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Sidelobe suppression in chirp radar systems

Stephen N. Honickman
New Jersey Institute of Technology

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SIDELOBE SUPPRESSION IN CHIRP RADAR SYSTEMS

BY

STEPHEN HONICKMAN

A DISSERTATION
PRESENTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE
OF
DOCTOR OF ENGINEERING SCIENCE
AT
NEWARK COLLEGE OF ENGINEERING

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Newark, New Jersey
1971
APPROVAL OF DISSERTATION

SIDELOBE SUPPRESSION IN CHIRP RADAR SYSTEMS

BY

STEPHEN HONICKMAN

FOR

DEPARTMENT OF ELECTRICAL ENGINEERING

NEWARK COLLEGE OF ENGINEERING

BY

FACULTY COMMITTEE

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NEWARK, NEW JERSEY
ABSTRACT

Pulse radars extend target range detection by increasing the transmitted pulse width. On the other hand, target resolution is enhanced by reducing the system pulse width. These dichotomous requirements led to the invention of chirp radar systems which achieve greater target resolution for a given detectable range by frequency modulating the carrier frequency of the transmitted pulse. Along with the advent of chirp radars came the extremely simple and reliable technique of chirp signal generation known as "passive generation". However, one of the undesirable features of "passive generation" lies in the infinite time required for transmission of the resultant pulse. This means that some chirp radar systems may require time gating before transmission of the pulse. Time gating becomes necessary when the time-bandwidth product (Dispersion Factor) is less than 60 because chirp radar systems with time-bandwidth products greater than 60 which do not employ time gating have provided satisfactory operation.

The waveform distortion introduced by those systems which employ time gating creates unwanted range sidelobes in the received signal since a mismatch now exists between transmitter and receiver. Range sidelobes are undesirable because when a number of targets appear close to each other the range sidelobes of the individual returns can give rise to a resultant structure which erroneously indicates a target to be present. However, past attempts at sidelobe reduction using the paired-echo technique have not provided satisfactory results because the nature of the response was unknown.
Therefore, this paper will present a derivation of the receiver response and a method for sidelobe suppression. The results of such a derivation clearly indicate that the received waveform possesses an F-M structure. It is this aspect of the results which can be used to possibly explain the reasons that the paired-echo technique has not provided satisfactory results because it is not possible with an F-M structure to produce advanced and delayed replicas of the radar return that have a phase structure which can be used to cancel the range sidelobes of the radar return.

Having ascertained a possible reason for the failure of the paired-echo method, a new method for sidelobe suppression had to be considered. The new technique requires modification of the known received signal to achieve a desired waveshape and a physically realizable sidelobe suppression filter. First of all, the filter must be limited in bandwidth to approximately that of the overall system. This implies that all the spectral shaping of the de-chirp output must be limited to that portion of the signal spectrum located within the system bandwidth. Since the shaping of the spectrum within the system bandwidth eliminated all the sidelobes except those near the main-lobe, one is led to consideration of pulse width widening because it reduces the signal bandwidth and incorporates the first and largest sidelobe into the main-lobe. The completion of this aforementioned task indicated the region of the altered spectrum most affected by the remaining sidelobes. With
this information at hand, successive attempts at smoothing this
portion of the spectrum led to range sidelobes levels of approximately
-34 db. The resultant filter, of course, is realizable in practice.
However, along with the desired reduction of sidelobes came such
undesirable features as increased pulse width and reduced signal-to-
noise ratio. For the example considered in the text, the increase
in pulse width amounted to approximately 18% and the decrease in
SNR was -.75 db. Finally, these effects are illustrated by graphs
depicting the resultant output waveform and the filter required to
achieve these results.
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1. The Sidelobe Level and Pulse Width of the Output Pulse as a Function of the Exponent of the Exponential Decay Function.
I. INTRODUCTION

In pulse radar systems, the separation between the target and radar is determined by calculating the distance electromagnetic energy can travel in the time between transmission and reception of the pulse. The maximum detectable separation or range between the target and radar depends primarily upon the energy, which is a function of pulse width and amplitude, possessed by the pulse. This implies that one can achieve increased range by increasing pulse width and/or pulse amplitude. However, neither of these can be increased without limit because of the peak power limitation in the transmitter section of the radar and the loss in range resolution due to increased pulse width.

Range resolution is a measure of the ability of a radar system to separate targets that are in close proximity to each other. Since range resolution is inversely proportional to pulse width, one is confronted with the inability to achieve long range detection and good range resolution.

This dichotomy in the characteristics of a pulse radar led to the creation of a chirp radar system. Chirp radar systems employ rectangular pulses whose carrier frequency is frequency-modulated in some prescribed fashion to achieve more desirable range resolution characteristics than a standard pulse radar. Although the chirp radar system achieves more desirable range resolution characteristics than a comparable pulse radar, its receiver introduces undesirable range
sidelobes which under certain circumstances make it appear as though an additional target or targets are present. The authors of reference 1 describe, in their paper, techniques for range sidelobe suppression in chirp radar systems which do not time gate the passively generated pulses.

This paper considers chirp radar systems which time gate the passively generated pulses and presents the derivation of the hitherto unknown receiver response and a technique for the evaluation of a filter which will achieve range sidelobe suppression.
II. PULSE RADAR SYSTEMS

Pulse radar systems are able to ascertain the range of a target by measuring the time elapsed, \( \Delta t \), between transmission and reception of the transmitted pulse and then using the following expression

\[
R = \frac{c\Delta t}{2}
\]

where \( R \) is the range and \( c \) the velocity of light. The factor of two accounts for the two-way transmission.

Since maximum detectable range depends upon the width and amplitude of the pulse, one might attempt to increase this range by amplifying the pulse amplitude and/or expanding pulse width. These increases are limited by the peak and average power capacity of the transmitting devices and the accompanying loss in target (range) resolution. Since there are numerous instances in which target resolution is a necessity, one must consider the relationship between the pulse characteristics and target resolution. This relationship can be seen with the aid of the following diagram.

Figure 1a depicts the wave fronts associated with the transmitted pulse. Figure 1b illustrates the conditions prevalent when the leading edge of the transmitted pulse has reached the second target, the trailing edge of the transmitted has not reached the first target and the leading edge of the transmitted pulse has produced a return from the first target. The conditions illustrated in Figure 1c exist when
Figure 1. Wavefront structure from two targets that are illuminated by a radar pulse. I.e.I represents the leading edge return from target I, etc. $t_1 < t_2 < t_3 < t_4$.

The leading edge return from target II has reached target I but the trailing edge of the transmitted pulse has not arrived at target I. The next figure describes the situation when the leading and trailing edges of the transmitted pulse have produced a return from target I and target II has produced a leading edge return. It is obvious from this diagram that the radar is unable to ascertain that two targets are present because only one extended return exists. This analysis indicates that two targets are resolvable if their range separation,
\( \Delta R \), is greater than the round-trip distance traveled by a pulse of width \( T \); i.e., \( \Delta R > cT/2 \).

The preceding discussion clearly indicates the inability of a standard pulse radar to achieve good target (range) resolution and long range detection. This dilemma could be solved by using a radar system that employs search and tracking subsystems. The search subsystem in this application acts, because of its wide pulse width, as a long range detection device that also provides a minimal tracking capability if the target range exceeds the capacity of the tracking system. In this way the search subsystem is not required to provide any target resolution. When the target enters the range coverage of the tracking system, the tracking radar assumes target responsibility. Since the tracking radar by definition uses narrower pulses than the search radar, the tracking system provides the desired range resolution. However, it is obvious that such a system is extremely complex and expensive.

This dichotomy on the requirements of a pulse system led to the introduction of chirp radars.
III. CHIRP RADARS

In a chirp radar, the carrier frequency of the transmitted pulse is frequency modulated before transmission. The equation describing the transmitted pulse is given by

\[ \varepsilon_o(t) = \text{rect}(t/T) \exp \left[ 2\pi i \left( f_0 t + \frac{kt^2}{2} \right) \right] \]  (2)

where

\[ \text{rect}(t/T) = \begin{cases} 1 & t \leq |T/2| \\ 0 & t > |T/2| \end{cases} \]  (3)

A graph of equation (2) is shown in Figure 2. The exponential form of the waveform has been introduced in order to facilitate the use of the equations. The quantity \( k \) is defined by the following expression

\[ k = \Delta/T \]  (4)

where \( \Delta \) is the maximum shift in the carrier frequency. \( k \) has the units of \((1/\text{sec})^2\). \( T \) is the system pulse width.

The type of modulation employed can be ascertained by considering the expression for the instantaneous frequency. Using the standard definition, the instantaneous frequency is given by the following equation
\[w_1(t) = \frac{d\theta}{dt} = \frac{d}{dt} \left[ 2\pi(f_0 + kt^2)/2 \right]\]

\[= 2\pi(f_0 + kt)\]

in rad/sec or

\[f_1(t) = f_0 + kt\]

in Hz

Equations (5) and (6) display the fact that the transmitted waveform possesses a linear modulation. As will be seen shortly, this characteristic of chirp pulses allows a chirp radar to achieve greater range resolution than a standard pulse radar with the same pulse width. Stated in other words, a chirp radar system achieves the same range resolution as a standard pulse radar having a narrower pulse width. These remarks point out the ability of a chirp radar system to achieve greater range detection for a given range resolution. It is obvious that this increased capacity is attained at the cost of a more complex radar system; i.e., a system that frequency modulates and demodulates as well as amplitude modulates and demodulates.

The final item of interest about the signal in equation (2), the frequency spectrum, can be found by taking the Fourier Transform of the equation; i.e.,

\[\mathcal{F} [\epsilon_o(t)] = \int \epsilon_o(t) \exp(2\pi if_0 t) dt = \mathcal{E}_o(f)\]
For convenience, only the results of the integral will be presented here. For a more detailed discussion see reference 1. Therefore,

\[ E_0(f) = \sqrt{\frac{T}{2\Delta}} \left[ \exp\left(-i\pi[f-f_0]^2\right) \right] (Z(u_2)-Z(u_1)) = A(f)\exp i\theta \]  

(8)

where \( Z(u) \) is the complex Fresnel Integral:

\[ Z(u) = C(u) + i S(u) = \int_0^u \exp(i\pi\alpha^2/2)d\alpha \]  

(9)

The arguments \( u_2 \) and \( u_1 \) are defined by

\[ u_2 = -2(f-f_0) \sqrt{\frac{T}{2\Delta}} + \sqrt{\frac{T\Delta}{2}} \]  

(10)

\[ u_1 = -2(f-f_0) \sqrt{\frac{T}{2\Delta}} + \sqrt{\frac{T\Delta}{2}} \]  

(11)

The amplitude spectrum, \( A(f) \), of the transmitted pulse is given by the absolute value of \( E_0(f) \).

\[ |E_0(f)| = A(f) = \sqrt{\frac{T}{2\Delta}} |Z(u_2)-Z(u_1)| \]  

(12)

A plot of equation (12) can be seen in Figure 3 where the variable \( D = T\Delta \), known as Dispersion Factor, has been introduced. The phase spectrum of \( E_0(f) \), \( \theta(f) \), is seen to be quadratic in nature, i.e.,

\[ \theta(f) = \pi(f-f_0)^2 \]  

(13)
Since the chirp and standard pulse radars both transmit pulses of width \( T \), the improvement in range resolution must be obtained by the electronic equipment in the chirp radar receiver. This equipment makes use of the linear modulation, introduced into the signal in the transmitter section, to compress the pulse width by delaying one portion of the signal more than the other. This variable delay function is obtained by the network characteristic shown in Figure 4 and described by equation (14). This network is known as a de-chirp filter.

\[
H_d(f) = \exp\left(i \pi \frac{(f-f_o)^2}{k}\right) ; \quad f > 0 \tag{14}
\]

This filter has a unity amplitude spectrum and quadratic phase spectrum; i.e.,

\[
\varphi = \frac{\pi (f-f_o)^2}{k} \tag{15}
\]

Time delay as a function of frequency can be found as follows,

\[
t_d = -\frac{1}{2\pi} \frac{d\varphi}{df} = -(f-f_o)/k \tag{16}
\]

Now, assume that the instantaneous frequency of a given signal is represented by the following expression

\[
f_1 = f_o + kt \quad 0 \leq t \leq T \quad \text{i.e.,} \quad k = \Delta T \tag{17}
\]

when this signal appears at the input to the de-chirp filter, the signal finds itself being delayed less and less as the frequency increases or, in other words, as time progresses. Therefore, if a summing
device is placed at the de-chirp filter output, the trailing portions of the incoming signal find themselves being added with the earlier arriving portions of the signal. Thus, one is able in this fashion to compress the time base of the reflected pulse. Figure 5 illustrates the pulse after the compression process and indicates that the new pulse width equals $2/\Delta$. Hence, the range resolution improvement factor is seen to be

$$RIF = \frac{\text{Minimum range separation in standard radar systems}}{\text{Minimum range separation in chirp radar systems}} = \frac{T/2}{2\Delta/2} = \frac{D}{2}$$

Figure 5 also displays the decaying oscillations which are characteristic of the $\sin x/x$. These oscillations which have been termed "range sidelobes" are undesirable because nonexistent targets can be created when they appear in conjunction with other nearby targets. This effect becomes evident by referring to Figure 6. The figure illustrates the returns from two nearby targets plotted individually and then one which displays the actual return. It has been assumed that an additive phase structure exists. The small but rather significant pulse located between the two main lobes in the combined return could be mistaken for a target even though no such target exists. Hence, range sidelobes could impair system performance and generally do.

As already indicated, this range resolution improvement is achieved only at the expense of increased system complexity, and
appears in the form of a frequency modulator and demodulator. Frequency modulation can be done "actively" and "passively". "Active" modulation requires use of a variable frequency oscillator as is used in many standard F-M systems. A second method, introduced by the authors of reference 1, is known as "passive generation". The beauty of this method lies in the fact that no active devices are required to produce the frequency-modulation, only passive networks. Figure 7 depicts a typical chirp radar transmitter and receiver that employs "passive generation" to achieve the frequency-modulated transmitted pulse and the equipment necessary for demodulation.

The input to this system is the delta function, $\delta(t)$; i.e.,

$$\delta(t) = \begin{cases} 0 & \text{for } t \neq 0 \\ \infty & \text{for } t = 0 \end{cases} \quad (19)$$

and

$$\int_{-\infty}^{\infty} \delta(t) \, dt = 1 \quad (20)$$

Now, the ideal bandpass filter, $H_b(f)$, which is defined by,

$$H_b(f) = \frac{1}{\Delta}, \quad f_o - \Delta/2 \leq |f| \leq f_o + \Delta/2$$

$$= 0, \quad \text{elsewhere} \quad (21)$$

has an output, in response to the delta function, given by

$$E_2(f) = \delta(f) \cdot H_b(f) = H_b(f) \quad (22)$$
in the frequency domain and by

\[ e_2(t) = \mathcal{F}^{-1}[E_2(f)] = \frac{\sin \Delta t}{\pi \Delta t} \sin 2\pi f_o t \]  

(23)

in the time domain. This signal serves as the input to the chirp filter in the transmitter. As before, only the positive exponential will be used in the following mathematical operations. The results, however, will include the effects of both the positive and negative exponentials.

The output of the chirp filter can be determined in the usual manner by finding the Inverse Fourier Transform of the product of the Fourier Transforms of the chirp filter and the input signal. Following this procedure,

\[ E_3(f) = E_2(f) H_c(f) \]

\[ = \frac{1}{\Delta} \exp \left( -i\pi \frac{[f - f_o]^2}{k} \right) ; \quad f_o - \Delta/2 < f < f_o + \Delta/2 \]  

(24)

\[ = 0 \quad \text{otherwise} \]

and

\[ e_3(t) = \frac{1}{\Delta} \int_{f_o - \Delta/2}^{f_o + \Delta/2} \exp \left( -i\pi \frac{[f - f_o]^2}{k} \right) \exp(2\pi i f t) dt \]  

(25)

As before, only the results of this integral will be presented here. For a more detailed analysis refer to reference 1.
Therefore,

\[ \epsilon_3(t) = \frac{1}{\sqrt{2D}} \exp \left( 2\pi i \left[ f_0 t + kt^2/2 \right] \right) \left[ Z^*(v_2) - Z^*(v_1) \right] \]  

where

\[ Z(v) = \int_0^v \exp(i\pi \alpha^2/2) d\alpha \]  

and

\[ v_2 = \sqrt{2} \left[ \sqrt{\frac{\Delta}{T}} t + \frac{1}{2} \sqrt{D} \right] \]  

\[ v_1 = \sqrt{2} \left[ \sqrt{\frac{\Delta}{T}} t - \frac{1}{2} \sqrt{D} \right] \]  

By comparing equations (8), (9), (10) and (11) with equations (26), (27), (28) and (29), one notes a great deal of similarity. This similarity permits the use of Figure 3 to describe the envelope of \( \epsilon_3(t) \) if the variable \((f-f_0)/\Delta\) is changed to \( t/T \).

Since the envelope of \( \epsilon_3(t) \) depends upon the solution of two Fresnel Integrals, it can be shown that the tails of the envelope extend to infinity. However, this effect is more pronounced for small values of \( D \); i.e., \( D < 60 \). Therefore, for these cases, this type of pulse is not very useful.

