur. Scattering theory plays an especially important role in modern physical applications, especially acoustic remote sensing systems like SONAR (Sound Detection and Ranging). Acoustic scattering problems are focused on the effect that inhomogeneous media have on incident acoustic waves (Meury, 2007).

If we adopt the splitting of the total field u_{tot} into a prescribed incident part u_{inc} and a resulting scattered field u_{scat}, we obtain a *direct scattering problem*, where we want to find u_{scat} given the knowledge of u_{inc} and physical laws determining wave motion (Meury, 2007). This differs from an *inverse scattering problem*, i.e., given the scattered field u_{scat}, one wants to compute the obstacle and/or incoming wave. One type of direct scattering problem is the scattering of incident acoustic waves

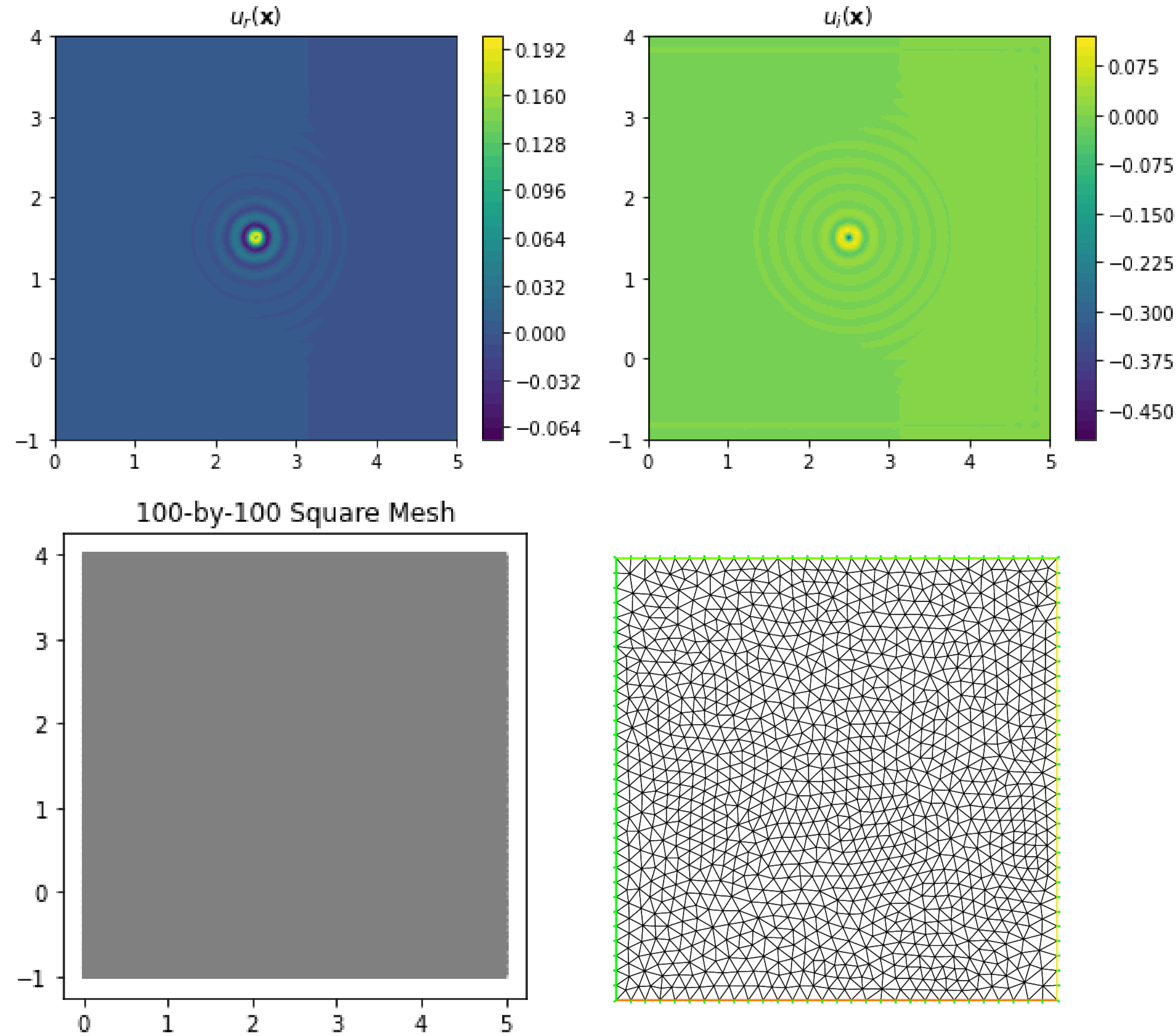

Figure 2.3 Solution to the Helmholtz problem with point source $f = \delta(\mathbf{x} - (1.5, 2.5))$ in square domain with zero boundary conditions on the square. Left-hand side is the **Real part of Solution**; Right-hand side is the **Imaginary part of the Solution**. Here the damping term $\alpha = 10$ and the square wavenumber $k = 10\pi$. The waves begin to "die out" before reaching the boundary. The square mesh with **N**=100 is the domain for the Helmholtz problem with point source, with the last figure illustrating the triangulation.

from impenetrable, homogeneous objects, i.e., *Helmholtz scattering problems.* In this chapter, we first formulate the problem for direct acoustic wave scattering. We then solve a specific Helmholtz scattering problem that will lay the groundwork for the rest of this thesis.

2.4 Problem Formulation

Consider the $d-$ *dimensional* problem of scattering a time-harmonic incident field by a *bounded soft obstacle* called Ω. Boundaries made up of sound-soft material have very low acoustic impedance compared to acoustic impedance of the carrier medium (Pedneault, 2018). What this entails is that when an incident wave advances over the sound soft material, a scattered wave of the same magnitude but opposite polarity is generated. In splitting up the total scattered field u into the incident field u_{inc} and u_{scat},i.e., $u(\mathbf{x}) = u_{scat}(\mathbf{x}) + u_{inc}(\mathbf{x})$, we obtain the following boundary value problem for the scattered field:

$$\begin{cases} \Delta u_{scat} + k^2 u = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ u_{scat} + u_{inc} = 0 \quad \text{on } \partial\Omega, \\ \frac{\partial u_{scat}}{\partial r} - iku_{scat} = o\Big(r^{(1-d)/2}\Big) \quad \text{uniformly for } r := |\mathbf{x}| \to \infty. \end{cases} \tag{2.15}$$

Another important problem is direct scattering from *sound-hard obstacles* where the normal velocity of the total field vanishes on $\partial\Omega$ (Meury, 2007). This occurs when the surrounding medium has much lower acoustic impedance than the boundary of the object (Pedneault, 2018). This is represented by the following exterior Neumann problem for the scattered field:

$$\begin{cases} \Delta u_{scat} + k^2 u = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ \frac{\partial u_{scat}}{\partial \mathbf{n}} + \frac{\partial u_{inc}}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \\ \frac{\partial u_{scat}}{\partial r} - iku_{scat} = o\Big(r^{(1-d)/2}\Big) \quad \text{uniformly for } r := |\mathbf{x}| \to \infty. \end{cases} \tag{2.16}$$

In summary, the direct scattering problem in $\mathbb{R}^d$ with $d = 2$ (or $d = 3$) can be posed as follows: given the propagation direction $\hat{\theta} = (\cos\theta, \sin\theta) \in S^1 = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| = 1\}$ defined on the unit circle with incident angle $\theta \in [0, 2\pi]$, find $u = u(\mathbf{x})$ such that

$$u = u_{inc} + u_{scat} \text{ with incident field } u_{inc}(\mathbf{x}) = e^{ik\hat{\theta}\cdot\mathbf{x}},$$

and $u_{scat} \in C^2(\mathbb{R}^2 \setminus \overline{\Omega})$ satisfies the Helmholtz equation

$$\Delta u_{scat} + k^2 u_{scat} = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega},$$

Sommerfield radiation condition

$$\frac{\partial u_{scat}}{\partial r} - iku_{scat} = o\Big(r^{(1-d)/2}\Big) \quad \text{uniformly for } r := |\mathbf{x}| \to \infty,$$

uniformly with respect to $\mathbf{x}^* := \frac{\mathbf{x}}{|\mathbf{x}|}$, and the boundary condition

$$u_{scat} + u_{inc} = 0 \quad \text{on } \partial\Omega,$$

or

$$\frac{\partial u_{scat}}{\partial \mathbf{n}} + \frac{\partial u_{inc}}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega.$$

2.4.1 Approximate/Absorbing Boundary Conditions (ABCs)

Again, consider the direct scattering exterior Helmholtz problem to a d-dimensional body:

$$
\begin{aligned}
\Delta u + k^2 u &= 0 \quad \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\
u = u_{scat} + e^{ik\cdot\mathbf{x}\cdot\hat{\theta}} &= 0 \quad \text{on } \partial\Omega \quad \text{(soft body)} \\
\text{or } \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial}{\partial \mathbf{n}}\Big(u_{scat} + e^{ik\cdot\mathbf{x}\cdot\hat{\theta}}\Big) &= 0 \quad \text{on } \partial\Omega \quad \text{(hard body)} \\
\lim_{r\to\infty} r^{\frac{n-1}{2}}\Big(\frac{\partial u_{scat}(\mathbf{x})}{\partial r} - iku_{scat}(\mathbf{x})\Big) &= 0 \quad \text{(Sommerfield radiation condition)}
\end{aligned}
$$

Note that $u_{inc} = e^{ik\hat{\theta}\cdot\mathbf{x}}$ and $r = |\mathbf{x}|$. The Helmholtz equation exterior to a body is well-posed only when one adds a Sommerfield radiation condition which models the behavior of the solution as the domain tends to infinity (Medvinsky et al., 2008). For a numerical solution, one needs to truncate the unbounded domain and introduce an artificial surface with a boundary condition (Medvinsky et al., 2008). That is, to solve the Helmholtz equation numerically, we replace the Sommerfield radiation condition with a boundary condition on a surface at a finite distance. We can derive approximations of the Sommerfield radiation condition called the *absorbing boundary conditions* (ABCs), which are typically derived from asymptotic expansions of the solution at large distances from the origin and become more accurate the larger the radius r of the boundary $\partial\Omega$ is. Higher accuracy of such approximations can be achieved by increasing the size of the computational domain.

- First Order ABC:

$$
\frac{\partial u}{\partial \mathbf{n}} - i\omega u = 0 \quad \text{on } \Gamma,
$$

where Γ is an artificial boundary. Absorbing boundary conditions (ABCs) are used when the computational domain of the exterior problem is infinite or too large to discretize numerically. We cut down the domain to a smaller size and introduce an artificial boundary. We then apply the ABCs at this boundary (Runberg, 2012-04).

2.5 Finding the Scattered Wave via Boundary Integral Equations

In the exterior problem, approximating the problem numerically due to the infinite size of the solution domain Ω is an especially difficult task (Runberg, 2012-04). It is not possible to discretize an unbounded domain. One approach we discussed and will prove important soon was truncating the infinite domain and implementing an absorbing boundary condition (ABC) at the new boundary Γ. In this section, we discuss another approach to solving an exterior problem.

In exterior problems, the Helmholtz equation is set in an unbounded domain. Assuming a constant index of refraction, it is possible to rewrite the Helmholtz equation,

$$\Delta u(\mathbf{x}) + \omega^2 u(\mathbf{x}) = 0, \quad \mathbf{x} \neq \bar{\Omega},$$

where $\bar{\Omega}$ denotes the closure of the domain, as an integral equation set on the boundary of Ω. Namely, we find the scattered solution outside Ω as an integral equation. Starting with the Dirichlet case, i.e., $g(\mathbf{x}) = -u_{inc}(\mathbf{x})$, let $\Phi(\mathbf{x})$ denote the fundamental solution for the Helmholtz equation in d dimensions. That is,

$$\Phi(\mathbf{x}) := \begin{cases} \frac{i}{4} H_0(\omega|\mathbf{x}|), & d = 2, \\ \frac{\exp(i\omega|\mathbf{x}|)}{4\pi|\mathbf{x}|}, & d = 3. \end{cases} \tag{2.17}$$

Remember that H_0 denotes the Hankel function of the first kind of order zero. Then, the scattered solution is subsequently given by evaluating the *single layer potential* integral equation

$$u_{scat}(\mathbf{x}) = \int_{\partial\Omega} \Phi(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\, ds(\mathbf{y}) = -u_{inc}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \setminus \bar{\Omega}, \tag{2.18}$$

where ψ is a continuous function on $\partial\Omega$.