As a result of this problem, time gating of the transmitted pulse must be employed when \( D \) is small in order to achieve more desirable pulse widths and repetition rates. However, the time
gating of the transmitted pulse creates a mismatch between transmitter and receiver. This point can be illustrated by the following equations. From equation (24)

$$E_3(f) = E_2(f) H_c(f)$$ \hspace{1cm} (24)

Since, in the receiver,

$$H_d(f) = H_c^*(f)$$ \hspace{1cm} (30)

It follows that

$$E_0(f) = E_2(f)H_c(f)H_d(f) = E_2(f)$$ \hspace{1cm} (31)

Since this result is valid only for these conditions, any network placed between the chirp and de-chirp filters creates a mismatched transmitter and receiver.

This mismatch has been found to produce additional range side-lobes at the de-chirp filter output which, until now, could not be removed. The remainder of this paper will be devoted to the solution of this range sidelobe problem. The first step in the solution will be the evaluation of the hitherto unknown de-chirp filter output waveform followed by a technique which permits evaluation of a filter for range sidelobe suppression.
IV. TIME GATED CHIRP RADAR SYSTEMS

In those systems employing time gating, the time limiting device is placed directly after the chirp filter as shown in Figure 8. The time limiting device acts upon the output of the chirp filter to produce the transmitting envelope shown in Figure 9. By comparing Figures 3 and 9, it is readily apparent that the time gating produces greater range sidelobe problems for small time-bandwidth products (D = $\Delta T$) than for large time-bandwidth products because it eliminates a larger portion of the chirp filter output waveform. In other words, the lower the Dispersion Factor (time-bandwidth product) the greater the mismatch created between transmitter and receiver by the time gating process; i.e., mismatch inversely related to Dispersion Factor.

A method for elimination of the range sidelobes for these cases (low D) had not been available because the nature of the de-chirp response was not known. Therefore, the first step in the solution of the sidelobe suppression problem will be the derivation of the form of the de-chirp filter output.

The determination of filter response begins with the chirp filter output as given by equation (26). This signal is altered by the time gating network whose characterization is given by the following equation in the time domain

$$r(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & \text{elsewhere} \end{cases}$$

(32)
and in the frequency domain by

\[
R(f) = \left[ r(t) \right] = \int_{-T/2}^{T/2} \exp(-2\pi j ft) dt
\]

\[= T \frac{\sin \pi f T}{\pi f T} \quad (33)\]

The transmitted waveform can be found by taking the product of equations (26) and (32). This results in the following equation in the time domain

\[
\epsilon_4(t) = \epsilon_3(t) \cdot r(t)
\]

\[= \frac{1}{\sqrt{2D}} \exp \left(2\pi i \left[ \int_0^{t+kT} \frac{2}{2} \right] \right) \left( Z^*(v_2) - Z^*(v_1) \right) \quad |t| < T/2 \quad (34)\]

\[= 0 \quad \text{elsewhere}\]

and in the frequency domain the result is

\[
E_4(f) = \mathcal{F} \left[ \epsilon_3(t) \cdot r(t) \right] = E_3(f) \otimes R(f)
\]

\[= \left[ \frac{1}{\Delta} \exp \left(-i\pi \left[ f - f_o \right]^2 / k \right) \right] \otimes T \frac{\sin \pi f T}{\pi f T} \quad \text{all } f \quad (35)\]

where the symbol \( \otimes \) designates the convolution of the two terms in the equation.

The next quantity of interest is the output of the de-chirp filter. Since its derivation is rather long and complex, only the results will be presented. The entire derivation, however, is presented in Appendix A where the form of the input in equation (35) was used to evaluate the results.
\[ e_d(t) = \sqrt{2A} \frac{\sin \pi t}{\pi t} \cos 2\pi \left( f_0 t - k t^2 / 2 + a \right) \]
\[ + \cos 2\pi f_0 t \left( \frac{\cos \pi t}{\pi t} (S - V') + \frac{\sin \pi t}{\pi t} S' \right) \]
\[ + \sin 2\pi f_0 t \left( \frac{\cos \pi t}{\pi t} (S' - V) - \frac{\sin \pi t}{\pi t} \right) \]

where

\[ Z(\sqrt{2D}) = A e^{-ic} = \int_0^{\sqrt{2D}} \exp(-i\pi x^2/2)dx; \quad a = \frac{1}{8} - c'; \quad c' = c/\pi/2 \]

\[ S, S' = \frac{1}{2} \left( R + X \right) \]

\[ V, V' = P \pm Q \]

\[ P + iQ = \int_{t'}^{t''} \exp(i\pi x^2/2)dx \]

\[ R + iX = \int_{t'}^{t''} \exp(i\pi x^2/2)dx \]

and

\[ t' = \sqrt{2k} t \]

\[ T' = \sqrt{2k} T \]

A plot of a typical waveform can be seen in Fig. 10.
As one would expect, when the width of the time gate is increased to infinity, in the limit, equation (36) yields the result shown in reference 1; i.e., $e_d(t) = \frac{\sin\Delta t}{\pi \Delta t} \cos 2\pi f_0 t$. This analysis and its results are also shown in Appendix A.

This result is the first significant item of interest because, for the first time, an exact description of the waveform now exists. It will now be used to describe a possible reason for the failure of the paired-echo theory to provide sidelobe suppression.

Equations (40) and (41) clearly indicate that the integrals give rise to terms which are functions of $t^2$. Therefore, the real and imaginary parts of these integrals are sinusoidal functions of $t^2$. This implies that $S, S', V$ and $V'$ are also sinusoidal functions of $t^2$ since these terms are composed of sums and differences of sinusoidal functions of $t^2$. The completion of these integrals can be shown to yield sinusoidal terms which are functions of $t^2$. When these four terms are combined with the sinusoidal terms at the carrier frequency, the results are sinusoidal terms which are functions of $t$ and $t^2$. This implies that the frequency of these terms possess a time dependence which, in turn, means that these terms contain an F-M modulation. Consequently, equation (36) is an F-M waveform with all terms possessing a similar F-M structure; i.e. $\Theta' \sim at + bt^2$. 

With this type of structure, it can be shown that any advanced or delayed version of this signal which would be in phase opposition to the original signal at some point, \( t_1 \), will be in-phase with the original signal at some other point, \( t_2 \). This implies that the sum of these two waveforms will produce a smaller signal at \( t_1 \) but a larger signal at \( t_2 \). This argument can be extended to include any number of advanced or delayed versions of the original signal. Therefore, one can conclude from these arguments, that the waveform which is the sum of the original signal and an arbitrary number of advanced and/or delayed versions of the original signal will contain an amplitude which is smaller than the original signal at \( t_1 \) but larger at \( t_2 \). This fact has also been borne out by experimental evidence. Since this is the method employed by the paired-echo technique to reduce sidelobes in the constant frequency or A-M case, one can conclude that it will not achieve the desired result in the variable frequency or F-M case.

Before leaving this section, it should be pointed out that the derivation in Appendix A represents a major contribution not only because of its length and complexity, but because for the first time the exact nature of the response is known.
V. DERIVATION OF THE DE-CHIRP OUTPUT SPECTRUM

Due to the complexity of the equation describing the de-chirp output waveform, it is, first of all, necessary to study the time and frequency domain pictorial structure of the waveform before attempting sidelobe suppression. However, this is not possible unless some particular values are chosen for the parameters $D$, $T$, $\Delta$, and $f_0$. The value for $D$ should be less than 60 and the values for $T$, $\Delta$ and $f_0$ should result in a realizable system. These considerations led to the following choices: $D = 30$, $T = 3 \mu\text{sec}$, $\Delta = 10 \text{ MHz}$ and $f_0 = 30 \text{ MHz}$. Although this approach can only produce a sidelobe suppression filter for this particular case, it will later be shown that the technique used to evaluate the filter for this particular case can be generalized for use with any combination of parameters. Using the aforementioned values in equation (36), one obtains the waveform structure shown in Figure 10. The original plot was done by a computer because of the complexity of the terms comprising equation (36). The program used for evaluating the original plot has been carefully outlined in Appendix B.

The diagram indicates that the waveform possesses numerous sidelobes with the largest one being only 13 dB below the amplitude of the main lobe. Sidelobes of this magnitude can easily give rise to the false target indications mentioned in previous discussions. These false target indications require that a major objective of this paper be the elimination or, at least, substantial reduction of these undesirable range sidelobes.
With the time domain structure available, attention can now be turned to the evaluation of the frequency spectrum. This task requires somewhat more discussion, in light of the sampling theorem, (i.e., aliasing) than was presented for the evaluation of the time structure. The idea of the sampling theorem has been mentioned at this point since the computer, which operates in the digital world, will be called upon to ascertain the frequency spectrum. This will be accomplished via use of the now very popular Fast Fourier Transform or FFT. However, before use can be made of the FFT some basic questions must be answered. First, what sampling rate should be used? Or another way of stating this is, what is the maximum frequency component? Second, given a sampling rate, how do we know that the required spectrum is that which is produced? Each of these questions is very difficult to answer; however, enough information is available in order to answer them. First, from figure 10, one can conclude that the envelope of the waveform resembles a \( \sin x/x \) structure. This implies that the spectrum envelope might well approximate a rectangular structure. Next, with the carrier frequency known to be 30 MHz and the system bandwidth 10 MHz, one can reasonably conclude, from these remarks, that the spectrum is centered at 30 MHz with a spectral width of approximately 10 MHz. In other words, the maximum frequency component lies around 35 MHz. Therefore, from the sampling theorem, the sampling frequency is given by

\[ f_s = 2f_m = 70 \text{ MHz} \] (44)
or

\[ T_s = \frac{1}{f_s} = \frac{1}{70 \text{ MHz}} = \sqrt{2} \times 10^{-8} \text{ sec.} = 14.1 \text{ ns} \]  

(45)

Since this sampling rate will not yield the number of samples required by the FFT; i.e.,

\[ \text{Number of Samples} = 2^N, \]  

(46)

it is more desirable to increase the sampling rate to meet this requirement rather than decrease it because more samples are available for processing. Therefore, for the first trial, the sampling interval is

\[ T_s = 6.25 \text{ ns} \]  

(47)

The result in equation (47) answers the first of the two aforementioned questions. As expected, this particular value of \( T_s \) far exceeds the Nyquist rate; i.e., \( f_s \geq 2f_m \). The answer to the second question has been given by such prominent people in the field of spectral analysis as Tukey, Cooley, and etc. They contend that by halving the sampling rate and comparing the two resultant spectra, corresponding to the two different sampling rates, one decides that a particular result or spectrum is sufficient when the two spectra are negligibly different. It has been assumed, of course, that each sampling rate exceeds the Nyquist Rate; i.e., \( f_s \geq 2f_m \).

Therefore, in order to satisfy the above mentioned requirements, the following sampling rates were used,
Having tentatively answered all the previous questions, a computer program was written to evaluate the spectrum for each of these sampling rates in order to ascertain the validity of the previous remarks. Due to its length and complexity the program is presented in Appendix C. The results of the programs were studied very carefully and they clearly indicated that each of the sampling rates produced satisfactory or negligibly different spectra. The results of the analysis are presented in Figure 11. The diagram, which indicates that all frequency components of the signal lie below 45 MHz, implies a sampling interval of

\[ T_s = 1/90 \text{ MHz} = 11.1 \text{ ns} \]  

which is considerably larger than the three rates used in evaluating the spectra and clearly bears out all of the previous remarks.

To summarize, the time and frequency domain pictorial structures of the de-chirp filter output have now been evaluated and, in the next section, will be used to evaluate a filter which suppresses the sidelobes of the time domain structure.
VI. DERIVATION OF THE SIDELOBE SUPPRESSION FILTER

The method described in this section for sidelobe suppression and the derivation of the filter necessary to achieve this characteristic have never been formalized in the literature until now. The method makes use of the known relationships between the time and frequency domain characteristics of waveforms and utilizes these characteristics to alter signals by a new and novel method. At the end of this section, the method will be formalized so that it can be used for a general class of signals. The first step requires determining the region of the frequency spectrum which can be altered. In most cases, as in the present one, only the region inside the system's operating bandwidth should contain spectral components of any significance. Thus, the spectral components near the band edges should diminish to zero as distance from the edge increases and those far removed from the edges should be zero. This implies that only those components within the operating bandwidth should be altered to achieve the desired result.

These comments indicate that for the present case only the spectrum between 25 MHz-to-35 MHz can be shaped as necessary and that in the transition region the spectrum should approach zero. For the region well below 25 MHz and well above 35 MHz the spectrum should be approximately zero.
At this point, by actual alteration of the de-chirp output time signal, one can ascertain which regions of the time signal give rise to the various spectral components because it is these sections of the spectrum which must ultimately be altered. A region of this time signal which is of considerable interest lies outside the main lobe because it is this region which contains all of the undesirable sidelobes. A reasonable alteration would be to reduce all those sidelobes to a more desirable level, say, -40 dB below the main lobe. The filter which produces these changes in the de-chirp output spectrum is then evaluated. These two steps will yield the desired information about the time signal and the spectrum. The altered waveform, $e_A(t)$, is obtained by reducing the amplitude of the original time signal in all those places outside the main-lobe where the amplitude exceeds -40dB. Then, using this new signal and the FFT, one can obtain the spectrum, $E_A(f)$, of this waveform. The ratio of $E_A(f)$ and the de-chirp output spectrum, $E_B(f)$, will yield the required filter. This filter is shown in figure 12A. The spikes represent locations where the input signal contains zero or negligible energy, however, for programming purposes, they were limited to the values shown. Therefore, one can conclude from this diagram that this filter is non-realizable, and, thus one cannot achieve this level of sidelobe reduction.

However, all is not lost because the filter components within the system bandwidth are realizable. This fact prompted application of only this portion of the filter to the de-chirp output spectrum, $E_B(f)$. The
new spectrum, $E_A'(f)$, can then be inverted to yield a time domain signal, $e_A'(t)$. This approach proved to be quite productive because it indicated as shown in figure 12B that only the near-in sidelobes still remained at undesirable levels. Even though this section which is shown in figure 12C, of the filter, did not reduce all of the sidelobes, it does represent the first stage in the development of a sidelobe suppression filter.

Since the waveform in figure 12B still possesses undesirable near-in sidelobes, one can eliminate them by incorporating them into the main lobe. This can be achieved by increasing the amplitude of $e_A'(t)$ that lies between the main-lobe and the sidelobe such that the envelope of the new time, $e_A''(t)$, has its first null point on the the right side of the first sidelobe of $e_A'(t)$. The spectrum of $e_A''(t)$, $E_A''(f)$, is shown in figure 12D. The ratio of $E_A''(f)$ and $E_B(f)$, the de-chirp output spectrum, produces a filter whose characteristics are shown in figure 12E. Although $E_A''(f)$ possesses some undesirable hard-edge spectral components, the filter required to produce this spectrum is still realizable and represents the second stage in the development of a sidelobe suppression filter. It should be pointed out, however, that this improvement in sidelobe level came at the expense of a wider pulse width. The amount of increase in pulse width is determined by comparing the new pulse width with the unequalized pulse width at the zero crossings of the envelopes.

As mentioned in the previous paragraph, $E_A''(f)$ possesses some undesirable spectral components at the band-edges. These spectral components can be reduced to more desirable levels through the use of a negative exponential.
Numerous trials, with various exponents, were performed and the resulting frequency spectrums, $E_A(f)$, inverted and the resulting time functions compared on the basis of sidelobe level and pulse width. The results of the trials clearly indicate that the exponential function given in equation (52) yields, as shown in table 1, the smallest sidelobe level and the narrowest pulse;

\[
m(f) = \exp(-.11[f-30\text{ mhz}]); \quad f > 35\text{ mhz} \\
= \exp(.11[f-30\text{ mhz}]); \quad f < 25\text{ mhz} \\
= 1 \quad ; \text{ elsewhere} 
\]

To verify the uniqueness of this result, the trials were performed with exponents both larger and smaller than the one shown in equation (52) and, in each case, the exponents produced larger values of both sidelobe level and pulse width.

By taking the ratio of

\[
E_A(f) = \frac{m(f) \cdot E_A(f)}{E_B(f)} \quad (53)
\]

and $E_B(f)$, the final form of the sidelobe suppression filter can be determined and is displayed in figure 12F.