Alternatively, we can solve the *double layer potential* integral equation

$$\frac{1}{2}\psi(\mathbf{x}) - \int_{\partial\Omega} \frac{\partial\Phi}{\partial\mathbf{n}}(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\, ds(\mathbf{y}) = -u_{inc}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \tag{2.19}$$

In this case, the scattered solution outside Ω then is given by

$$u_{scat}(\mathbf{x}) = -\int_{\partial\Omega} \frac{\partial\Phi}{\partial\mathbf{n}}(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\, ds(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^d \setminus \bar{\Omega} \tag{2.20}$$

(Runberg, 2012-04). We can denote the *single layer operator*, with $\psi \in L^2(\partial\Omega)$, by S:

$$(S\psi)(\mathbf{x}) := \int_{\partial\Omega} \Phi(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\, ds(\mathbf{y}), \quad \mathbf{x} \in \partial\Omega. \tag{2.21}$$

We can think of ψ as the density of acoustic sources generating the field $S\psi$ (Moiola, n.d.). The single-layer operator S is a type of *boundary integral operator*. The other type of boundary integral operator discussed is the *double layer potential* D:

$$(D\psi)(\mathbf{x}) := \int_{\partial\Omega} \frac{\partial\Phi}{\partial\mathbf{n}(\mathbf{y})}\psi(\mathbf{y})\, ds(\mathbf{y}), \quad \mathbf{x} \in \partial\Omega. \tag{2.22}$$

2.6 The Water-Sediment Model

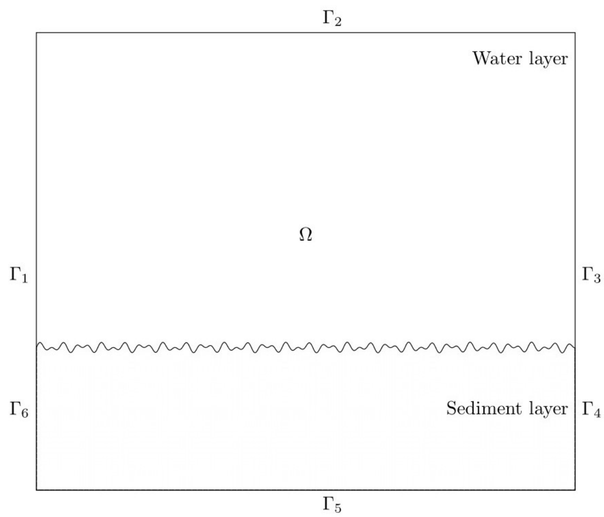

Figure 2.4 The domain for the water-sediment model. Similar to the domain for the point source Helmholtz problem but with new boundaries created by splitting the left and right sides of the boundary. The boundary conditions include periodic boundary conditions for the top and bottom subdomains.
.

Consider the damped Helmholtz equation $\Delta u + \omega^2 u + i\omega\alpha u = 0$ with the following boundary conditions:

$$\begin{aligned} \text{Bottom } \Gamma_5 : u = 0 \quad &\text{(Dirichlet)} \\ \text{Top } \Gamma_2 : \frac{\partial u}{\partial \mathbf{n}} - i\omega u = 0 \quad &\text{(ABC)} \\ \text{Left and Right} : u\Big|_{\Gamma_1} = u\Big|_{\Gamma_3} & \\ u\Big|_{\Gamma_6} = u\Big|_{\Gamma_4}. & \end{aligned}$$

Consider the ABC boundary condition, which is an impedance condition for the subdomain Γ_2: $\frac{\partial u}{\partial \mathbf{n}} - i\omega u = 0$. We separate the incoming and scattered waves so we can solve for u_{scat}. The solution $u = u_{inc} + u_{scat}$ and we substitute u into the ABC, obtaining

$$\begin{aligned} \frac{\partial u_{inc}}{\partial \mathbf{n}} + \frac{\partial u_{scat}}{\partial \mathbf{n}} - i\omega u_{inc} - i\omega u_{scat} = 0 \\ \implies \frac{\partial u_{scat}}{\partial \mathbf{n}} - i\omega u_{scat} = -\frac{\partial u_{inc}}{\partial \mathbf{n}} + i\omega u_{inc}. \end{aligned}$$

Note that $u_{inc} = \exp\Big(i\omega\hat{\theta}\cdot\mathbf{x}\Big) = \exp\Big(i\omega(\cos\theta x_1 + \sin\theta x_2)\Big)$, where $\hat{\theta} = (\cos\theta, \sin\theta)$ and $\mathbf{x} = (x_1, x_2)$. Note also that

$$\nabla u_{inc}\cdot\mathbf{n} = u_{inc}\cdot\Big(i\omega\cos\theta, i\omega\sin\theta\Big)\cdot(0,1) \quad = i\omega\sin\theta\cdot\exp\Big(i\omega(\cos\theta x_1 + \sin\theta x_2)\Big).$$

Hence

$$\begin{aligned} -\frac{\partial u_{inc}}{\partial \mathbf{n}} + i\omega u_{inc} &= -i\omega\sin\theta\cdot\exp\Big(i\omega(\cos\theta x_1 + \sin\theta x_2)\Big) + i\omega\exp\Big(i\omega(\cos\theta x_1 + \sin\theta x_2)\Big) \\ &= \exp\Big(i\omega(\cos\theta x_1 + \sin\theta x_2)\Big)\cdot(i\omega(1-\sin\theta)). \end{aligned}$$

Variational Form Derivation

Figure 3.1 refers to the water-sediment domain Ω. The incident plane wave $u_{inc} = \exp\Big(i\omega\hat{\theta}\cdot\mathbf{x}\Big)$; therefore, $u = u_{scat} + u_{inc} = u_{scat} + \exp\Big(i\omega\hat{\theta}\cdot\mathbf{x}\Big) = u_{scat} + \exp\Big(i\omega(\cos\theta x_1 + \sin\theta x_2)\Big)$. We again split the solution u into real and imaginary parts, i.e., $u = u_r + iu_i$, where $\text{Re}(u) = u_r$ and $\text{Im}(u) = u_i$. Consider the Helmholtz equation with a damping term with $n \equiv 1$, the index of refraction.

Substituting $u_r + iu_i$ for u, we obtain

$$\Delta(u_r + iu_i) + \omega^2(u_r + iu_i) + i\alpha\omega(u_r + iu_i) = 0$$

Again, we can separate the terms further into real and imaginary parts

$$\Delta u_r + \omega^2 u_r - \alpha\omega u_i + i(\Delta u_i + \omega^2 u_i + \alpha\omega u_r) = 0$$

We obtain the coupled system

$$\begin{cases} \Delta u_r + \omega^2 u_r - \alpha\omega u_i = 0, \\ \Delta u_i + \omega^2 u_i + \alpha\omega u_r = 0. \end{cases}$$

However, for a domain $\Omega \subset \mathbb{R}^2$ with boundary Γ_2 as seen in Figure 3.1, the Helmholtz equation with attenuation and absorbing boundary condition reads:

$$\begin{aligned} \Delta u + \omega^2 u + i\omega\alpha u = f \quad \text{in } \Omega, \\ i\omega(1 - \sin\theta)\exp\Big(i\omega(\cos\theta x_1 + \sin\theta x_2)\Big) = \nabla u \cdot \mathbf{n} - i\omega u \quad \text{on } \Gamma_2. \end{aligned}$$

We additionally split the boundary condition to real and imaginary parts. Letting $\psi = \cos\theta x_1 + \sin\theta x_2$, we split the absorbing boundary condition into real and imaginary parts:

$$
\begin{aligned}
\nabla u \cdot \mathbf{n} - i\omega u &= i\omega(1-\sin\theta)\exp\Big(i\omega\psi)\\
&= i\omega(1-\sin\theta)\Big[\cos\omega\psi + i\sin\omega\psi\Big]\\
&= \omega(1-\sin\theta)\Big[-\sin\omega\psi\Big] + i\omega(1-\sin\theta)\cos\omega\psi\\
&= \nabla u_r\cdot\mathbf{n} + \nabla u_i\cdot\mathbf{n} - i\omega u_r + i\omega u_i.
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\nabla u_r\cdot\mathbf{n} + \omega u_i = \omega(1-\sin\theta)(-\sin\omega\psi)\\
&\implies \nabla u_r\cdot\mathbf{n} = -\omega u_i + \omega(1-\sin\theta)(-\sin\omega\psi).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\nabla u_i\cdot\mathbf{n} - \omega u_r = \omega(1-\sin\theta)\cos(\omega\psi)\\
&\implies \nabla u_i\cdot\mathbf{n} = \omega u_r + \omega(1-\sin\theta)\cos(\omega\psi).
\end{aligned}
$$

Thus, after splitting the ABC to real and imaginary parts, we obtain Neumann boundary conditions

$$
\begin{cases}
\nabla u_r\cdot\mathbf{n} = -\omega u_i + \omega(1-\sin\theta)(-\sin\omega\psi) = g_1,\\
\nabla u_i\cdot\mathbf{n} = \omega u_r + \omega(1-\sin\theta)\cos(\omega\psi) = g_2.
\end{cases}
\tag{2.23}
$$

We obtain the weak form by multiplying each equation by a test function $v_r \in H^1(\Omega)$ and $v_i \in H^1(\Omega)$ for each respective equation, integrate by parts, and apply Green's first identity. We then obtain the following variational form: i.e., we found a $u = u_r + iu_r \in H^1(\Omega) \times H^1(\Omega)$ such that

$$\begin{cases} a_r(u,v) & = \int_\Omega (\nabla u_r \cdot \nabla v_r - \alpha\omega u_i v_r + \omega^2 u_r v_r)\, d\mathbf{x} \\ & = \int_\Omega f v_r \, d\mathbf{x} + \int_{\Gamma_2} g_1 v_r \, ds = L_r(v) \\ a_i(u,v) & = \int_\Omega (\nabla u_i \cdot \nabla v_i + \alpha\omega u_r v_i + \omega^2 u_i v_i)\, d\mathbf{x} = \int_{\Gamma_2} g_2 v_i \, ds = L_i(v), \end{cases}$$

where f is the source term and g_1 and g_2 are given by (3.9).

Alternatively, one can derive the weak form without splitting u into real and imaginary parts. That is, we focus on the Helmholtz equation with attenuation and impedance boundary condition on Γ_2:

$$\begin{cases} \Delta u + \omega^2 u + i\omega\alpha u = f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} - i\omega u = h \quad \text{on } \Gamma_2, \end{cases} \tag{2.24}$$

where $h = \nabla u_{inc} \cdot \mathbf{n}$. Since the boundary condition involves the imaginary unit, the variational formulation of this BVP involves complex-valued Sobolev spaces, a sesquilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{C}$, and an antilinear function $L(\cdot) : V \to \mathbb{C}$. Multiplying (3.9) by $\overline{v}$, where $\overline{v}$ is the complex conjugate of v, and integrating over Ω while using Green's first identity and the impedance boundary condition yields the following variational form without the split:

$$
\begin{cases}
a(u,v) := \displaystyle\int_\Omega (-\nabla u \cdot \overline{\nabla v} + \omega^2 u\overline{v} + i\omega\alpha u)\, d\mathbf{x}, \\
L(u,v) := \displaystyle\int_\Omega f\overline{v}\, d\mathbf{x} + \int_{\Gamma_2} h\overline{v}\, ds.
\end{cases}
\tag{2.25}
$$