Since the filter depicted in figure 12F is only a practical realization of the filter shown in figure 12A, only a -34db sidelobe level was attainable. However, this result may be satisfactory in many radar applications.
At the same time, however, the filter introduced a degradation in SNR and an increase in pulse width. The degradation in SNR ratio, which is defined in equation (54), is primarily attributable to the exponential smoothing process because as the spectral components were reduced so was the total energy contained in the spectrum reduced.

$$\text{SNR} = 10 \log_{10} \left[ \frac{(\text{Peak Amplitude})^2}{\text{Mean Noise Power}} \right]$$

(54)

where

$$\text{Mean Noise Power} = N = N_c B \quad (55)$$

where \( B \) is the equivalent noise bandwidth and \( N_c \) the noise power density in watts/cycle. On the other hand, the smoothing process also produced a 4% reduction in the equivalent noise bandwidth. By taking these changes into account, the degradation in SNR can be evaluated as follows:

$$\frac{(SNR)_o}{(SNR)_i} = 10 \log_{10} \left[ \frac{(\text{Peak Amp.})^2_o}{(\text{Peak Amp.})^2_i} \frac{B_o}{B_i} \right] \quad (56)$$

$$= -.75 \text{ db}$$

$$\frac{(\text{Peak Amp.})_o}{(\text{Peak Amp.})_i} = (.898/1)^2 = .806 \quad (57)$$

and

$$\frac{N_c B_o}{N_c B_i} = \frac{B_o}{B_i} = .96 \quad (58)$$
The increase in pulse width is due primarily to the increase which was made in \( e_A(t) \) in the third step of the filter derivation. The effect of these changes on system performance will depend upon system requirements.

It should be mentioned that although the only reduction considered was \(-40\) db below the main-lobe, larger reductions were tried and in each case the only differences between the altered spectrums were in the number of spikes outside the system bandwidth.

To summarize, this new procedure has yielded a sidelobe suppression filter capable of reducing the de-chirp filter's output sidelobes to \(-34\) db below the main-lobe while, at the same time, introducing a \(-.75\) db degradation in its SNR and an increase of \(18\)% in its pulse width.

The zero phase angle required by the sidelobe suppression filter which is shown in figure 12F represents a very significant item because it implies that the de-chirp filter and the sidelobe suppression can be combined into one filter by simply altering the amplitude spectrum of the de-chirp filter in accordance with figure 12F. By performing the aforementioned task, one produces a single filter that possesses the amplitude spectrum of the sidelobe suppression filter and the phase angle spectrum of the de-chirp filter.
A second interesting item is the shape of the amplitude spectrum. The shape resembles the Taylor weighting filter which is the sidelobe suppression filter presented in reference 1, with the only difference lying in the amplitude of the filter sidelobes. Those of the Taylor filter are smaller than those of the new sidelobe suppression filter.

To ascertain the limits of application of this approach, a study was made in order to evaluate a family of these filters for various values of D. A representative sample of the results of that analysis can be seen in Figures 13-16. As expected, the filters approach the Taylor structure as Dispersion Factor increases because as the Dispersion Factor increases the amount of distortion introduced by the time gating decreases and, as a result, the situation begins to approach the one discussed in reference 1 where the Taylor filter is shown to produce the required sidelobe suppression. In addition, the resemblance becomes more pronounced as bandwidth increases, for a given Dispersion Factor, because fewer spectral components are altered as the bandwidth expands.

The derivation of this family of filters also permitted a comparison of the various range sidelobe levels produced by each of the filters. These results are shown in Figure 17. The curve indicates that for a small Dispersion Factor and small bandwidth one is unable to achieve desirable sidelobe levels. However, as one might expect, by increasing the bandwidth one is able to reduce the sidelobes to somewhat more desirable levels. This result stems from the fact
that fewer spectral components near the band-edges undergo radical changes to satisfy the bandwidth requirements because they are now included within the larger system bandwidth and, therefore, these components find themselves being smoothed rather than being reduced to zero by the sidelobe suppression filter. As pointed out previously, it was the spectral changes brought about at the band edges that also gave rise to reduced sidelobe levels. These comments can also be used to explain the reasons for the lower sidelobe levels that occur for all the larger values of Dispersion Factor.

For large Dispersion Factors, Figure 17 indicates that lower sidelobe levels can be achieved by any combination of bandwidth and pulse width. This conclusion seems quite reasonable because for large Dispersion Factors the time gating of the pulse introduces smaller amounts of distortion and, thus, spectral alteration is minimized and in the limit one is able to achieve the filter and sidelobe levels outlined in reference 1.

Figure 17 also illustrates that sidelobe level reductions become smaller as bandwidth increases for a given Dispersion Factor. This implies, as expected, that fewer important components are included within the bandwidth as it increases.

Figures 18 and 19 describe the degradation in pulse width and in the signal-to-noise ratio caused by the sidelobe reduction filter.
The shape of each figure can be easily explained as follows: as the size of the gate width in the transmitter increases, smaller amounts of distortion are introduced into the transmitted pulse. Consequently, fewer spectral components lie outside the system bandwidth. Therefore, increases in pulse width become smaller as gate width increases because fewer sidelobes have to be included in the main lobe in order to reduce the spectral aberrations at the band-edges. As the system bandwidth increases, smaller increases in pulse width will allow the remaining band-edge components to be included within the system bandwidth.

Losses in SNR become smaller as the gate width and bandwidth increase because fewer spectral components lie outside the system bandwidth. This implies that the loss in signal energy becomes smaller, thereby reducing the loss in SNR.

Having completed the derivation of the sidelobe suppression filter and the associated discussions it seems quite apropos at this point to outline the generalized technique employed for its derivation. It is assumed that a diagram of the waveshape being altered is available. One then proceeds as follows:

1. If the spectrum of the output waveform has been specified, then the filter required for such a situation can be evaluated by taking the ratio of the output and input spectra, i.e., the input spectrum being found in the usual manner of taking the Fourier Transform of the input signal. The output waveform is then determined from the Inverse Fourier
Transform of the output spectrum. This task can be performed by using the IFFT (Interpolatory Fast Fourier Transform).

2. If the output waveform is known, then using the FFT yields the output spectrum and, then, one can follow the procedure outlined in Step 1.

In many cases however, only a certain characteristic has been specified and the detail form left to the discretion of the designer or the procedures outlined in steps 1 and 2 yield a non-realizable filter. In these situations, the following steps must be taken:

3. Divide the frequency spectrum into a region lying inside the system bandwidth and one outside.

   Since the amplitude spectrum lying outside the system bandwidth must always be reduced to zero, only the amplitude spectrum lying within the system bandwidth is open to arbitrary modification.

4. Divide the time domain signal into desirable and undesirable sections.

5. Perform a convenient alteration on the original time signal and then note the differences between the two frequency spectrums.

   The most obvious alteration being the reduction of all undesirable portions of the signal because the purpose of this
step is not to produce non-realizable changes but to ascertain which portions of the spectrum give rise to the undesirable portions of the time signal.

After deciding upon the change in the time signal, use of the FFT yields the new spectrum. One then examines the ratio of the new and original spectrums to ascertain the realizability of the required frequency filter. Should the filter possess any non-realizable components, then additional changes are still required.

In any event, the results of this step should depict those sections of the original spectrum which are responsible for the undesirable portions of the time domain signal and the frequency changes required to yield a more desirable waveshape. Therefore, it is imperative that the alteration be chosen with care.

6. Eliminate all changes in the filter which are non-realizable and those that lie outside the system bandwidth and apply the remaining components of the filter to the original spectrum. Then, take the resultant spectrum and evaluate the time signal via use of the IFFT.

7. In some cases, elimination of some spectral components outside the system bandwidth yields undesirable time domain terms. In these instances, reduction rather than elimination will yield a more desirable time domain structure.
The reduction function should be varied until the most desirable waveform is found.

Completion of the previous steps yields a waveform in the time domain which can be produced by a realizable filter. However, should some changes in the structure still be necessary, the previous steps can be repeated as an iteration process until a more satisfactory structure is obtained. Along with subsequent iterations, the following items should be considered.

1. Increase the time base of the signal to include some undesirable components. This procedure, in turn, reduces the system bandwidth so that now some undesirable frequency components near the band edges can be reduced or eliminated by the reduction function.

2. The system bandwidth can be increased if it has been ascertained in some of the previous steps that reduction of spectral components outside the system bandwidth produce undesirable components of the time signal.

Although the previous procedure began by altering the time signal, it is obvious that one could just as easily have begun by altering the frequency spectra. In either case, all of the previous remarks are still valid.

It should be pointed out that the preceding procedure is most effectively employed in situations where an analytical analysis cannot be performed.
VII. SUMMARY AND CONCLUSIONS

This paper analyzes and solves the range sidelobe problem that is encountered in all chirp radar systems that employ time gating of "passively generated" chirp pulses. Time gating may be employed to remove the tails of the pulses.

The paper presents the derivation of the hitherto unknown chirp receiver response to the time gated transmitted pulses which are "passively generated". Since the results of the derivation clearly indicate that the receiver response has an F-M structure, one is now able to understand a possible reason for the failure of the paired-echo theory to achieve range sidelobe reductions.

The paper then goes on to describe a method for the derivation of a filter which achieves range sidelobe suppression at a minimal loss in SNR and increase in pulse width and then presents a representative group from the family of these filters. One significant result of this derivation lies in the fact that the sidelobe suppression filter can be easily combined with the de-chirp filter, because of its zero phase angle, to yield a single filter whose amplitude characteristics are those of the sidelobe suppression filter and whose phase characteristics are those of the de-chirp filter.

This family of filters clearly indicates that as the effect of the time gating diminishes the required filter approaches the Taylor filter which is shown in reference 1 to be the required filter when
time gating is not employed in the transmitter section.

Finally, figures 17, 18 and 19 are presented in order to display the relationships between sidelobe level, pulse width, SNR and dispersion factor, respectively, of the resultant output pulse. These curves indicate that the sidelobe level, pulse width increase and loss in SNR all decrease as dispersion factor increases and, in addition, these decreases become even larger as bandwidth increases.

It should be pointed in closing that in those chirp systems which employ large time-bandwidth products, time gating is not a necessity because the pulse is nearly rectangular. Therefore, the results of this paper are primarily useful in those systems employing moderate time-bandwidth products.
REFERENCES


APPENDIX A

Derivation of the De-Chirp Response

The starting point will be to define the de-chirp output, in general, as

\[ E_d(f) = E_4(f)H_d(f) \]  \hspace{1cm} (A)

\( H_d(f) \) was given in equation (14) of Section 3, and is repeated here for convenience.

\[ H_d(f) = e^{\frac{i\pi(f-f_0)^2}{k}} ; \quad f > 0 \] \hspace{1cm} (14)

\( E_4(f) \) was stated in equation (35) of Section 4 to be

\[ E_4(f) = E_3(f)^*R(f) ; \quad \text{all } f \] \hspace{1cm} (35)

\( E_3(f) \) is the output of the chirp filter which is given by

\[ E_3(f) = \frac{1}{\Delta} \exp\left(-\frac{i\pi}{\Delta}(f-f_0)^2\right) ; \quad f_0 - \frac{\Delta}{2} \leq f \leq f_0 + \frac{\Delta}{2} \] \hspace{1cm} (1)

This is of course the result of taking the product of the transforms of the bandpass filter, given in equation (21) of Section 3 and the chirp filter, given in equation (24) of Section 3.
\( R(f) \) is the Fourier Transform of \( r(t) \); see equation (33) of Section 4

\[
R(f) = F \left[ r(t) \right] = F \left[ \text{rect} \left( \frac{t}{T} \right) \right] = \frac{T \sin \frac{\pi Tf}{T}}{\frac{\pi Tf}{T}} \tag{2}
\]

Equation (35) can be rewritten as

\[
E_4(f) = \int E_3(\alpha)R(f-\alpha)d\alpha = \int E_3(\alpha)R(\alpha)d\alpha. \tag{3}
\]

Now, from equations (24) of section 3

\[
E_3(f-\alpha) = \frac{1}{\Delta} \exp \left( -\frac{\pi}{\Delta}(f-f_0-\alpha)^2 \right); \quad +f_0 - \frac{\Delta}{2} \leq f - \alpha \leq f_0 + \frac{\Delta}{2}
\]

and equation (2) above

\[
R(\alpha) = T \frac{\sin \frac{\pi T \alpha}{T}}{\frac{\pi T \alpha}{T}} \tag{5}
\]

\( E_4(f) \) can be written as follows because \( f_0 >> 1/T \)

\[
E_4(f) = \frac{T}{\Delta} \left[ \left( \frac{\sin \frac{\pi T \alpha}{T}}{\frac{\pi T \alpha}{T}} \right) \exp \left( -\frac{\pi}{\Delta}(f-f_0-\alpha)^2 \right) \right] d\alpha; \quad f > 0 \tag{6}
\]

where \( \alpha_1 = \left[ f - (f_0 + \Delta/2) \right] \) and \( \alpha_2 = \left[ f - (f_0 - \Delta/2) \right] \).
With $f_0 > 1/T$, one can neglect the contribution of the negative frequency terms. The values for the limits can be best explained with the help of figure 1 on pages 4 and 5. The product of $E_s(f-a)$ and $R(a)$ will exist only for a width $\Delta$ in the $a$-plane. As noted in Figure 1 d, the two endpoints are given by $a_1$ and $a_2$, respectively; i.e., $(a_2 - a_1 = \Delta)$. It should be apparent that for any value of $f$, the area exists for only for a width $\Delta$. Thus, the expressions for $a_1$ and $a_2$ are valid for any $f$ because they represent the limits on the region in the $a$-plane for which the product exists. Hence, equation (6) correctly describes the chirp filter output Fourier Transform.

Having obtained the chirp filter output Fourier Transform, equation (A) dictates that the de-chirp filter output Fourier Transform should be given by

$$F_d(\beta + f_0) = \frac{2}{\Delta} e^{i\pi/\kappa^2} \int_{a_1}^{a_2} \frac{\sin \pi Ta}{\pi Ta} e^{i\pi/k(\beta-a)^2} da$$

(7)

where $\beta = f - f_0$. As a result, $a_1 = \beta - \frac{\Delta}{2}$ and $a_2 = \beta + \frac{\Delta}{2}$. The substitution was made because of the ease of handling the resulting equations.

One notes, at this point, that the exponential factor in equation (7) can be expanded very simply.
Figure 1. Diagram illustrating the CONVOLUTION used in equation 3.
\[ \exp[-i\pi/k(\beta - \alpha)^2] = \exp[-i\pi/k(\beta^2 - 2\alpha + \alpha^2)] \]  \hspace{1cm} (8)

But the term \( \exp[-i\pi/k\beta^2] \) is independent of the variable of integration and, consequently, can be taken outside of the integral sign. When taken outside the integral, it can be combined with \( \exp(i\pi/k\beta^2) \) to produce unity. As a result, equation (7) can be rewritten as

\[
E_d(f) = \frac{T}{A} \int_{\alpha_1}^{\alpha_2} \frac{\sin \pi T\alpha}{\pi T\alpha} \exp\left[2\pi i/k(\beta\alpha - \alpha^2/2)\right] d\alpha \]  \hspace{1cm} (9)

because of the two remaining items on the right-hand side of equation (8).

Various tables were searched in an effort to find the inverse Fourier Transform of equation (9). However, this equation has no standard inverse Fourier Transform. As a result, many techniques for integration were considered and after numerous attempts the following method proved to be useful. The method requires that equation (9) be differentiated with respect to \( \beta \). The rule for performing such a differentiation is given by
\[
\frac{\partial}{\partial \beta} \int_{f_1(\beta)}^{f_2(\beta)} F(\alpha, \beta) d\alpha = \int_{f_1(\beta)}^{f_2(\beta)} \frac{\partial F(\alpha, \beta)}{\partial \beta} d\alpha + F(f_2(\beta), \beta) \frac{\partial f_2(\beta)}{\partial \beta} \\
- F(f_1(\beta), \beta) \frac{\partial f_1(\beta)}{\partial \beta}
\]

(10)

This equation, as stated, can be readily applied to equation (9).