2.6.1 Solution to Water-Sediment Model Problem

In summary, the water-sediment model refers to wave propagation in an underwater acoustic environment consisting of a water-sediment layer. Scattering effects close to the seafloor require simulations of the solution to the Helmholtz equation. Now again for the Helmholtz equation (which has attenuation term $i\omega\alpha u$ for the bottom sediment layer but no attenuation for the top water layer), we have the following boundary conditions:

$$
\begin{aligned}
u = 0 \text{ on } \Gamma_5 \quad &\text{(Dirichlet)} \\
\frac{\partial u}{\partial \mathbf{n}} - i\omega u = i\omega(1 - \sin\theta)u_{inc} \text{ on } \Gamma_2 \quad &\text{(ABC)} \\
u\big|_{\Gamma_1} = u\big|_{\Gamma_3}, \quad u\big|_{\Gamma_6} = u\big|_{\Gamma_4}
\end{aligned}
$$

(We assume that the attenuation of sound in the water is negligible, so we set $\alpha = 0$ in the water domain). Then we separated the incident and scattered waves u_{inc} and u_{scat} respectively, deriving the following variational form of the BVP for the water-sediment model:

$$
\begin{cases}
a(u,v) := \displaystyle\int_\Omega (-\nabla u \cdot \overline{\nabla v} + \omega^2 u\overline{v} + i\omega\alpha u)\, d\mathbf{x}, \\
L(u,v) := \displaystyle\int_{\Gamma_2} h\overline{v}\, ds, \\
V := H^1(\Omega),
\end{cases}
$$

where $h = \nabla u_{inc} \cdot \mathbf{n}$. The equations above refer to the wave propagation in the bottom sediment layer. The variational form for the topwater layer part does not contain the attenuation term and so would not include the term $i\omega\alpha u$.

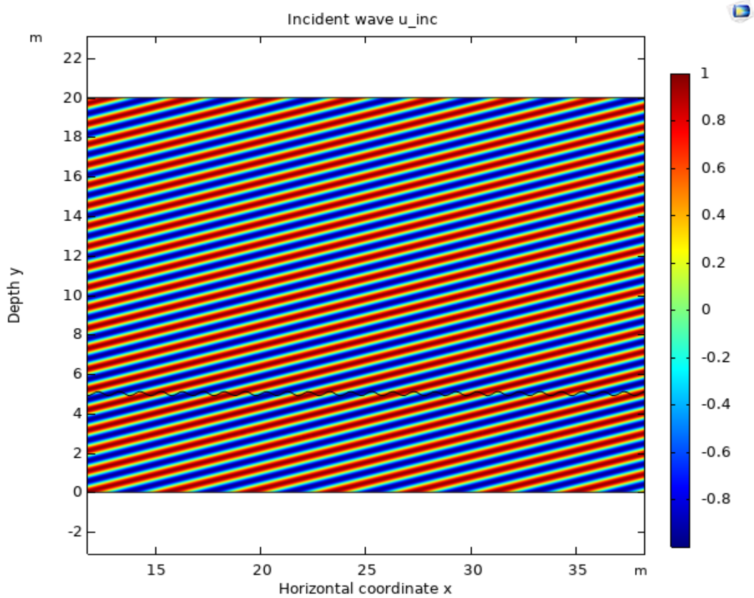

Figure 2.5 Plot of the incident plane wave $u_{inc} = \exp\Big(i\omega(\cos\theta x + \sin\theta y)\Big)$. The incident angle is $\theta = 7\pi/12$ and frequency $\omega = 2\pi$. The computational domain is denoted by a parametric curve $x = s, y = 0.1\sin(2\pi s) + 5$.

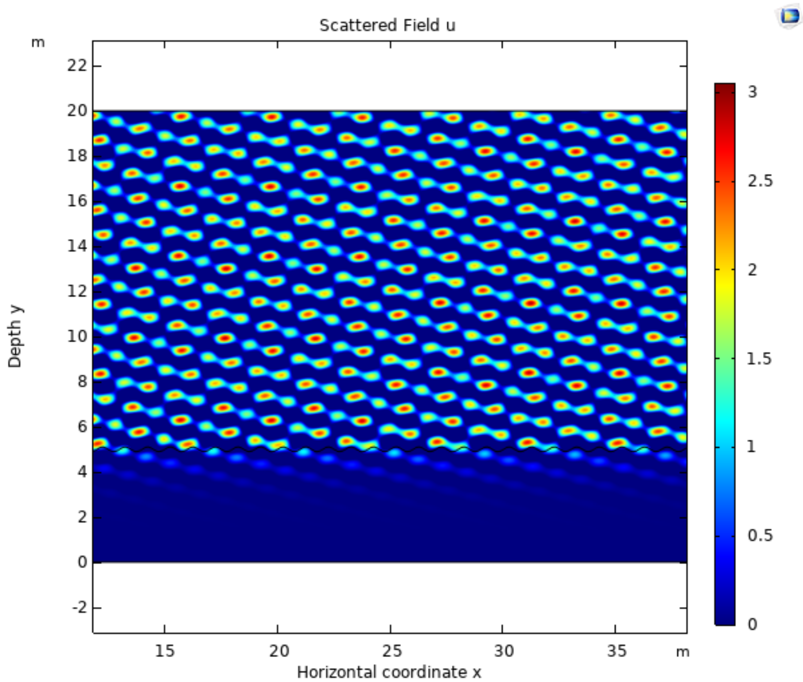

Figure 2.6 Plot of the Helmholtz solution u to the water-sediment problem with frequency $\omega = 2\pi$. The attenuation $\alpha = 5$ and we can see damping occurs at the bottom sedimentary layer part. The computational domain is denoted by a parametric curve $x = s, y = 0.1\sin(2\pi s) + 5$.

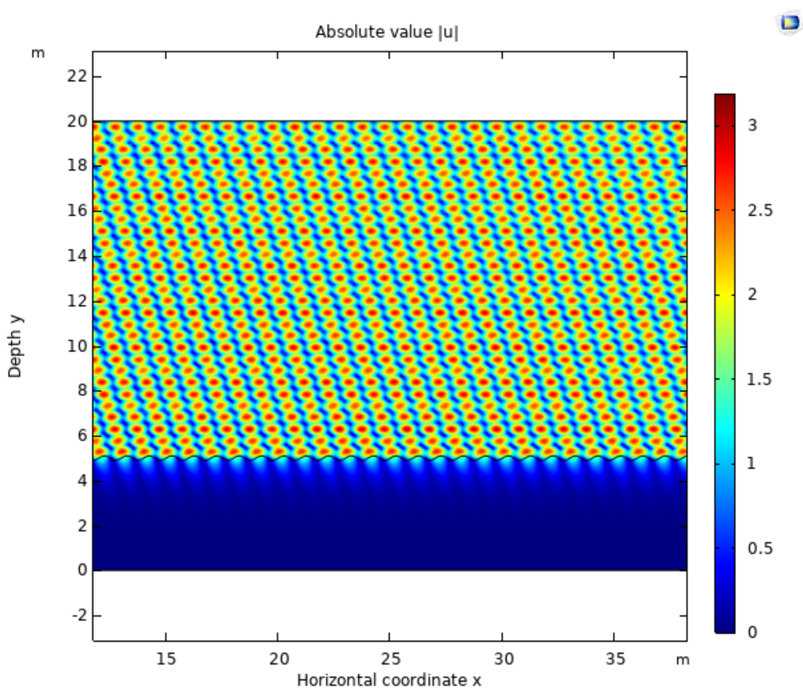

Figure 2.7 Plot of the absolute value solution $|u|$. The implementation of the scattering field u and $|u|$ was done on COMSOL Multiphysics Software.

CHAPTER 3

DOMAIN DECOMPOSITION METHODS FOR FORWARD MODELING OF UNDERWATER ACOUSTIC ENVIRONMENTS

3.1 Introduction to the Domain Decomposition Method

One of the main difficulties with the finite element method is that when especially considering large wavenumbers (in other words solving high-frequency scattering problems), FEM often leads to a large, complex-valued, and highly indefinite sparse matrix for larger-sized problems. Furthermore, the exterior Helmholtz problem renders the use of finite elements alone often inefficient on account of the size of the unknown vector. In particular, the use of finite elements proves especially inefficient in dealing with the Sommerfield radiation condition on the boundary of the truncated domain of interest (Pedneault, 2018). Hence we have to approximate the Sommerfield radiation condition with absorbing boundary conditions to mimic the phenomenon that only holds at infinity, yielding approximation errors in addition to the issue of the inefficacy of FEM for too large-sized domains.

In this chapter, we introduce another approach to direct scattering problems for especially large-sized domains, namely *domain decomposition methods*. Domain decomposition methods (DDMs) can sometimes reduce the computational complexity of the underlying solution method. Additionally, DDMs help enhance the localized treatment of complex and irregular geometries (Chan, Mathew, et al., 1994). DDMs are iterative methods for solving PDEs based on the decomposition of the spatial domain of the BVP into several subdomains (in our case, we will first split one large rectangular domain into two smaller rectangular subdomains). The large-sized problem is then broken down into subproblems that can be solved more efficiently with existing methods (in our case we will use FEM) (Discacciati, 2004).

3.1.1 How Domains Are Decomposed: Overlapping vs. Non-overlapping Regions

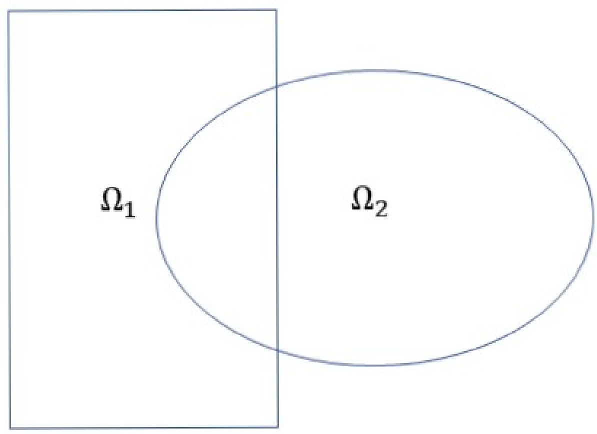

Figure 3.1 Example domain with two overlapping regions: $\Omega = \Omega_1 \cup \Omega_2$.

The main geometric issue arising in domain decomposition concerns how the domains are to be decomposed into subregions as well as how the region is to be discretized using some form of mesh. Domain decomposition methods divide broadly into either being overlapping or nonoverlapping methods. When there is overlap, the methods are sometimes referred to as Schwarz methods; when there is no overlap, the methods are sometimes known as substructuring (Edelman, 2005). We will especially focus on non-overlapping domain decomposition methods for the Helmholtz equation, and specifically apply the *Lions-Després DDM*, a method that combines the continuity conditions on the artificial interfaces between subdomains to obtain absorbing boundary conditions and solve the overall problem by iterating over subdomains (Boubendir et al., 2012).

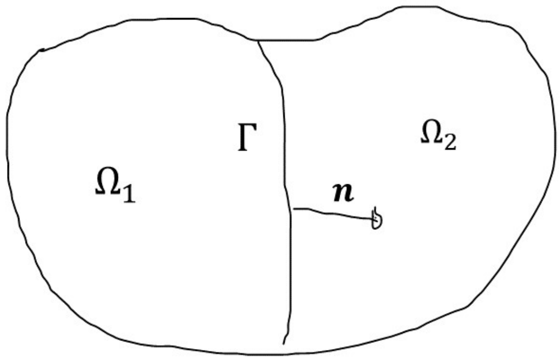

Figure 3.2 Example of a two-dimensional non-overlapping partition of the computation domain Ω.

3.2 Lions-Després DDM

Consider the two-dimensional sound-soft Helmholtz scattering problem of an incident acoustic wave by an obstacle Ω:

$$\begin{cases} \Delta u + \omega^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \Omega, \\ u = -u_{inc} \quad \text{on } \Gamma = \partial\Omega, \\ \lim_{r\to\infty} r\Big(\partial_r u - iku\Big) = 0. \end{cases} \tag{3.1}$$

Here $u_{inc} = \exp\Big(ik\hat{\theta}\cdot\mathbf{x}\Big)$ is a plane wave with $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $r = |\mathbf{x}|$. $\hat{\theta}$ is the incident angle normalized on the unit circle. The Sommerfield radiation condition in (4.1) imposes that the scattered wave is outgoing.