Thus,

(11)

\[
\frac{\partial E_d(f)}{\partial \beta} = \frac{T}{\Delta} \left[ \frac{\sin \pi T \alpha}{\pi T \alpha} \exp \left( \frac{2\pi i}{k} \left[ \beta \alpha - \alpha^2/2 \right] \right) \left( \frac{2\pi i \alpha}{k} \right) d\alpha 
+ \frac{\sin \pi T(\beta + \Delta/2)}{\pi T(\beta + \Delta/2)} \exp \left( \frac{2\pi i}{k} \left[ \beta (\beta + \Delta/2) - (\beta + \Delta/2)^2/2 \right] \right) 
- \frac{\sin \pi T(\beta - \Delta/2)}{\pi T(\beta - \Delta/2)} \exp \left( \frac{2\pi i}{k} \left[ \beta (\beta - \Delta/2) - (\beta - \Delta/2)^2/2 \right] \right) \right]
\]

Having the derivative of the Fourier Transform of a function still permits one to obtain the inverse Fourier Transform via use of the following method. Using the definition of the Fourier Transform

\[
E_d(f) = \int_{-\infty}^{\infty} e(t)e^{-12\pi ft} dt
\]

(12)
and differentiating (12) with respect to \( f \) gives

\[
\frac{\partial E_d(f)}{\partial f} = -12\pi t E_d(f)
\]  

(13)

now, equation (13) can be rewritten into a more useable form. That is,

\[
\frac{\partial E_d(f)}{\partial f} = -12\pi t E_d(f)
\]  

(14)

where equation (12) was used once again. Now, the inverse Fourier Transform of equation (14) can be written as

\[
F^{-1}\left[\frac{\partial E_d(f)}{\partial f}\right] = F^{-1}\left[-12\pi t E_d(f)\right] = \frac{2\pi t}{i} F^{-1}\left[E_d(f)\right]
\]

(15)

\[
= \frac{2\pi t}{i} e_o(t).
\]

where use was made of the definition of the inverse Fourier Transform. Solving for \( e_o(t) \) in equation (15) yields

\[
e_o(t) = \frac{1}{2\pi t} F^{-1}\left[\frac{\partial E_d(f)}{\partial f}\right]
\]

(16)
now that the basic idea has been presented, the task of finding
the inverse Fourier Transform of equation (11) will now be
undertaken. Consider the second term on the right hand side of
equation (11).

\[
\frac{\sin \pi T(\beta + \Delta/2)}{\pi T(\beta + \Delta/2)} \exp \left( \frac{2\pi i}{k} \left[ \beta (\beta + \Delta/2) - (\beta + \Delta/2)^2/2 \right] \right)
\]  

(17)

Expanding the exponent of the exponential produces

\[
\frac{\sin \pi T(\beta + \Delta/2)}{\pi T(\beta + \Delta/2)} \exp \left( \frac{2\pi i}{k} \left[ \beta (\beta + \Delta/2) - (\beta + \Delta/2)^2/2 \right] \right) = \sin \pi T(\beta + \Delta/2) \exp \left( \frac{2\pi i}{k} \times \left[ \frac{\beta^2}{2} - \frac{\Delta^2}{8} \right] \right).
\]

(18)

In a similar fashion, the third term on the right hand side of
equation (11) can be seen to equal

\[
\frac{\sin \pi T(\beta - \Delta/2)}{\pi T(\beta - \Delta/2)} \exp \left( \frac{2\pi i}{k} \left[ \beta^2 - \frac{\Delta^2}{8} \right] \right).
\]

(19)

The first term in equation (11) can be simplified by cancellation
of the \( \pi \alpha \) terms in numerator and denominator and, from equation (4)
of Section 3, by substituting \( T/\Delta = 1/k \). Thus, equation (11) can
be restated as,
\[
\frac{\partial E(f)}{\partial \beta} = \frac{1}{k} \left[ \int_{\beta - \Delta/2}^{\beta + \Delta/2} \sin \pi T \alpha \exp \left( \frac{2\pi i}{k} \left[ \beta \alpha - \alpha^2/2 \right] \right) d\alpha \right. \\
\left. + \frac{\sin \pi T (\beta + \Delta/2)}{\pi T (\beta + \Delta/2)} \exp \left( \frac{\pi}{k} \left[ \beta^2 - \Delta^2/4 \right] \right) \right)
\]

For ease of handling these three terms, let

\[
\frac{\partial E(f)}{\partial \beta} = E_{01} + E_{02} + E_{03}
\]

and then, consider each term separately. Let us consider \( E_{01} \) first.

\[
E_{01} = \frac{2i}{k^2 T} \int_{\beta - \Delta/2}^{\beta + \Delta/2} \sin \pi T \alpha \exp \left( \frac{2\pi i}{k} \left[ \beta \alpha - \alpha^2/2 \right] \right) d\alpha
\]

\[
= \frac{2i}{k^2 T} \left[ \exp(\pi i T \alpha) - \exp(-\pi i T \alpha) \right] \exp \left( \frac{2\pi i}{k} \left[ \beta \alpha - \alpha^2/2 \right] \right) d\alpha
\]
In equation (22), the complex expression for the sinusoid was used in the expansion, and the \((2\text{i})\) term in the numerator cancels the \((2\text{i})\) term in the denominator. The exponential terms can be combined as follows:

\[
\exp(\pi T\alpha)\exp\left(\frac{2\pi k}{\Delta} [k + \alpha^2/2]\right) = \exp\left(\frac{\pi k}{2} \left[2(\beta+\Delta/2)\alpha - \alpha^2\right]\right)
\]

where use has been made of the fact that \(k = \Delta/T\), i.e.,

\[
\frac{\pi}{k} (\Delta\alpha) = \frac{\pi}{k} \cdot kT\alpha = \pi T\alpha.
\]

In a similar fashion, the second exponential term can be combined to yield the following:

\[
\exp(-\pi T\alpha)\exp\left(\frac{2\pi k}{\Delta} [k - \alpha^2/2]\right) = \exp\left(\frac{\pi k}{2} \left[2(\beta-\Delta/2)\alpha - \alpha^2\right]\right).
\]

Using equations (23) and (24) permits \(E_{01}\) to be written as

\[
E_{01} = \frac{1}{k^2 T} \int_{\beta-\Delta/2}^{\beta+\Delta/2} \exp\left(\frac{\pi k}{2} \left[2(\beta+\Delta/2)\alpha - \alpha^2\right]\right) d\alpha
\]  

\[
- \frac{1}{k^2 T} \int_{\beta-\Delta/2}^{\beta+\Delta/2} \exp\left(\frac{\pi k}{2} \left[2(\beta-\Delta/2)\alpha - \alpha^2\right]\right) d\alpha.
\]
The next step in the analysis requires completing the square for each of the terms in brackets in equation (25). For the first term, with a minus sign placed in front of the exponent, the following expression used

\[-2(\beta+\Delta/2) + \alpha^2 = (\beta+\Delta/2)^2 - 2(\beta+\Delta/2) + \alpha^2 - (\beta+\Delta/2)^2\]

\[= \left[\alpha - (\beta+\Delta/2)\right]^2 - (\beta+\Delta/2)^2 \quad (26)\]

In a similar way, the following expression is obtained

\[\alpha^2 - 2(\beta+\Delta/2) = \left[\alpha - (\beta-\Delta/2)\right]^2 - (\beta-\Delta/2)^2 \quad (27)\]

The expressions derived in equations (26) and (27) should be substituted in equation (25) to yield

\[E_{ol} = \frac{1}{kT} \int_{\beta-\Delta/2}^{\beta+\Delta/2} \exp\left[-\frac{\pi^2}{k} \left(\alpha - [\beta+\Delta/2]\right)^2\right] \exp\left[\frac{\pi^2}{k} (\beta+\Delta/2)^2\right] d\alpha\]

\[-\int_{\beta-\Delta/2}^{\beta+\Delta/2} \exp\left[-\frac{\pi^2}{k} \left(\alpha - [\beta-\Delta/2]\right)^2\right] \exp\left[\frac{\pi^2}{k} (\beta-\Delta/2)^2\right] d\alpha\]

\[= \quad (28)\]
For ease of handling, some appropriate substitutions will be made. First let,

\[ \theta' = \sqrt{\frac{2}{k}} \alpha - [\beta + A/2] \]  

and then, let \( n = \beta + A/2 \). Now

\[ \theta' = \sqrt{\frac{2}{k}} (\alpha - n) \]  

Now, the exponents, for the first term in equation (28) are given by

\[ - \frac{\pi^1 \theta'^2}{2} \]  

and

\[ \frac{\pi^1}{k} n^2 \]  

Substituting these into the first term yields

\[ \int_{-\sqrt{\frac{2}{k}} \Delta}^{0} \exp\left(- \frac{\pi^1}{k} \theta'^2 \right) \exp\left(\frac{\pi^1}{k} n^2 \right) d\theta' \]  

(33)
The upper and lower limits for the integral were determined in the following manner.

\[ \alpha_1 = \beta + \Delta/2 \]  \hspace{1cm} (34)

and

\[ \alpha_2 = \beta - \Delta/2 \]  \hspace{1cm} (35)

but

\[ \theta_1' = \sqrt{\frac{\rho}{k}} \left( \alpha_1 - [\beta + \Delta/2] \right) = 0 \]  \hspace{1cm} (36)

and

\[ \theta_2' = \sqrt{\frac{\rho}{k}} \left( \alpha_2 - [\beta + \Delta/2] \right) = \sqrt{\frac{\rho}{k}} (-\Delta) = -\sqrt{\frac{\rho}{k}} \Delta \]  \hspace{1cm} (37)

And the constant in front of the integral and the variable of integration were obtained by use of equation (29); i.e., differentiate equation (29)

\[ d\theta' = \sqrt{\frac{\rho}{k}} \, d\alpha, \]  \hspace{1cm} (38)

or

\[ d\alpha = \sqrt{k/\rho} \, d\theta'. \]  \hspace{1cm} (39)
Equations (36), (37), (38), and (39) were substituted into the first integral of equation (28) to yield equation (33). For the second integral in equation (28), the following substitutions will be made. First let,

$$\sigma = \beta - \Delta/2$$  \hspace{1cm} (40)

then let,

$$\phi = \sqrt{k} (\alpha - \sigma)$$  \hspace{1cm} (41)

Differentiating (41) yields

$$d\phi = \sqrt{k} \alpha$$  \hspace{1cm} (42)

The upper and lower limits can be found as before,

$$\phi_1 = \sqrt{k} (\alpha_1 - \sigma) = \sqrt{k} (\Delta)$$  \hspace{1cm} (43)

and

$$\phi_2 = \sqrt{k} (\alpha_2 - \sigma) = 0.$$
Now, the exponents of the second integral in equation (28) can be written as

\[ \frac{\pi i \phi^2}{2} \]  \hspace{1cm} (44)

and

\[ \frac{\pi i \sigma^2}{k} \] \hspace{1cm} (45)

Substituting, equations (40) through (45) into the second integral in equation (28) yields

\[ -\frac{\sqrt{k/2}}{k^2 T} \int_0^{\frac{\sqrt{2}}{k} \Delta} \exp\left(-\frac{\pi i}{k} \sigma^2\right) \exp\left(\frac{\pi i \phi^2}{2}\right) d\phi \] \hspace{1cm} (46)

Before combining equations (33) and (46), one more substitution is necessary that one will be,

\[ \theta = -\theta' \] \hspace{1cm} (47)

then

\[ \theta_1 = \sqrt{\frac{2}{k}} \Delta \] \hspace{1cm} (48)
and

$$\theta_2 = 0 \quad (49)$$

and

$$d\theta = -d\theta' \quad (50)$$

Substituting equations (47) through (50) into equation (33) yields

$$- \frac{\sqrt{k/2}}{k^2T} \int_{\sqrt{2k} \Delta}^{0} \exp \left( \frac{\pi n^2}{k} \right) \exp \left( - \frac{\pi}{2} \theta^2 \right) d\theta \quad (51)$$

The minus sign can be eliminated by interchanging the upper and lower limits in equation (51), i.e.,

$$\frac{\sqrt{k/2}}{k^2T} \int_{0}^{\sqrt{2k} \Delta} \exp \left( \frac{\pi n^2}{k} \right) \exp \left( - \frac{\pi}{2} \theta^2 \right) d\theta \quad (52)$$

The next step will be to combine equations (46) and (52).
\[ E_{01} = \frac{1}{\sqrt{2k^3T^2}} \left[ \int_0^{\theta_1} \exp\left(\frac{\pi}{k} n^2\right)\exp\left(-\frac{\pi}{k} \theta^2\right)d\theta \right. \]

\[ \left. - \int_0^{\phi_1} \exp\left(\frac{\pi}{k} \sigma^2\right)\exp\left(-\frac{\pi}{k} \phi^2\right)d\phi \right] \]

(53)

where \( \theta_1 = \phi_1 \). Also,

\[ \frac{\sqrt{k/2}}{k^2T} = \frac{\sqrt{k/2}}{k^4\Delta^2} = \frac{1}{\sqrt{2k^3T^2}} \]

(54)

Also,

\[ \sqrt{\frac{2}{k}} \Delta = \sqrt{\frac{2}{\Delta/T}} \Delta = \sqrt{\frac{2\Delta}{\Delta}} \Delta = \sqrt{2\Delta^2} = \sqrt{2D} = \theta_1 \]

(55)

The reason for these substitutions may not appear obvious, but, if one refers to the definition of the Fresnel Integral the reason should become obvious.

Equation (53) can also be written as

\[ E_{01} = \frac{\pi}{k} \frac{n^2}{\sqrt{2k} \Delta} \left[ \int_0^{\theta_1} e^{-\frac{\pi}{2} \theta^2} d\theta - e^{\frac{\pi}{2} \sigma^2} \right] \]

(56)
The two integrals in equation (56) are identical. Hence,

\[
E_{01} = \frac{1}{\sqrt{2\pi} \Delta} \left( \int_{0}^{\theta_1} e^{-\frac{\pi i}{2} \theta^2} d\theta \right) \left[ e^{\frac{\pi i}{k} n^2} - e^{\frac{\pi i}{k} \sigma^2} \right]
\]  

(57)

One can recognize that the integral expression in equation (57) is in the form of the Fresnel Integral and from the definition of the Fresnel Integral

\[
\int_{0}^{\theta_1} e^{-\frac{\pi i}{2} \theta^2} d\theta = Z^*(\theta_1) = Z^*(\sqrt{2D})
\]

(58)

More important than the beautiful form is the fact that \(Z(\sqrt{2D})\) is a constant independent of frequency. This fact will permit us to reduce equation (57) to one that permits its inverse Fourier Transform to be taken quite easily. Let us continue, with this idea in mind, and rewrite equation (57) as follows,

\[
E_{01} = \frac{Z^*(\sqrt{2D})}{\Delta \sqrt{2\pi}} \left[ e^{\frac{\pi i}{k} n^2} - e^{\frac{\pi i}{k} \sigma^2} \right]
\]

(59)

Now,

\[
n = \beta + \Delta/2 = f - f_o + \Delta/2
\]

(60)
where use has been made of some previous definitions. This has been done here to illustrate the dependence of $E_{ol}$ on frequency. Now, consider the expression,

$$\frac{\pi^2}{k} n^2$$

and let

$$n = -\frac{i}{k}.$$  \hspace{1cm} (62)

Then,

$$e^{\frac{\pi^2}{k} n^2} = e^{-\pi n^2}.$$ \hspace{1cm} (63)

The inverse Fourier Transform of equation (63) will now be sought. According to Campbell and Foster,

$$F^{-1}\left[e^{-\pi n^2}\right] = \frac{1}{\eta^{1/2}} \exp\left(-\frac{\pi t}{\eta}\right).$$ \hspace{1cm} (64)

Substituting equation (62) into equation (64) yields
The only item that remains is to determine how $e_0'(t)$ is affected by the results of equation (60). To determine the effect, consider the following

$$F^{-1}\left[F(f - u)\right] = e^{2\pi iut}f(t).$$  \hspace{1cm} (66)

Now, let use be made of equation (66) by making

$$f - u = f - f_0 + \Delta/2 = f - [f_0 - \Delta/2].$$  \hspace{1cm} (67)

Therefore,

$$u = f_0 - \Delta/2.$$  \hspace{1cm} (68)
Making the appropriate substitution in equation (66) produces

$$F^{-1} \left[ F \left( f - \left[ f_0 - \Delta/2 \right] \right) \right] = e^{2\pi i \left( f_0 - \Delta/2 \right) t} f(t). \quad (69)$$

The results of equation (69) can be applied to equation (65) if we rewrite equation (65) as follows.

$$F^{-1} \left[ \exp \left( -\pi \eta^2 \right) \right]^2 = F^{-1} \left[ \exp \left( -\pi \eta \left[ f - (f_0 - \Delta/2) \right] \right) \right]^2 \quad (70)$$

Then, by comparing with equation (66)

$$F(f) = e^{-\pi \eta f^2} \quad (71)$$

and

$$F(f-u) = e^{-\pi \eta (f-u)^2} \quad (72)$$

where $u = f_0 - \Delta/2$. Continuing in this fashion gives

$$F^{-1}[F(f-u)] = F^{-1} \left[ \exp \left( -\pi \eta [f-u]^2 \right) \right] = \sqrt{1k} \exp(2\pi i ut) \exp(-\pi k t)$$

$$= \sqrt{1k} \exp \left( 2\pi i \left[ f_0 - \Delta/2 \right] t \right) \exp(-\pi k t^2)$$

$$= \sqrt{1k} \exp \left( 2\pi i \left[ (f_0 - \Delta/2) t - kt^2/2 \right] \right) \quad (73)$$
In a similar way, the inverse Fourier Transform of the second term in equation (59) will be determined. First,

$$\sigma = \beta - \Delta/2 = f - f_0 - \Delta/2 = f - [f_0 + \Delta/2].$$ (74)

Comparing this with equation (60) indicates that the only difference between them is the form of the term $u$; i.e.,

$$u = f_0 + \Delta/2$$ (75)

for this term. In any event, the form of the resulting equations is unaltered. Therefore, in an analogous fashion, one can write, for the second term in equation (59), the following inverse Fourier Transform.

$$F^{-1}\left[\exp\left(\frac{\pi i}{k} \sigma^2\right)\right] = \sqrt{\frac{1}{k}} \exp\left(2\pi i (f_0 + \Delta/2)t - kt^2/2\right).$$ (76)

Having evaluated the inverse Fourier Transform of the components of equation (59), the complete inverse Fourier Transform can be written by inspection.

$$F^{-1}[E_{01}] = \frac{Z^2(\sqrt{2D})}{\Delta \sqrt{2k}} \left[\sqrt{\frac{1}{k}} \left(\exp\left[2\pi i \left(f_0 - \Delta/2\right)t - kt^2/2\right]\right) - \exp\left[2\pi i \left(f_0 + \Delta/2\right)t - kt^2/2\right]\right].$$

$$= \varepsilon'_{01}(t)$$ (77)
Thus can be simplified by combining like expression in the two exponentials. This simplification yields,

\[ \varepsilon'_{o1}(t) = \frac{Z^*(\sqrt{2D})}{\Delta \sqrt{-21}} \exp\left(2\pi i \left[ f_0 t - kt^2 / 2 \right]\right) \left[ \exp(-\pi i A) t - \exp(\pi i A) t \right] \]

\[(78)\]

where

\[ \frac{\sqrt{1k}}{\sqrt{2k}} \]

\[(79)\]

were combined to give

\[ \frac{1}{\sqrt{-21}} \]

\[(80)\]

The next simplification can be obtained if the expression for \( \sin x \) is considered; i.e.,

\[ \sin x = \frac{e^{ix} - e^{-ix}}{2i} \]

\[(81)\]

The two exponentials inside the brackets can be combined to yield
\[ \epsilon'_0(t) = \frac{Z^*(\sqrt{2D})}{\Delta} \exp \left( 2\pi i \left[ f_0 t - k t^2 / 2 \right] \right) \left[ -2i \left( \exp(+\pi \Delta) - \exp(-\pi \Delta) \right) \right] \]

\[ = \frac{(-2i)}{\sqrt{-21}} \frac{Z^*(\sqrt{2D})}{\Delta} \exp \left( 2\pi i \left[ f_0 t - k t^2 / 2 \right] \right) \sin \pi \Delta t \]

\[ = \sqrt{-21} \frac{\pi Z^*(\sqrt{2D})}{\Delta} \exp \left( 2\pi i \left[ f_0 t - k t^2 / 2 \right] \right) \frac{\sin \pi \Delta t}{\pi \Delta} \quad (82) \]

This can be further simplified by realizing that

\[ \sqrt{-1} = \sqrt{-\frac{\pi}{2}} i = \left( e^{-\frac{\pi i}{2}} \right)^{1/2} = e^{-\frac{\pi i}{4}} = \cos(-\pi/4) + i \sin(-\pi/4) \]

\[ = \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} . \quad (83) \]

Substituting the exponential form of \( \sqrt{-1} \) into equation (82) gives

\[ \epsilon'_0(t) = \sqrt{2} \frac{\pi Z^*(\sqrt{2D})}{\Delta} \exp \left( 2\pi i \left[ f_0 t - k t^2 / 2 - \frac{1}{8} \right] \right) \frac{\sin \pi \Delta t}{\pi \Delta} . \quad (84) \]

Refering to equation (20) reveals that \( \epsilon'_0(t) \) is actually one component of the inverse Fourier Transform of \( \frac{\partial E_d(t)}{\partial \beta} \). This implies that the actual inverse Fourier Transform of this component will have to be found using the results of equation (16); i.e.,
Using equations (84) and (85) permits the desired inverse Fourier Transform to be obtained. Hence,

\[ \varepsilon_{01}(t) = \frac{1}{2\pi t} F^{-1}\left[ E_{01} \right] = \frac{1}{2\pi t} \varepsilon'_1(t) \]  

(85)

Simplifying (86) by combination of a few terms gives

\[ \varepsilon_{01}(t) = \frac{1}{2\pi} \sqrt{2} \pi Z^* (\sqrt{2D}) \exp\left(2\pi i \left[ f_0 t - k t^2/2 - \frac{1}{8} \right]\right) \frac{\sin \frac{\pi \Delta t}{\pi \Delta}}{\pi \Delta} \]  

(86)

As before, the term "i" can be eliminated by expressing "im" as an exponential.

\[ i = e^{\pi i/2} \]  

(88)

Substituting this into (87) gives

\[ \varepsilon_{01}(t) = \frac{Z^* (\sqrt{2D})}{\sqrt{2}} \exp\left(2\pi i \left[ f_0 t - k t^2/2 + 1/8 \right]\right) \frac{\sin \frac{\pi \Delta t}{\pi \Delta}}{\pi \Delta} \]  

(89)
At this point, using the expression

\[
\text{sinc } \Delta t = \frac{\sin \pi \Delta t}{\pi \Delta t} \tag{90}
\]

It is possible to rewrite equation (89) to read as follows

\[
e_{01}(t) = \frac{Z^*(\sqrt{2D})}{\sqrt{2}} \exp\left(2\pi i\left[\frac{f_0 t - kt^2}{2} + \frac{1}{8}\right]\right) \text{sinc } \Delta t
\]  

\(\tag{91}\)

The signal given by equation (91) is still in complex form. In order to see the actual form of this component of the de-chirp filter output, it is necessary to take the real part of equation (91). Thus,

\[
e_{01}(t) = \text{Re}\left[e_{01}(t)\right] = \text{Re}\left[\frac{Z^*(\sqrt{2D})}{\sqrt{2}} \exp\left(2\pi i\left[\frac{f_0 t - kt^2}{2} + \frac{1}{8}\right]\right) \text{sinc } \Delta t\right]
\]  

\(\tag{92}\)

Since \(Z^*(\sqrt{2D})\) is a complex quantity, let it be written as

\[
Z^*(\sqrt{2D}) = Ae^{-ic} \tag{93}
\]

where \(A\) and \(c\) are real constants and depend only on the upper limit of \(Z^*(\sqrt{2D})\). That is, they depend only on \(D\), which is a constant for a particular system. This allows equation (92) to be written as follows,
where \( a = \frac{1}{8} - c \) and is a constant. This equation will be used in the final summation of terms because of its simplicity. It is interesting to consider a comparison between equation (92) and the equation in figure 5, which is reproduced here for convenience.

\[
e_{o1}(t) = \frac{A}{\sqrt{2}} \text{sinc} \Delta t \cos 2\pi \left[ f_o t - kt^2/2 + a \right] \quad (94)
\]

These two equations have exactly the same form. The only differences lie in the amplitude and the phase angle of the cosine term. Equation (B) of figure 5 was the de-chirp filter response to a chirp waveform that was "actively generated" at the transmitter, whereas equation (94) is only part of the de-chirp filter response to the gated waveform. However, the similarities lead us to some conclusions. First, the remaining terms, that are to be presently evaluated, will distort the response and cause a new system to be necessary for demodulation. Second, it would seem reasonable that if these two terms could be eliminated with one or two pieces of additional equipment, then the previous chirp radar systems could still be used and a tremendous saving could result. However, this previous suggestion will depend upon the form of these two terms.
Returning to equation (16)

\[ E_{o2} = \frac{1}{k} \frac{\sin \pi T(\beta + \Delta/2)}{\pi T(\beta + \Delta/2)} \exp \left( \frac{\pi i}{k} \left[ \beta^2 - \frac{\Delta^2}{4} \right] \right) \]  \quad (95)

Evaluation of the inverse Fourier Transform will start by making the following substitutions. Let

\[ y = \beta + \frac{\Delta}{2} \]  \quad (96)

or

\[ \beta = y - \frac{\Delta}{2} \]  \quad (97)

Squaring equation (97) gives

\[ \beta^2 = y^2 - \Delta y + \frac{\Delta^2}{4} \]  \quad (98)

Now, substitute equations (96), (97), and (98) into equation (95). This yields

\[ E_{o2} = \frac{1}{k} \frac{\sin \pi Ty}{\pi Ty} \exp \left( \frac{\pi i}{k} \left[ y^2 - \Delta y + \frac{\Delta^2}{4} - \frac{\Delta^2}{4} \right] \right) \]

\[ = \frac{1}{k} \frac{\sin \pi Ty}{\pi Ty} \exp \left( \frac{\pi i}{k} \left[ y^2 - \Delta y \right] \right). \]  \quad (99)
This equation can be restated if the sine term is expanded in exponential form. If this is done, equation (99) appears as

\[ E_{02} = \frac{1}{k} \left( \frac{1}{\pi Ty} \left[ \exp \left( \frac{i \pi Ty}{k} \right) - \exp \left( \frac{-i \pi Ty}{k} \right) \right] \right) \exp \left( \frac{\pi i}{k} \left[ y^2 - \Delta y \right] \right) \]

(100)

Combining exponential terms gives

\[ E_{02} = \frac{1}{k} \left[ \exp \left( \frac{\pi i}{k} \left[ y^2 - \Delta y + k Ty \right] \right) - \exp \left( \frac{\pi i}{k} \left[ y^2 - \Delta y - k Ty \right] \right) \right] \frac{1}{2\pi Ty} \]

(101)

Noting that \( k = \Delta/T \) and using this in the two exponents of equation (101) simplifies the exponents as shown below. For the first exponent,

\[ y^2 - \Delta y + k Ty = y^2 - \Delta y + \Delta/T Ty \]

(102)

\[ = y^2 - \Delta y + \Delta y = y^2 \]

and for the second

\[ y^2 - \Delta y - k Ty = y^2 - \Delta y - \Delta/T Ty \]

(103)

\[ = y^2 - \Delta y - \Delta y = y^2 - 2\Delta y. \]
Putting the results of equations (102) and (103) into equation (101) yields

$$E_{o2} = \frac{\exp\left(\frac{\pi i}{k} \left[ y^2 \right]\right) - \exp\left(\frac{\pi i}{k} \left[ y^2 - 2\Delta y \right]\right)}{2\pi i k T y}.$$  \hspace{1cm} (104)

As before these substitutions were done in order to put $E_{o2}$ into a form which lends itself toward being inversly transformed. Once again, after searching various texts, a similar form has been obtained. Consider the first term on the right-hand side of equation (104), the inverse Fourier Transform is given by

$$\mathcal{F}^{-1}\left[ e^{\frac{\pi i}{k} \frac{y^2}{kT\rho}} \right] = \frac{1}{kT} \frac{1}{\sqrt{-1/k}} \int_{-\infty}^{t} e^{-i\pi k t^2} dt = \mathcal{E}''_{o2}(t) \hspace{1cm} (105)$$

where $\rho = 2\pi iy$.

The form of equation (95) does not appear to be readily useful. However, there are similar terms yet to be evaluated and judgment should be reserved until then. It should also be noted that the integral is in a Fresnel integral form so at worst the integral will be available in tabulated form.

Returning to equation (104), the inverse Fourier Transform of the second term will now be calculated. The second term can be written in the following form by completing the square of the exponent.
The second term was rewritten in the form because it lends itself to the same form as equation (105). However, some additional changes will have to be made in order to permit equation (105) to be used. First of all it should be noted that the term \( \exp\left(\frac{\pi i}{k} \Delta^2\right) \) is independent of frequency. Therefore, it will be treated as a constant while performing the inverse Fourier Transform. The change will be in the form of a substitution, so as to permit equation (106) to appear as equation (105).

Therefore, let

\[ Z = y - \Delta. \quad (107) \]

Then making the appropriate substitution in equation (106) yields

\[
\frac{\exp\left(\frac{\pi i}{k} \left[ y - \Delta \right]^2\right) \exp\left(-\frac{\pi i}{k} \Delta^2\right)}{2\pi i k T y} = \frac{\exp\left(\frac{\pi i}{k} Z^2\right) \exp(\pi i \Delta T)}{2\pi i (z+\Delta) k T}
\]

\[ (108) \]
where the substitution $k = \Delta/T$ was made in the second exponential term. To make equation (108) appear more useful rewrite as follows.

$$\exp\left(\frac{\pi i}{k} Z^2\right)\exp(-\pi i\Delta T) = \exp\left(\frac{\pi i}{k} Z^2\right)\exp(-\pi i\Delta T)$$

$$\frac{2\pi i(z+\Delta)kT}{(\rho-\rho_0)kT}$$

(109)

where $\rho = 2\pi iz$ and $\rho_0 = -2\pi i\Delta$

Although this is not the same as equation (105), it does lend itself to a standard form available in the tables. The inverse Fourier Transform is given by the following expression

$$F^{-1}\left[\frac{\exp(-\pi i\Delta T)\exp\left(\frac{\pi i}{k} Z^2\right)}{kT(\rho-\rho_0)}\right] = \frac{\exp(-\pi i\Delta T)}{kT \sqrt{-1/k}} \int_{-\infty}^{t} e^{+2\pi i\Delta \tau} e^{-1\pi k\tau^2} d\tau$$

$$= \varepsilon_{02}''(t)$$

(110)

The variable of integration, $\tau$, was used so that the terms inside the integral would not be confused with the variable $t$ outside the integral.
Some additional things can still be done with equation (110) to simplify the picture. Consider the three exponential terms

$$\exp(-\pi i \Delta T) \exp(2\pi i \tau) \exp(-\pi k \tau^2).$$  \(\text{(111)}\)

These can be combined in the following fashion

$$\exp(-\pi i \Delta T) \exp(2\pi i \tau) \exp(-\pi k \tau^2) = \exp\left(-\pi k i \left[\tau^2 - 2\tau T + T^2\right]\right),$$  \(\text{(112)}\)

Where use has been made of the fact that \( k = \Delta / T \). Now, the exponent in equation (112) is a perfect square; i.e.,

$$\exp(-\pi k i [\tau - T]^2).$$  \(\text{(113)}\)

This allows equation (110) to be written as

$$\varepsilon_{02}(t) = \frac{\exp(-2\pi i \Delta t)}{kT \sqrt{-i/k}} \int_{-\infty}^{t} \exp(-\pi k i [\tau - T]^2) d\tau$$  \(\text{(114)}\)

By substituting
\[ \tau = \tau - T \]  

(115)

and

\[ d\tau = d\tau \]  

(116)

and

\[ \tau_1 = -\infty \]  

(117)

and

\[ \tau_2 = t - T \]  

(118)

into equation (114) will give

\[ \varepsilon_{02}(t) = \frac{\exp(-2\pi i\Delta t)}{kT \sqrt{-1/k}} \int_{-\infty}^{t-T} \exp(-\pi k\tau^2) d\tau \]  

(119)

Before we can complete the inverse Fourier Transform, some additional factors must be considered. First, by definition

\[ y = \beta + \Delta/2 = f - f_o + \Delta/2. \]  

(120)
Second, again by definition

\[ z = y - \Delta = \beta + \Delta/2 - \Delta = \beta - \Delta/2 = f - f_o - \Delta/2 \quad (121) \]

These two equations require the alteration of equations (105) and (119). However, the alteration is not particularly difficult one to obtain. In fact a similar alteration was done to \( \epsilon_0'(t) \). The basis for the change can be seen in equation (66). To make the terms more compatible, let

\[ y = f - f_o + \Delta/2 = f - [f_o - \Delta/2] \quad (122) \]

\[ = f - u \]

where \( u = f_o - \Delta/2 \).

Then, equation (105) can be rewritten as follows

\[ P^{-1} \left[ \frac{\pi i}{k} \frac{y^2}{kT\rho} \right] = P^{-1} \left[ \frac{\pi i}{k} \frac{(f-u)^2}{2\pi i kT(f-u)} \right] \quad (123) \]

which according to equation (66) can also be written as

\[ P^{-1} \left[ \frac{F(f-u)}{kT} \right] \quad (124) \]
where

$$F(f-u) = \frac{\exp \frac{\pi i}{k} (f-u)^2}{(P-u)}$$  \hspace{1cm} (125)

then,

$$F(f) = \frac{\exp \frac{\pi i}{k} f^2}{P}$$  \hspace{1cm} (126)

With these definition, equation (105) takes on the following form

$$\varepsilon''_o(t) = \exp \left( \frac{2\pi i[f_o-A/2]t}{kT} \right) \cdot \int_{-\infty}^{t} e^{-i\pi k\tau^2} d\tau$$  \hspace{1cm} (127)

where for simplicity $\tau$ has again been used as the variable of integration.