In this section, we combine absorbing boundary conditions with Lions-Després' non-overlapping domain decomposition method to solve (4.1). As discussed last chapter, implementing ABCs involves truncating the infinite domain by introducing an artificial boundary Γ' to get a bounded computational region. Thus, we can approximate (4.1) by

$$\begin{cases} \Delta u + \omega^2 u = 0 \quad \text{in } \Omega, \\ u = -u_{inc} \quad \text{on } \Gamma, \\ \partial_{\mathbf{n}} u + \mathcal{B} u = 0 \quad \text{on } \Gamma', \end{cases} \tag{3.2}$$

where Ω is the bounded domain enclosed by the artificial boundary Γ' and Γ and the operator $\mathcal{B}$ represents an approximation of the *Dirichlet to Neumann* (DtN) operator (here $\mathcal{B} = ik$ on Γ') (Boubendir et al., 2012). Note that the DtN operator and the ABCs are related and the DtN operator maps the Dirichlet data u to the Neumann data $\partial u / \partial \mathbf{n}$ with $\mathbf{n}$ pointing outward, i.e., it is the operator D satisfying

$$\frac{\partial u}{\partial \mathbf{n}} = Du, \quad x \in \partial\Omega. \tag{3.3}$$

Next we discuss the iterative Lions-Després' non-overlapping DDM:

Algorithm 1 Lions-Després DDM

1: Split Ω into N_{dom} subdomains $\Omega_i,\ i = 1, 2, \ldots, N_{dom}$, such that

- $\overline{\Omega} = \bigcup_{i=1}^{N_{dom}} \overline{\Omega}_i$.
- $\Omega_i \cap \Omega_j = \emptyset$, if $i \neq j$ $(i, j = 1, \ldots, N_{dom})$.
- $\partial\Omega_i \cap \partial\Omega_j = \overline{\Sigma}_{ij} = \overline{\Sigma}_{ji}$ $(i, j = 1, \ldots, N_{dom})$ is the artificial interface separating Ω_i and Ω_j as long as its interior $\Sigma_{ij} \neq \emptyset$.
- $\Gamma_i = \Gamma \cap \partial\Omega_i$ and $\Gamma_i' = \Gamma' \cap \partial\Omega_i$ for $i = 1, \ldots, N_{dom}$.

2: Reduce the solution of (4.2) by solving the *local transmission problems* for $i = 1, \ldots, N_{dom}$

$$\begin{cases} \Delta u_i^{(n+1)} + \omega^2 u_i^{(n+1)} = 0 \quad \text{in } \Omega_i, \\ u_i^{(n+1)} = -u_{inc} \quad \text{on } \Gamma_i, \\ \partial_{\mathbf{n}_i} u_i^{(n+1)} + \mathcal{S} u_i^{(n+1)} = g_{ij}^{(n)} \quad \text{on } \Sigma_{ij} \ . \end{cases}$$

Forming the quantities to be transmitted through the interfaces yields

$$g_{ji}^{(n+1)} = -\partial_{\mathbf{n}_i} u_i^{(n+1)} + \mathcal{S} u_i^{(n+1)} = -g_{ij}^{(n)} + 2\mathcal{S} u_i^{(n+1)} \quad \text{on } \Sigma_{ij}.$$

Here $u_i = u|_{\Omega_i}$. $\mathcal{S}$ is the *transmission operator* (Boubendir et al., 2012).

3.3 Application of Lions-Després DDM to the Direct Scattering Model Problem

We divide the water-sediment model from the last chapter into two domains yielding the following local transmission problems:

Water Domain

$$\begin{aligned} \Delta u_1 + \omega^2 u_1 &= 0 \quad \text{on } \Omega_1 \\ (\partial_{\mathbf{n}_1} u_1 - i\omega u_1)^{(n+1)} &= g_{21}^{(n)} \\ \frac{\partial u_1}{\partial \mathbf{n}} - i\omega u_1 &= i\omega \exp\Big(i\omega(\cos\theta x + \sin\theta y)\Big) \quad \text{(ABC)} \\ u_1\big|_{\Gamma_1} &= u_1\big|_{\Gamma_3} \quad \text{(periodic boundary conditions)} \end{aligned} \tag{3.4}$$

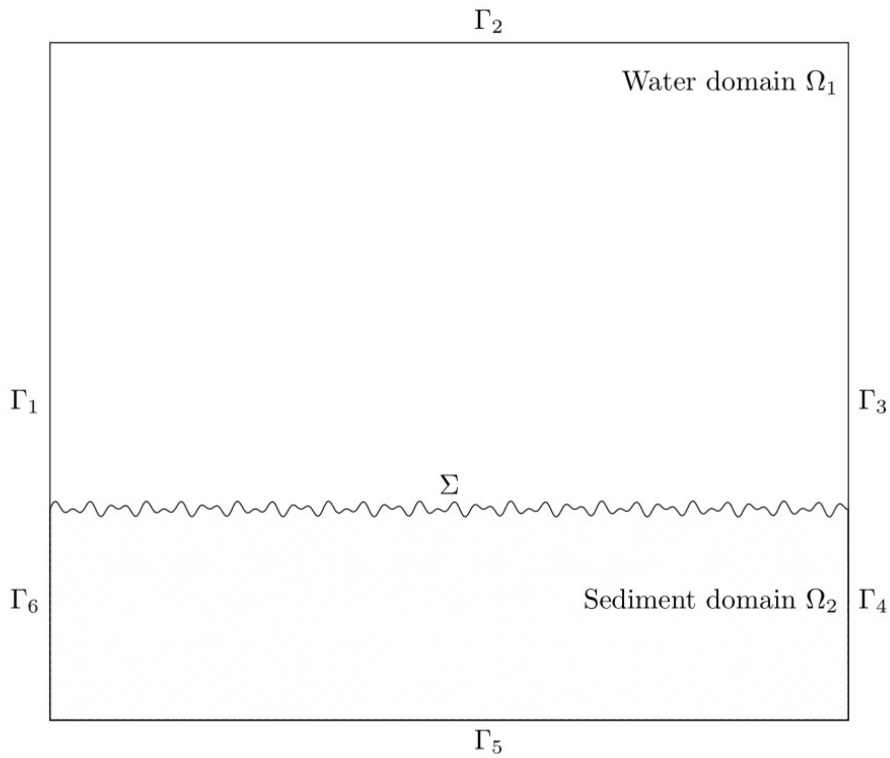

Figure 3.3 The rectangular two-dimensional non-overlapping partition of the computation domain $\Omega = \Omega_1 \cup \Omega_2$ for the water-sediment model. Ω_1 refers to the water domain and Ω_2 refers to the sediment domain.