In a similar fashion, the same can be done with equation (119) with the aid of equation (121). Thus,

$$z = f - f_o - \Delta/2 = f - [f_o + \Delta/2]$$  \hspace{1cm} (128)

$$= f - u$$
where

\[ u = f_o + \Delta / 2 \]

Now, from equation (110)

\[
\mathcal{F}^{-1}\left[ \frac{\exp(-\pi i \Delta T)}{kT} \left( \frac{\exp\left(\frac{\pi i}{k} z^2\right)}{(p-p_0)} \right) \right] = \mathcal{F}^{-1}\left[ \frac{\exp(-\pi i \Delta T)}{kT} \left( \frac{\exp\left(\frac{\pi i}{k} [f-u]^2\right)}{(p-u-p_0)} \right) \right]
\]

which has the standard form

\[
\mathcal{F}^{-1} \frac{\exp(-\pi i \Delta T)}{kT} \mathcal{F}(f-u)
\]

where

\[
\mathcal{F}(f-u) = \frac{\exp\left(\frac{\pi i}{k} [f-u]^2\right)}{(p-u-p_0)}
\]

then

\[
\mathcal{F}(f) = \frac{\exp \frac{\pi i}{k} f^2}{(p-p_o)}
\]

Using the previous definitions permits equation (119) to take on the following form
This can be simplified by combining the two exponential terms in front of the integral. For these two terms

\[
\exp(2\pi i[f_0 + \Delta/2]t)\exp(-2\pi i\Delta t) = \exp(2\pi i[f_0 - \Delta/2]t).
\]

Replacing this expression in equation (133) yields

\[
\varepsilon_{o2}^m(t) = \frac{\exp(2\pi i[f_0 - \Delta/2]t)}{kT \sqrt{-1/k}} \int_{-\infty}^{t-T} \exp(-\pi kir^2)dr
\]

Since \(E_{o2}\) is comprised of two terms, its inverse Fourier Transform is given by the difference of equations (127) and (135). The results are,

\[
\varepsilon'_{o2}(t) = \frac{1}{kT} \frac{\exp(2\pi i[f_0 - \Delta/2]t)}{\sqrt{-1/k}} \left[ \int_{-\infty}^{t} e^{-\pi kir^2}dr - \int_{-\infty}^{t-T} e^{-\pi kir^2}dr \right]
\]

(136)
Equation (136) can still be simplified even further if one considers combining the two integrals. Each integral has the same integrands but different limits. The first integral extends from \((\infty - t_0 - t)\) and the second integral extends from \((\infty - t_0 - t - T)\); therefore, the second integral will subtract, from the first, the contribution from \((\infty - t_0 - t - T)\) and have only the contribution from \((t - T - t_0 - t)\). Thus, equation (136) will take on the following form,

\[
\epsilon_{o2}'(t) = \frac{\exp\left(2\pi i[f_0 - \Delta/2]t\right)}{kT \sqrt{-1/k}} \int_{t-T}^{t} \exp(-\pi ki\tau^2) d\tau \tag{137}
\]

Equation (20) indicates that \(\epsilon_{o2}'(t)\) is the second component that comprises the inverse Fourier Transform of \(\frac{\partial E_{d}(f)}{\partial f}\). As a result, the inverse Fourier Transform of this component of \(E_{d}(f)\) can be found by reference to equation (16), which is repeated for convenience

\[
\epsilon_o(t) = \frac{1}{2\pi t} F^{-1}\left[\frac{\partial E_{d}(f)}{\partial f}\right] \tag{16}
\]

Equation (137) is the term in brackets, i.e.,

\[
F^{-1}\left[\frac{\partial E_{d}(f)}{\partial f}\right] = \frac{\exp\left(2\pi i[f_0 - \Delta/2]t\right)}{kT \sqrt{-1/k}} \int_{t-T}^{t} \exp(-\pi ki\tau^2) d\tau \tag{138}
\]
Using equation (138) allows $\epsilon_{o2}(t)$ to be evaluated with the aid of equation (16). That is,

$$
\epsilon_{o2}(t) = \frac{1}{2\pi t} \exp\left(\frac{2\pi i[f_0 - \Delta/2]t}{kT}\right) \int_{t-T}^{t} \exp\left(-\pi ki^2\right) d\tau \quad (139)
$$

Now, it would seem reasonable that the terms in "i" should be combined in exponential form with the already existent exponential term. That is,

$$
\frac{1}{\sqrt{-1}} = \frac{e^{i\pi/2}}{\sqrt{e^{-i\pi/2}}} = \left(e^{i\pi/2}/e^{-i\pi/2}\right)^{1/2} = e^{i\pi/2}e^{i\pi/4} = e^{i3\pi/4} = e^{13\pi/4} \quad (140)
$$

Expression (140) can be placed in equation (139) and simplify it to read as follows,

$$
\epsilon_{o2}(t) = \exp\left(2\pi i\left([f_0 - \Delta/2]t + 3/8\right)\right) \int_{t-T}^{t} \exp\left(-\pi ki^2\right) d\tau \quad (141)
$$

One more simplification can still be made in equation (141) by combining the items $kT \sqrt{\frac{1}{k}}$. This yields
\[ kT \sqrt{\frac{T}{k}} = \sqrt{\frac{k^2 T^2}{k}} = \sqrt{kT^2} \]  \hspace{1cm} (142)

But, \( k = \Delta/T \). Thus,

\[ \sqrt{kT^2} = \sqrt{\frac{\Delta}{T} T^2} = \sqrt{\Delta T} = \sqrt{D} \]  \hspace{1cm} (143)

Using the results of equation (143) in equation (141) produces the following result,

\[ \varepsilon_{\infty}(t) = \exp \left( \frac{2\pi i \left[ (f_0 - \Delta/2) t + 3/8 \right]}{2\pi \sqrt{D} t} \right) \int_{t-T}^{t} \exp(-\pi\kappa t^2) \, d\tau \]  \hspace{1cm} (144)

A few items of interest can be seen in equation (144). First of all, the denominator has a term "t"; normally this would present a problem at \( t = 0 \), however, when the third term comprising the inverse Fourier Transform of \( E_{d}(f) \) is found a compensating term will become evident; therefore, discussion of this item will be left till that time. The next item of interest is the integral expression. This can be rewritten as follows,
\[ \int_{t-T}^{t} \exp(-\pi k_{t} \tau^{2}) d\tau = \int_{0}^{t} \exp(-\pi k_{t} t') d\tau - \int_{0}^{t-T} \exp(-\pi k_{t} t') d\tau \]

(145)

With the integral rewritten as so, it should be noted that each integral has the form of a Fresnel integral; i.e., equation (9) of Section 3 which is repeated here for convenience.

\[ Z(v) = \int_{0}^{v} e^{\frac{i \pi \alpha^{2}}{2}} d\alpha = C(v) + i S(v) \]

(9)

To apply equation (9) of Section 3 to equation (145) means that in the first term "v" is synonymous with "t" and "\( \alpha^{2}/2 \)" is synonymous with "\( k\tau^{2} \)" and also the complex conjugate of equation (9) of Section 3 must be taken. For the second term, "v" is synonymous with "\( t-T \)" and "\( \alpha^{2}/2 \)" is synonymous with "\( k\tau^{2} \)" and, as before, the complex conjugate must be taken because of the minus sign in the exponent. Thus, it can be concluded that all the items in equation (144) can be evaluated even if a computer is necessary for the evaluation.

The final point to be made concerning equation (144) stems from the fact that \( \epsilon_{o2}(t) \) is due to a complex input signal and, in order to determine the response to a real input signal, the real part of equation (144) must be taken. That is,
\[ e_{o2}(t) = \text{Re} \, \epsilon_{o2}(t) = \text{Re} \left[ \frac{\exp \left( 2\pi i \left[ (f_o - \Delta/2) t + 3/8 \right] \right)}{2\pi \sqrt{D} \, t} \int_{t-T}^{t} \exp(-\pi k \tau^2) d\tau \right] \]

\[ = \text{Re} \left[ \frac{\exp \left( 2\pi i \left[ (f_o - \Delta/2) t + 3/8 \right] \right)}{2\pi \sqrt{D} \, t} \right] \left( z^*(\sqrt{2k} t) - z^*(\sqrt{2k} [t-T]) \right) \left( \frac{1}{\sqrt{2k}} \right) \] (146)

where use has been made of the discussion above and equation (145);

In equation (145) let

\[ k\tau^2 = \frac{\alpha^2}{2} \] (147)

or

\[ \tau^2 = \frac{\alpha^2}{2k} \] (148)

or

\[ \tau = \frac{\alpha}{\sqrt{2k}} \] (149)
Then, by differentiating equation (149)

\[ \frac{d\tau}{d\alpha} = \frac{\alpha}{\sqrt{2k}} \]  

(150)

The lower limit for each integral on the left-hand side of equation (145) is still zero. However, the upper limit for the first integral changes to

\[ \alpha_2 = \sqrt{2k} \tau_2 = \sqrt{2k} t \]  

(151)

and, for the second integral, the upper limit becomes

\[ \alpha_2 = \sqrt{2k} \tau_2 = (t-T)\sqrt{2k} \]  

(152)

Using the above mentioned substitution, equation (145) becomes

\[
\int_{t-T}^{t} e^{-\pi kt^2} d\tau = \int_{0}^{t} e^{-\pi k\tau^2} d\tau - \int_{0}^{t-T} e^{-\pi k\tau^2} d\tau
\]

\[
= \frac{1}{\sqrt{2k}} \int_{0}^{\sqrt{2k} t} e^{-\pi \alpha^2/2} d\alpha
\]

\[
- \frac{1}{\sqrt{2k}} \int_{0}^{\sqrt{2k} (t-T)} e^{-\pi \alpha^2/2} d\alpha
\]  

(153)
Using the notation of equation (9) of Section 3, equation (154) takes on the following form,

\[
\int_{t-T}^{t} \exp(-\pi k \tau^2) d\tau = \frac{1}{\sqrt{2k}} \left[ z^*(\sqrt{2k} t) - z^*(\sqrt{2k} (t-T)) \right]
\]

(154)

In equation (146), the Fresnel integrals are noted to be functions of time. Also, the results of actually taking the real part of equation (146) will put off until the final term, corresponding to \( E_d(f) \), has been evaluated.

Thus, the only remaining task to be completed is the evaluation of \( E_{o3} \); i.e.,

\[
E_{o3} = -\frac{1}{k} \sin \frac{\pi T(\beta-\Delta/2)}{\pi T(\beta-\Delta/2)} \exp \left( \frac{\pi i}{k} \left[ \beta^2 - \frac{\Delta^2}{4} \right] \right)
\]

(155)

Since \( E_{o3} \) and \( E_{o2} \) are very similar in form, the same method for evaluation of the inverse Fourier Transform will be followed here. Therefore, let

\[
x = \beta - \Delta/2
\]

(156)

then,

\[
\beta = x + \Delta/2
\]

(157)
Squaring equation (157) yields

$$\beta^2 = (x + \Delta/2)^2 = x^2 + \Delta x + \Delta^2/4$$  \hspace{1cm} (158)

Now, the exponent of equation (155) can be determined by evaluating

$$\beta^2 - \Delta^2/4 = x^2 + \Delta x + \Delta^2/4 - \Delta^2/4 = x^2 + \Delta x$$  \hspace{1cm} (159)

Substituting the results of equations (156) and (159) into equation (155) yields the following form,

$$E_{03} = -\frac{1}{k} \frac{\sin \pi T x}{\pi T x} \exp\left(\frac{\pi i}{k} \left[ x^2 + \Delta x \right] \right)$$  \hspace{1cm} (160)

The next step will be to expand the sinusoidal term in its exponential form, i.e.,

$$\frac{\sin \pi T x}{\pi T x} = \frac{e^{i\pi T x} - e^{-i\pi T x}}{2i\pi T x}$$  \hspace{1cm} (161)

This allows equation (160) to be put into the following form
\[
E_{03} = - \frac{1}{k} \exp\left(\frac{\pi i}{k} (x^2 + \Delta x)\right) \left[ \frac{\exp(i\pi Tx) - \exp(-i\pi Tx)}{2\pi ikTx} \right]
\]  

\[
= - \exp\left(\frac{\pi i}{k} [x^2 + \Delta x]\right) \left[ \frac{\exp(i\pi Tx) - \exp(-i\pi Tx)}{2\pi ikTx} \right]
\]

(162)

It is obvious that the next step is to combine the exponential terms. Therefore,

\[
E_{03} = - \left[ \frac{\exp\left(\frac{\pi i}{k} \left[x^2 + \Delta x + kTx\right]\right) - \exp\left(\frac{\pi i}{k} \left[x^2 + \Delta x - kTx\right]\right)}{2\pi ikTx} \right]
\]

(163)

where the exponential terms inside the brackets of equation (162) were rewritten as

\[
\exp(i\pi Tx) = \exp\left(\frac{\pi i}{k} kTx\right)
\]

(164)

and

\[
\exp(-i\pi Tx) = \exp\left(\frac{-\pi i}{k} kTx\right)
\]

(165)

before they were combined into equation (163). Now, equation (163) can be simplified even further if kT is replaced by \(\Delta\). Using this equivalence in equation (163) permits it to be written in the following way,
The expression for $E_{o3}$ is in a form which allows its inverse Fourier Transform to be taken with just a few more manipulations. Toward this end, let us operate on the first term of $E_{o3}$ in equation (166) by completing the square of the exponent of the exponential term. That is, by adding a $\Delta^2$ and subtracting a $\Delta^2$ from the exponent it is possible to complete the square, i.e.,

$$x^2 + 2\Delta x = x^2 + 2\Delta x + \Delta^2 - \Delta^2$$

$$= (x^2 + 2\Delta x + \Delta^2) - \Delta^2 \quad (167)$$

$$= (x + \Delta)^2 - \Delta^2$$

Substituting the results of equation (167) into the exponent of the first term in equation (167) allows the first term to take on the following form,
\[
\exp\left(\frac{\pi i}{k} \left[ x^2 + 2Ax \right]\right) = \exp\left(\frac{\pi i}{k} \left[ x^2 + 2Ax + \Delta^2 - \Delta^2 \right]\right)
\]

\[
= \frac{\exp\left(\frac{\pi i}{k} [x+\Delta]^2 - \Delta^2\right)}{2\pi i\Delta x}
\]

Note that the second exponential term of equation (168); i.e., \(\exp\left(\frac{\pi i}{k} \Delta^2\right)\) is independent of frequency and will be carried along as a constant as the inverse Fourier Transform is performed. The first exponential term is a function of frequency and its inverse Fourier Transform will now be found. First some substitution will be made in order to facilitate the work. Therefore, let

\[
\omega = x + \Delta \quad \text{(169)}
\]

or

\[
\omega - \Delta = x \quad \text{(170)}
\]

Substituting this into equation (168) yields
\[
\exp \left( \frac{\pi^2}{k} [x+\Delta]^2 \right) \exp \left( -\frac{\pi \Delta^2}{k} \right) = \frac{\exp \left( -\frac{\pi \Delta^2}{k} \right) \exp \left( \frac{\pi}{k} \omega^2 \right)}{2\pi i \Delta (\omega - \Delta)}
\]

(171)

\[
= \frac{\exp \left( -\frac{\pi \Delta^2}{k} \right) \exp \left( \frac{\pi}{k} \omega^2 \right)}{\Delta (\rho - \rho_o)}
\]

where the expressions for \( \rho \) and \( \rho_o \) in equation (171) are

\[
\rho = 2\pi i \omega 
\]

(172)

and

\[
\rho_o = 2\pi i \Delta 
\]

(173)

As written, equation (171) is very similar to equation (110) and its inverse Fourier Transform will be given by a very similar expression. Therefore, using the ideas from equation (110), it is possible to write down the inverse Fourier Transform of equation (171) as,

\[
p^{-1} \left[ \exp \left( -\frac{\pi \Delta^2}{k} \right) \exp \left( \frac{\pi}{k} \omega^2 \right) \right] = \frac{\exp \left( -\frac{\pi \Delta^2}{k} \right)}{\Delta \sqrt{-1/k}} e^{-2\pi i \Delta t} \int_{-\infty}^{t} e^{-2\pi i \Delta \tau} e^{-i\pi k \tau^2} d\tau
\]

\[
= \varepsilon''_0(t) 
\]

(174)
The variable of integration used is $t$ in order to avoid any confusion between the terms outside and inside the integral. In order to be able to rewrite equation (174) in a more useful form consider the following three terms,

$$\exp\left(-\frac{\pi \Delta^2}{k}\right)\exp(-2\pi i\Delta t)\exp(-i\pi k t^2) \quad (175)$$

These three terms are under consideration because it appears that these terms might form a perfect square in the exponents. To see this, consider the first exponential term,

$$\exp\left(-\frac{\pi \Delta^2}{k}\right) = \exp\left(-\frac{\pi (kT)^2}{k}\right) = \exp(-i\pi k T^2) \quad (176)$$

where the substitution $\Delta = kT$ was made. Now, it is apparent that the next term to consider is

$$\exp(-2\pi i\Delta t) = \exp(-2\pi i k T t). \quad (177)$$

Again $\Delta = kT$ was used.

All three exponents have the common factor of $\pi ik$ which can be factored to give
\[
\exp(-\pi k T^2) \exp(-2\pi i k T \tau) \exp(-i\pi k \tau^2) = \exp(-\pi k i [\tau^2 + 2T \tau + T^2])
\]
\[
= \exp(-\pi k i [\tau + T]^2)
\]
(178)

The results of equation (178) permit the following simplification of equation (174) to be made

\[
\varepsilon''_0(t) = \frac{\exp(-2\pi i \Delta t)}{\Delta \sqrt{-i/k}} \int_{-\infty}^{t} \exp(-\pi k i [\tau + T]^2) d\tau
\]
(179)

Still further simplifications can be made to the integral if the following substitutions are made,

\[
\tau = \tau + T
\]
(180)

then by taking the derivative of equation (180) the following equation results,

\[
d\tau = d\tau
\]
(181)

As far as the limits are concerned, the following are seen to follow. For the lower limit
and for the upper limit

\[ \tau_2 = t + T \]  \hspace{1cm} (183)

Making these substitutions in equation (179) produces the following result,

\[ \varepsilon''_0(t) = \frac{\exp(-2\pi i \Delta t)}{\Delta \sqrt{-1/\kappa}} \int_{-\infty}^{t+T} \exp(-\pi k_1 t^2) dt \] \hspace{1cm} (184)

Referring to equation (174) indicates that the frequency variable used in the inverse Fourier Transform was \( \omega \). Therefore, in order to complete the inverse Fourier Transform of equation (184), it will necessary to see how the frequency variable depends upon \( f \). First, from equation (169),

\[ \omega = x + \Delta. \] \hspace{1cm} (185)

Then, using equation (156), where \( x \) was defined, and substituting this equation (185) gives,

\[ \omega = \beta - \Delta/2 + \Delta \] \hspace{1cm} (186)
the only remaining quantity to be evaluated in terms of  \( f \) is  \( \beta \). 