Sediment Domain

$$\begin{aligned} \Delta u_2 + (\omega^2 + ia\omega)u_2 &= 0 \quad \text{on } \Omega_2 \\ u_2\big|_{\Gamma_5} &= 0 \quad \text{(Dirichlet BC)} \\ u_2\big|_{\Gamma_6} &= u_2\big|_{\Gamma_4} \quad \text{(periodic boundary conditions)} \\ (\partial_{\mathbf{n}_2} u_2 - i\omega u_2)^{(n+1)} &= g_{12}^{(n)} \end{aligned} \tag{3.5}$$

3.3.1 Weak Form of the Split Domains

Here we derive the weak forms of the local transmission problems for the water domain and the sediment domain. First, we recall the integration by parts formula

$$\int_{\Omega} \Delta u v = \int_{\Omega} \nabla \cdot (\nabla u) v = \int_{\partial\Omega} (\nabla u \cdot n) v - \int_{\Omega} \nabla u \cdot \nabla v. \tag{3.6}$$

Domain 1 : Water Domain Weak Form

Observing that

$$\int_{\partial\Omega_1} (\nabla u_1 \cdot n) v = \int_{\Gamma_2} (i\omega u) v + \int_{\Sigma} (g_{21} + i\omega u_1 v,$$

we obtain the weak form for the water domain:
set $\mathcal{V} = \left\{ v \in H^1(\Omega) \ : \ v\big|_{\Gamma_1} = v\big|_{\Gamma_3} \right\}$. We multiplied by a test function $v \in \mathcal{V}$ and found $u_1 \in \mathcal{V}$ such that

$$\begin{cases} \mathcal{V} = \left\{ v \in H^1(\Omega) \ : \ v\big|_{\Gamma_1} = v\big|_{\Gamma_3} \right\}, \\ a(u_1, v) = \displaystyle\int_{\Omega_1} \nabla u_1 \cdot \nabla v - \omega^2 u_1 v \, d\mathbf{x} - i\omega \int_{\Sigma} u_1 v \, dS, \\ L(v) = \displaystyle\int_{\Gamma_2} hv \, dS + \int_{\Sigma} g_{21} v \, dS, \end{cases}$$

where $h = \nabla u_{inc} \cdot \mathbf{n} = i\omega \exp\Big(i\omega(\cos\theta x + \sin\theta y)\Big)$.

Domain 2 : Sediment Domain Weak Form

We note that

$$\int_{\partial\Omega_2} (\nabla u_2 \cdot n)v = \int_{\Sigma} (g_{12} + i\omega u_2 v).$$

Hence, we obtain the weak form for the sediment domain:
Set $\mathcal{W} = \Big\{v \in H^1(\Omega) \ : \ v\big|_{\Gamma_6} = v\big|_{\Gamma_4} \text{ and } v\big|_{\Gamma_5} = 0\Big\}$. We multiplied by a test function $v \in \mathcal{W}$ and found $u_2 \in \mathcal{W}$ such that

$$\begin{cases} \mathcal{W} = \Big\{v \in H^1(\Omega) \ : \ v\big|_{\Gamma_6} = v\big|_{\Gamma_4} \text{ and } v\big|_{\Gamma_5} = 0\Big\}, \\ a(u_2, v) = \displaystyle\int_{\Omega_2} \nabla u_2 \cdot \nabla v - (\omega^2 + i\omega\alpha)u_2 v \, d\mathbf{x} - i\omega \int_{\Sigma} u_2 v \, dS \\ L(v) = \displaystyle\int_{\Sigma} g_{12} v \, dS. \end{cases}$$

Hence the large-sized problem is broken down into two subproblems that can be solved more efficiently and ideally reduce the computational complexity of the numerical solution.

CHAPTER 4

CONCLUSION AND FUTURE DIRECTIONS FOR THIS WORK

Forward modeling of underwater acoustic environments via finite element modeling remains an important challenge in computational ocean acoustics. This thesis addressed how the properties of finite element solutions of the Helmholtz equation can lead to forward simulations of the underwater seafloor environment.

Thus, a finite element approach has been presented for forward modeling of the underwater seafloor. Firstly, a rigorous treatment of both the Helmholtz equation and finite element method was developed. Then we applied finite element modeling to solving the boundary value problem consisting of the Helmholtz equation with and without a point source in the domain and zero boundary conditions in the square to showcase both the plane and circular solutions in 2D. We observed that a wave generated by a localized source function, or from inhomogeneous boundary conditions on a bounded scatterer, will propagate outwards from the source, and as distance increases will assume the form of a circular wave (Runberg, 2012-04). This forward model incorporates simulations of circular waves seen in sonar imaging and is valuable for analyzing the seafloor. This also motivates forward modeling resulting in simulations of spatial scales in the seafloor geometry.

We then motivated the necessary background on time-harmonic acoustic scattering problems defined on exterior domains. In the exterior problem, the infinite size of the solution domain provides difficulties with numerical approximation of the solution. We thus couple absorbing boundary conditions, where we truncate the domain to a manageable size and introduce an artificial boundary. In doing this, we simulated a two-layer seafloor with wave scattering eventually damping past the computational domain, thus incorporating a model-based approach to underwater acoustic wave scattering simulation. Finally, we concluded with an introduction to a non-overlapping domain decomposition method for our forward model. The method consists of combining continuity conditions on the artificial interfaces between subdomains to obtain absorbing boundary conditions and iterate over the subdomains (Boubendir et al., 2012).

We can extend this work by investigating the impact of the domain decomposition method on computational efficiency results, such as accounting for the number of iterations necessary to improve the detailed simulation of underwater acoustic wave scattering in the seafloor. We successfully derived the weak formulation of both the water subdomain and sediment subdomain. However, there is still a need to perform numerical tests. In particular, we can incorporate more complex seafloor features into the water-sediment model, including modeling within the water column, absorption, and $3D$ simulations.

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