And from equation (7)

\[
\omega = f - f_o + \Delta/2 \quad (187)
\]

\[
= f - [f_o - \Delta/2].
\]

Having evaluated the dependence of  \( \omega \) are  \( f \), there remains only the task of using these ideas to change the results of equation (184). To do this, reference is once again made to equation (66) which is repeated here for convenience.

\[
F^{-1}[F(f - u)] = e^{2\pi iuf(t)}. \quad (66)
\]

For equation (66) to be useful it will be necessary to rewrite equation (171) in a form similar to equation (66), therefore,

\[
\exp\left(-\frac{\pi i \Delta^2}{k}\right)\exp\left(\frac{\pi i}{k} u^2\right) = \frac{\exp\left(-\frac{\pi i \Delta^2}{k}\right)\exp\left(\frac{\pi i}{k} [f-u]^2\right)}{\Delta [p-p_o] - \rho_o} \quad (188)
\]

where equation (187) was used in place of  \( \omega \) and

\[
u = f_o - \Delta/2 \quad (189)
\]
The expression $\rho_u$ was obtained in the following fashion. From equation (172),

$$\rho = 2\pi i \omega$$  \hspace{1cm} (190)

Therefore, using equation (174) yields

$$\rho = 2\pi i \omega = 2\pi i [f - (f_o - \Delta/2)]$$

$$= 2\pi i[f - u] = \rho - \rho_u$$  \hspace{1cm} (191)

where $\rho_u = 2\pi i u$

From equation (188) it becomes apparent that

$$\exp(-S^*) \exp(f^2)$$

The form of the inverse Fourier Transform of equation (192) is given by equation (174). All that remains is to now make use of equation (66). Hence, equation can be now be written as,

$$\varepsilon''_0(t) = \frac{\exp(2\pi i ut)\exp(-2\pi i \Delta t)}{\Delta \sqrt{-i/k}} \int_{-\infty}^{t+T} \exp(-\pi k i \tau^2) d\tau$$  \hspace{1cm} (193)
The first two exponentials outside the integral in equation (193) can be combined in the following way,

\[ \exp(2\pi i \omega t) \exp(-2\pi i \Delta t) = \exp(2\pi i [f_0 - \Delta/2] t) \exp(2\pi i \Delta t) \]

\[ = \exp(2\pi i [f_0 + \Delta/2] t). \quad (194) \]

This result should now be substituted into equation (193), i.e.,

\[ \varepsilon''(t) = \frac{\exp(2\pi i [f_0 + \Delta/2] t)}{\Delta \sqrt{-1/k}} \int_{-\infty}^{t+T} \exp(-\pi k \tau^2) d\tau \quad (195) \]

now that the inverse Fourier Transform for the first exponential has been completed, the next step will be to evaluate the inverse Fourier Transform for the second exponential term that comprises \( E_{03} \) in equation (166). That is, we are interested in the inverse Fourier Transform of

\[ \exp\left(\frac{\pi i}{k} x^2\right) \frac{1}{2\pi i \Delta x} \quad (196) \]

To aid this in this problem, let

\[ \rho = 2\pi i x \quad (197) \]
then, it is possible to rewrite equation (196) as follows,

\[
\frac{\exp \left( \frac{\pi x^2}{k} \right)}{2\pi i \Delta x} = \frac{\exp \left( \frac{\pi x^2}{k} \right)}{\Delta \rho}
\]  

(198)

It should be noted that equation (198) is in a form very similar to equation (105). Therefore, using the same basic idea the inverse Fourier Transform of equation (198) can be written as follows,

\[
F^{-1} \left[ \frac{\exp \left( \frac{\pi x^2}{k} \right)}{\Delta \rho} \right] = \frac{1}{\Delta \sqrt{-1/k}} \int_{-\infty}^{t} \exp \left( -\pi k \tau^2 \right) d\tau = \varepsilon^{\prime\prime}_{o3}(t) \]  

(199)

It is obvious that the frequency variable \( x \) used in equation (199) is not equal to \( f \). Therefore, it will be necessary to find the relation between \( x \) and \( f \) in order to determine the actual inverse Fourier Transform of \( \varepsilon^{\prime\prime}_{o3}(t) \). Referring to equation (156) indicates that

\[
x = \beta - \Delta/2
\]  

(200)

But from equation (7), it is possible to write

\[
x = f - f_o - \Delta/2
\]  

\[
= f - \left[ f_o + \Delta/2 \right].
\]
Now, let

\[ u = f_0 + \Delta/2 \]  \hspace{1cm} (202)

then,

\[ x = f - u. \]  \hspace{1cm} (203)

Now to make use of equation (16), substitute equation (203) into equation (198). This substitution yields the following result,

\[ \exp \left( \frac{\pi i x^2}{k} \right) = \frac{\exp \left( \frac{\pi i [f-u]^2}{k} \right)}{\Delta \rho \Delta (\rho - \rho u)} \]  \hspace{1cm} (204)

where

\[ \rho u = 2\pi i u. \]  \hspace{1cm} (205)

From equation (204), it is apparent that

\[ F(f) = \frac{\exp \left( \frac{\pi f^2}{k} \right)}{\Delta \rho} \]  \hspace{1cm} (206)
Now, utilizing the results of equation (66) permits the inverse Fourier Transform of equation (206) to take on the following form

\[
F^{-1}\left[ \frac{\exp\left(\frac{\pi i (f-u)^2}{k} \right)}{\Delta (\rho - \rho u)} \right] = \varepsilon_{03}(t) = \frac{\exp(2\pi i ut)}{\Delta \sqrt{-1/k}} \int_{-\infty}^{t} \exp(-\pi k \tau^2) d\tau .
\]

\[
= \frac{\exp(2\pi i[f_0 + \Delta/2]t)}{\Delta \sqrt{-1/k}} \int_{-\infty}^{t} \exp(-\pi k \tau^2) d\tau - \int_{-\infty}^{t+T} \exp(-\pi k \tau^2) d\tau
\]

Equation (207) represents the completion of the inverse Fourier Transform of the second term of \( E_{03} \). The inverse Fourier Transform of \( E_{03} \) can now be found using the difference of equations (195) and (207). Hence,

\[
F^{-1}[E_{03}] = - (\varepsilon''_{03}(t) - \varepsilon''_{03}(t)) = \varepsilon''_{03}(t) - \varepsilon''_{03}(t) = \varepsilon'_{03}(t)
\]

\[
= \frac{\exp(2\pi i[f_0 + \Delta/2]t)}{\Delta \sqrt{-1/k}} \int_{-\infty}^{t} \exp(-\pi k \tau^2) d\tau - \int_{-\infty}^{t+T} \exp(-\pi k \tau^2) d\tau
\]

(208)
where all the common exponential factors were placed in front of the integrals. The next step in the process will be to simplify equation (208) by combining the integrals because they have the same integrands. The difference between the two integrals be in the limits of integration. In order to combine the terms, consider rewriting the integrals in the following manner,

\[
\int_{-\infty}^{t} \exp(-\pi k \tau^2) \, d\tau - \int_{-\infty}^{t+T} \exp(-\pi k \tau^2) \, d\tau
\]

\[
= - \left[ \int_{-\infty}^{t+T} \exp(-\pi k \tau^2) \, d\tau - \int_{-\infty}^{t} \exp(-\pi k \tau^2) \, d\tau \right] \tag{209}
\]

With the integrals written as in equation (209) it's a simple matter to them because the first integrals extends from \((-\infty \rightarrow t_0 - t\)). Therefore, the result of the combination gives an integral which extends from \((t - t_0 - t+T)\). Therefore,

\[
\int_{-\infty}^{t} \exp(-\pi k \tau^2) \, d\tau - \int_{-\infty}^{t+T} \exp(-\pi k \tau^2) \, d\tau = \int_{t}^{t+T} \exp(-\pi k \tau^2) \, d\tau. \tag{210}
\]
The results of equation (210) should be substituted into equation (208) to yield the following result,

$$e_0(t) = \exp(2\pi i[f_0 + \Delta/2]t) \int_{-\frac{T}{\sqrt{-1/k}}}^{t+T} \exp(-\pi k\tau^2) d\tau. \quad (211)$$

By referring to equation (21), it is seen that equation (211) represents the inverse Fourier Transform of the third component $\frac{\partial E_d}{\partial \beta}$. Therefore, in order to determine the actual inverse Fourier Transform of this component, it is necessary to make use of equation (16) which is repeated here for convenience.

$$e_0(t) = \frac{1}{2\pi t} F^{-1}\left[\frac{\partial E_d(f)}{\partial \beta}\right] \quad (16)$$

Since $e_0'(t)$ represents the inverse Fourier Transform of the derivative of a frequency function, the actual inverse Fourier Transform can be found by multiplying equation (211) by the term $\left(\frac{1}{2\pi t}\right)$. That is,

$$e_0(t) = \left(e_0'(t)\right)\left(\frac{1}{2\pi t}\right) = -\frac{1}{2\pi t} \frac{\exp(2\pi i[f_0 + \Delta/2]t)}{\Delta \sqrt{-1/k}} \int_{t}^{t+T} \exp(-\pi k\tau^2) d\tau \quad (212)$$
The first step in simplifying equation (212) will be to represent
the terms in "1" by the exponential equivalent. Therefore,

\[ -1 = e^{-\pi i/2} \quad (213) \]

and

\[ \sqrt{-1} = \sqrt{e^{-i\pi/2}} = \left(e^{-i\pi/2}\right)^{1/2} \quad (214) \]

\[ = e^{-i\pi/4} \]

Since one term appears in the numerator and one in the denominator, they can be combined to give,

\[ \frac{-1}{\sqrt{-1}} = \frac{e^{-i\pi/2}}{e^{-i\pi/4}} = e^{-i\pi/2}e^{i\pi/4} \quad (215) \]

\[ = e^{-i\pi/4} = e^{+2\pi i\left(-\frac{1}{8}\right)}. \]

Thus, equation (215) can be substituted into equation (212) to yield the following result,

\[ \varepsilon_{o3}(t) = \frac{\exp\left(2\pi i\left(\left[f_0 + \Delta/2\right]t - 1/8\right)\right)}{2\pi \Delta t \sqrt{1/k}} \int_{t}^{t+T} \exp\left(-\pi k \tau^2\right)d\tau \quad (216) \]
The terms in the denominator can be combined to yield a further simplification. That is,

\[ \Delta \sqrt{1/k} = kT \sqrt{1/k} \]

\[ = \sqrt{k^2 T^2 / k} = \frac{k^2 T^2}{k} = kT^2 \]

or this can be written as

\[ \Delta \sqrt{1/k} = \sqrt{kT^2} = \sqrt{\frac{\Delta T^2}{T}} = \sqrt{\Delta T} = \sqrt{D}. \]

Substituting this result into equation (216) produces

\[ \varepsilon_{o3}(t) = \frac{\exp \left( 2\pi \left( [f_0 + \Delta/2]t - 1/8 \right) \right)}{2\pi \sqrt{D} t} \int_t^{t+T} \exp \left( -\pi k\tau^2 \right) d\tau \]

(218)

The integral in equation (218) appears to be a type of Fresnel integral. Thus, consider expanding the integral as follows

\[ \int_t^{t+T} \exp \left( -\pi k\tau^2 \right) d\tau = \int_0^{t+T} \exp \left( -\pi k\tau^2 \right) d\tau - \int_0^t \exp \left( -\pi k\tau^2 \right) d\tau \]

(219)
Compare this with equation (9) of section 3 which will be repeated for convenience,

\[ Z(v) = \int_{0}^{v} e^{i\pi a^2/2} da = C(v) + iS(v) \]  

(220)

If one were to compare these equations, it becomes apparent that they are very similar and by making a few simple substitutions they can be made equal. This shall be done presently. For the first integral, let

\[ a^2/2 = k\tau^2 \]  

(221)

then

\[ \tau^2 = a^2/2k \]  

(222)

Taking the square root of both sides yields

\[ \tau = a/\sqrt{2k} \]  

(223)

Now differentiate equation (223). This yields

\[ d\tau = \frac{d\alpha}{\sqrt{2k}} \]  

(224)
For the limits, the following is obtained

\[ a_1 = \sqrt{2k} \tau_1 = \sqrt{2k} (0) = 0 \]  \hspace{1cm} (225)

and for the upper limit

\[ a_2 = \sqrt{2k} \tau_2 = \sqrt{2k} (t+T). \]  \hspace{1cm} (226)

For the second integral equations (221) through (225) are still valid only the upper limit changes. That is,

\[ a_2 = \sqrt{2k} t. \]  \hspace{1cm} (227)

Making these substitutions in equation (219) will give the following result

\[ \int_{t}^{t+T} \exp(-\pi k \tau^2) \, d\tau \]

\[ = \frac{1}{\sqrt{2k}} \int_{0}^{\sqrt{2k} (t+T)} \exp(-\pi \alpha^2/2) \, d\alpha - \frac{1}{\sqrt{2k}} \int_{0}^{\sqrt{2k} t} \exp(-\pi \alpha^2/2) \, d\alpha \]  \hspace{1cm} (228)
Comparing the equation (220) with the integrals on the right-hand side of equation (228) indicates that the only difference is the sign of the exponents. However, this difference can be eliminated by taking the complex conjugate of equation (220).

\[ Z^*(v) = \int_{0}^{V} \exp\left(-\pi \alpha^2 / 2\right) d\alpha = C(v) - iS(v) \quad (229) \]

Now equations (228) and (229) are in a similar form and would be equal if \( v \) in the first integral is given by

\[ v = \sqrt{2k} (t+T) \quad (230) \]

and if \( v \) in the second integral is equal to the following

\[ v = \sqrt{2k} t. \quad (231) \]

With these expressions for \( v \), it is possible to write the integral on the left-hand side of equation (228) as

\[ \int_{t}^{t+T} \exp\left(-\pi k \tau^2\right) d\tau = \frac{1}{\sqrt{2k}} \left[ Z^*(\sqrt{2k} (t+T)) - Z^*(\sqrt{2k} t) \right] \quad (232) \]
This allows equation (218) to have the following form

\[
\varepsilon_{o3}(t) = \frac{\exp \left(2\pi i \left(\frac{f_0 + \Delta/2}{2}t - \frac{1}{8}\right)\right)}{2\pi \sqrt{D} \ t} \left[ \frac{Z^* \left(\frac{\sqrt{2k}}{t+T} \right) - Z^* \left(\sqrt{2k} \ t \right)}{\sqrt{2k}} \right] \]

(233)

Remembering that this is the response to a complex waveform, the response to the actual input signal will be given by the real part of \(\varepsilon_{o3}(t)\). That is,

\[
\varepsilon_{o3}(t) = \text{Re} \varepsilon_{o3}(t) = \text{Re} \left[ \frac{\exp \left(2\pi i \left(\frac{f_0 + \Delta/2}{2}t - \frac{1}{8}\right)\right)}{2\pi \sqrt{D} \ t} \left( \frac{Z^* \left(\frac{\sqrt{2k}}{t+T} \right) - Z^* \left(\sqrt{2k} \ t \right)}{\sqrt{2k}} \right) \right] = \text{Re} \left[ \frac{\exp \left(2\pi i \left(\frac{f_0 + \Delta/2}{2}t - \frac{1}{8}\right)\right)}{2\pi \sqrt{D} \ t} \left( \frac{Z^* \left(\frac{\sqrt{2k}}{t+T} \right) - Z^* \left(\sqrt{2k} \ t \right)}{\sqrt{2k}} \right) \right] \]

(234)

Since there are three components comprising the output signal, the actual output is the sum of the three terms. That is,

\[
e_o(t) = e_{o1}(t) + e_{o2}(t) + e_{o3}(t) = \text{Re} \left[ \varepsilon_{o1}(t) + \varepsilon_{o2}(t) + \varepsilon_{o3}(t) \right] = \text{Re} \left[ \frac{\exp \left(2\pi i \left(\frac{f_0 + \Delta/2}{2}t - \frac{1}{8}\right)\right)}{2\pi \sqrt{D} \ t} \left( \frac{Z^* \left(\frac{\sqrt{2k}}{t+T} \right) - Z^* \left(\sqrt{2k} \ t \right)}{\sqrt{2k}} \right) \right] \]

(235)

To obtain equation (235) take the sum of equations (94), (146), and (235).
Before going any further, this is a good point to recapitulate what has gone on already. The first step in the analysis produced the output of the ideal band pass filter. With this serving as the input to the chirp filter, the chirp filter output signal was obtained. The frequency spectrum of the chirp filter output was then convolved with the frequency spectrum of the gate circuit. The resulting spectrum was multiplied by the spectrum of the de-chirp filter. This final spectrum was inverted and this is what appears in equation (235). The form of equation (235), however, still needs some further simplification. However, this effort led to extremely difficult terms to evaluate and forced the author to seek additional means to find a more useful form of equation (235).

This approach is based upon the use of the output signal due an input which is the conjugate of the input used to evaluate equation (235).

In other words, evaluate the output of the de-chirp filter to two chirp filter input signals. The first one being

\[ e^{2\pi f_0 t} \sin \frac{\pi \Delta t}{\pi \Delta t} \quad (236) \]

and the second being

\[ e^{-2\pi f_0 t} \sin \frac{\pi \Delta t}{\pi \Delta t} \quad (237) \]
the sum of equations (236) and (237) is

\[ 2 \cos 2\pi f_0 t \frac{\sin \Delta t}{\Delta t} \]  

which is the actual input signal under consideration. The expression in equation (237) is the negative frequency component of the spectrum of the signal in equation (238). For this input waveform the chirp and de-chirp filter spectrums are the complex conjugates of the spectrums given by equations (26) and (8) of Section 3.

The evaluation of the de-chirp filter response will not require an extensive derivation similar to the one already given. Much of this work will be saved because of the following characteristic of the Fourier Transform. Since the input waveform and filter characteristics are the complex conjugates of the ones already considered, the spectrum, \( E_d(f) \), is the complex conjugate of the one given in equation (7). Therefore, consider the following equations. Given that

\[ C(t) = \int_{-\infty}^{\infty} C(f)e^{2\pi ift} df = F^{-1}[C(f)]. \]  

Then, the complex conjugate of \( C(t) \) will be given by the following
But, we are interested in the inverse Fourier Transform of \( C^*(f) \) only. Therefore, to achieve this result, consider the following

\[
C^*(t) = \int_{-\infty}^{\infty} C^*(f)e^{2\pi itf} df
\]

thus, this equation indicates that the inverse Fourier Transform of \( C^*(f) \) is \( C^*(t) \).

Since, the form of \( C(t) \) has already been evaluated, it will be a simple matter to find the response to the input given by equation (237). Then, the two signals can be added to evaluate the response to the actual input signal.

Before invoking the results of equation (241) some groundwork must first be laid. The groundwork involves considering the expression for chirp filter in the negative frequency range. In the negative frequency range, the chirp filter expression can be written as follows

\[
H_d(f) = e^{-\left(\frac{\pi}{k}(f+f_0)^2\right)}; \quad f < 0
\]
This expression also has one item which distinguishes from the previous case. In the previous case,

\[ \beta = f - f_0 \]  \hspace{1cm} (243)

whereas for the case at hand

\[ \beta = f + f_0 \]  \hspace{1cm} (244)

Therefore, before use is made of the above mentioned fact it will be important to see where it will change the resulting output waveform. While determining the inverse Fourier Transform of the aforementioned time signals \( \beta \) was used as frequency variable. Therefore, for these time signals, no effect will appear. However, the changes will occur when making use of equation (66), i.e.,

\[ F^{-1} F(f-u) = e^{2\pi iuf(t)}. \]  \hspace{1cm} (66)

Thus, before applying equation (244), this alteration in the meaning of \( \beta \) must be made. To see the alteration necessary, let's reconsider the various frequency variables used when determining the inverse Fourier Transforms.

In equation (96), repeated here for convenience,
\[ y = \beta + \Delta/2 = f - f_0 + \Delta/2 \quad \text{(96)} \]
\[ = f - (f_0 - \Delta/2) \]

The value of \( u \) is given by

\[ u = f_0 - \Delta/2 \quad \text{(245)} \]

where the \( \beta \) in equation (243) was used. Now, using the definition of \( \beta \) in equation (244)

\[ y = \beta + \Delta/2 = f + f_0 + \Delta/2 \]
\[ = f - (-f_0 - \Delta/2) \quad \text{(246)} \]

and

\[ u = -f_0 - \Delta/2 \quad \text{(247)} \]

Therefore, before using equation (241), the expression for \( \varepsilon_{o2}(t) \) in equation (139) should be changed as follows

\[ \varepsilon_{o2}(t) = \frac{1}{2\pi t} \exp\left(\frac{2\pi i[-f_0 - \Delta/2]t}{kT \sqrt{-1/k}}\right) \int_{t-T}^{t} \exp\left(-\pi k\tau^2\right) d\tau \quad \text{(248)} \]
the only change that was made reflected the change in \( u \) of equation (247).

The next step will be to apply the results of equation (241); i.e., take the complex conjugate and substitute \((-t)\) for \( t \). Therefore,

\[
\hat{e}_{o2}^*(-t) = \frac{1}{2\pi t} \frac{\exp(-2\pi i[f_0+\Delta/2]t)}{kT \sqrt{1/k}} \int_{-t}^{-t-T} \exp(\pi ki\tau^2) d\tau \quad (249)
\]

First thing that can be done to simplify equation (249) will be to interchange limits by placing a minus sign in front of equation (249). That is,

\[
\hat{e}_{o2}^*(-t) = -\frac{1}{2\pi t} \frac{\exp(-2\pi i[f_0+\Delta/2]t)}{kT \sqrt{1/k}} \int_{-t}^{-t-T} \exp(\pi ki\tau^2) d\tau \quad (250)
\]

Next, let

\[
\tau = -\tau \quad (251)
\]

or

\[
\mathrm{d}\tau = -\mathrm{d}\tau \quad (252)
\]
then, substituting these results into equation (250) allows us to write the following

\[ \varepsilon_{o2}^*(t) = \frac{i}{2\pi t} \frac{\exp(-2\pi i [f_o + \Delta/2] t)}{k T \sqrt{1/k}} \int_t^{t+T} \exp(\pi k i \tau^2) d\tau \]  

(253)

The two complex terms should now be combined into the exponential as follows

\[ i = e^{i\pi/2} \]  

(254)

and

\[ \sqrt{1} = \left(e^{i\pi/2}\right)^{1/2} = e^{i\pi/4} \]  

(255)

therefore,

\[ \frac{1}{\sqrt{1}} = e^{i\pi/2} - e^{i\pi/4} = e^{i\pi/4} \]  

(256)

Substituting these results into equation (253) yields the following results,

\[ \varepsilon_{o2}^*(t) = \frac{\exp(-2\pi i [f_o + \Delta/2] t - 1/8)}{2\pi k T \sqrt{1/k}} \int_t^{t+T} \exp(\pi k i \tau^2) d\tau \]  

(257)
Now, it is possible to simplify even a bit further. Consider the terms

\[ k_T \sqrt{1/k} = \sqrt{k^2 T^2/k} = \sqrt{kT^2} \quad (258) \]

But, \( k = \Delta/T \). Therefore,

\[ \sqrt{kT^2} = \sqrt{\Delta T} \quad (259) \]

Again, substituting gives

\[ f_{27\pi} \left[ (f_o + \Delta/2)t - 1/8 \right] \quad (260) \]

Now that the expression \( \varepsilon_{o2}^*(-t) \), a corresponding expression for \( \varepsilon_{o3}(t) \) should be derived. However, before that expression can be obtained a similar procedure that preceded the evaluation of \( \varepsilon_{o2}^*(-t) \) must be undertaken. Therefore, the variable used in the inverse Fourier Transform must be reconsidered. Hence,

\[ x = \beta - \Delta/2 = f - f_o - \Delta/2 \quad (261) \]
must be changed because now

\[ \beta = f + f_0. \quad (262) \]

Therefore,

\[ x = f + f_0 - \Delta/2 \]

\[ = f - (-f_0 + \Delta/2) = f - u. \quad (263) \]

where

\[ u = -f_0 + \Delta/2 \quad (264) \]

Therefore, the expression in equation (212) should be changed to read

\[ \varepsilon_{03}(t) = -\frac{1}{2\pi t} \frac{\exp(-2\pi i[f_0-\Delta/2]t)}{\Delta \sqrt{-i/k}} \int_t^{t+T} \exp(-\pi k\tau^2) d\tau \]  

(265)

where the only change, that was made, was due to the re-evaluation of \( u \).

With this change completed, the results of equation (241) can be used to complete the desired evaluation. Therefore,
Consider the following substitution,

\[ \tau = -\tau \]  \hspace{1cm} (267)

and

\[ d\tau = -d\tau \]  \hspace{1cm} (268)

then, making these substitutions into the integral in equation (266) gives the following results,

\[ \hat{e}_{o3}^*(t) = \frac{-i}{2 \pi t} \frac{\exp\left(-2 \pi i [f_o - \Delta/2] t\right)}{\Delta \sqrt{1/k}} \int_{-t-T}^{t-T} \exp(\pi k \tau^2) d\tau \]  \hspace{1cm} (269)

As before, the complex terms can be combined in exponential form as follows

\[ -i = e^{-i\pi/2} \]  \hspace{1cm} (270)

and

\[ i = e^{i\pi/2} \]  \hspace{1cm} (271)
then

\[ \sqrt{1} = \left( e^{\frac{i\pi}{2}} \right)^{1/4} = e^{\frac{i\pi}{4}} \]  

(272)

Hence,

\[ \frac{+1}{\sqrt{1}} = \frac{e^{+i\pi/2}}{e^{i\pi/4}} = e^{+i\pi/2} e^{-i\pi/4} = e^{+i\pi/4} \]

(273)

Replacing the left-hand side of equation (273) with the right-hand side in equation (269) yields

\[ *_{o_3}(-t) = \exp\left(\frac{-2\pi i \left( \frac{f_0 - \Delta/2}{\sqrt{1/k}} t - 1/8 \right) }{2\pi \Delta \sqrt{1/k} t} \right) \int_{t}^{t-T} \exp\left(\frac{+\pi k t^2}{2} \right) dt \]

(274)

Finally

\[ \Delta \sqrt{\frac{1}{k}} = \sqrt{kT^2} \quad \frac{\Delta}{\sqrt{k}} = \sqrt{\frac{k^2 T^2}{k}} \]

(275)

\[ = \sqrt{kT^2} = \sqrt{\frac{\Delta}{T} T^2} = \sqrt{\Delta T} \]

\[ = \sqrt{D} \]
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making the appropriate substitution in equation (274) produces
the following result,

\[ \tilde{e}_{03}^*(t) = \frac{\exp \left( -2\pi i \left( f_o - \frac{\Delta}{2} \right) t - \frac{1}{8} \right)}{2\pi \sqrt{D} t} \int_{t}^{t-T} \exp \left( \frac{+\pi k \tau^2}{\tau} \right) d\tau \]

(276)

The final term to be considered is \( \varepsilon_{01}(t) \). Once
again, it will be necessary to reconsider the variable of inte­
gration, i.e., equations (59) and (74)

\[ \sigma = \beta \pm \Delta/2 \]  \hspace{1cm} (74)

where

\[ \beta = f - f_o. \]  \hspace{1cm} (277)

The new expression for \( \beta \) is given by

\[ \beta = f + f_o \]  \hspace{1cm} (278)

therefore,

\[ \sigma = f + f_o \pm \Delta/2 = f - (\pm f_o \pm \Delta/2) \]  \hspace{1cm} (279)

\[ = f - u \]
where

\[ u = -f_0 \pm \Delta/2 \]  \hspace{1cm} (280)

Once again the only difference being the sign of \( f_0 \). Thus, this sign can be altered in equation (89) to give

\[
\hat{\varepsilon}_{o1}(t) = \frac{Z^*(\sqrt{2D})}{\sqrt{2}} \exp\left(2\pi I\left[-f_0 t - kt^2 + 1/8\right]\right) \frac{\sin \pi \Delta t}{\pi \Delta t}
\]  \hspace{1cm} (281)

Now, the results of equation (241) can be invoked to produce the following results,

\[
\hat{\varepsilon}_{o1}^*(-t) = \frac{Z(\sqrt{2D})}{\sqrt{2}} \exp\left(-2\pi I\left[f_0 t - kt^2/2 + \frac{1}{8}\right]\right) \frac{\sin \pi \Delta t}{\pi \Delta t}
\]  \hspace{1cm} (281')

Now, with all the terms necessary for evaluation of the final output determined, all that needs to be done is combine all the terms.

The first two terms to be combined will be \( \hat{\varepsilon}_{o1}(t) \) and \( \hat{\varepsilon}_{o1}^*(-t) \). The value of \( \varepsilon_{o1}(t) \) given in equation (91) will be combined with \( \hat{\varepsilon}_{o1}^*(-t) \) given in equation (281). Therefore,
\[ e_{o1}(t) = e_{o1}(t) + \hat{e}_{o1}(-t) \]

\[
= \frac{sinc \Delta t}{\sqrt{2}} \left[ Z* \sqrt{2D} \exp\left(2\pi i \left[f_o t - kt^2/2 + \frac{1}{8}\right]\right) + Z\sqrt{2D} \exp\left(-2\pi i \left[f_o t - kt^2/2 + \frac{1}{8}\right]\right) \right]
\]

\[
= \frac{A \sin \Delta T}{\sqrt{2}} \left[ \exp\left(2\pi i \left[f_o t - kt^2/2 + a\right]\right) + \exp\left(2\pi i \left[f_o t - kt^2/2 + a\right]\right) \right]
\]

where

\[ Z(\sqrt{2D}) = Ae^{1c} \]  

(283)

and

\[ a = 1/8 - c. \]  

(284)

These terms in brackets can be combined as follows

\[
e_{o1}(t) = \frac{2A \sin \Delta t}{\sqrt{2}} \cos\left[2\pi \left(f_o t - kt^2/2 + a\right)\right]
\]

\[
= \sqrt{2} A \sin \Delta t \cos\left[2\pi \left(f_o t - kt^2/2 + a\right)\right]
\]

(285)
The next terms to be combined will be $\epsilon_{o2}(t)$ and $\epsilon_{o3}(t)$. Then, $\epsilon_{o1}(-t)$ and $\epsilon_{o3}(-t)$ will be combined. This will result in two terms which themselves will be put together to yield the final form.

To combine $\epsilon_{o2}(t)$ and $\epsilon_{o3}(t)$ consider the values given in equations (137) and (211) written here for convenience.

\[
\epsilon_{o2}'(t) = \frac{\exp(2\pi i[f_o - \Delta/2]t)}{kT \sqrt{-i/k}} \int_{t-T}^{t} \exp(-\pi k \tau^2) d\tau \quad (137)
\]

and

\[
\epsilon_{o3}'(t) = \frac{\exp(2\pi i[f_o + \Delta/2]t)}{\Delta \sqrt{-i/k}} \int_{t}^{t+T} \exp(-\pi k \tau^2) d\tau \quad (211)
\]

When combining these two terms, the common terms will be factored as the first step. Thus,

\[
\epsilon'(t) = \frac{\exp(2\pi i f_o t)}{\Delta \sqrt{-i/k}} \left[ \exp(-\pi i \Delta t) \int_{t-T}^{t} \exp(-\pi k \tau^2) d\tau \right. \\
- \exp(\pi i \Delta t) \left. \int_{t}^{t+T} \exp(-\pi k \tau^2) d\tau \right]
\]

(286)
In order to combine these integrals, consider taking the two integrals and breaking them into parts as follows,

\[
\int_{t}^{t+T} \exp(-\pi k \tau^2) d\tau = \int_{0}^{t+T} \exp(-\pi k \tau^2) d\tau - \int_{0}^{t} \exp(-\pi k \tau^2) d\tau
\]  
\[
(287)
\]

This can still be changed even further by expanding the first integral, on the right-hand side, into two integrals as follows

\[
\int_{t}^{t+T} \exp(-\pi k \tau^2) d\tau
\]

\[
(288)
\]

\[
= \int_{t-T}^{t} \exp(-\pi k \tau^2) d\tau + \int_{t}^{t+T} \exp(-\pi k \tau^2) d\tau - \int_{0}^{t-T} \exp(-\pi k \tau^2) d\tau
\]

By combining the first and last integrals on the right-hand side of equation (288), the following form results,

\[
\int_{t}^{t+T} \exp(-\pi k \tau^2) d\tau = \int_{t-T}^{t} \exp(-\pi k \tau^2) d\tau - \int_{t}^{t+T} \exp(-\pi k \tau^2) d\tau
\]

\[
(289)
\]
Substituting these results for the integral into equation (287) yields the following

\[ \varepsilon'(t) = \frac{\exp(2\pi if_0 t)}{\Delta \sqrt{-1/k}} \left[ - \exp(+\pi i\Delta t) \int_{t-T}^{t+T} \exp(-\pi k\tau^2) d\tau \right. \]

\[ \left. + \exp(+\pi i\Delta t) \int_{t-T}^{t} \exp(-\pi k\tau^2) d\tau \right] + \exp(-\pi i\Delta t) \int_{t-T}^{t} \exp(-\pi k\tau^2) d\tau \]

\[ \varepsilon'(t) = \frac{\exp(2\pi if_0 t)}{\Delta \sqrt{-1/k}} \left[ \begin{array}{c}
t = t + T \\
t - T \end{array} \right] \]

(290)

The last two integrals on the right-hand side can be combined because they have the same integrals. Thus,

\[ \varepsilon'(t) = \frac{\exp(2\pi if_0 t)}{\Delta \sqrt{-1/k}} \left[ - \exp(+\pi i\Delta t) \int_{t-T}^{t+T} \exp(-\pi k\tau^2) d\tau \right. \]

\[ \left. + 2 \cos \pi \Delta t \int_{t-T}^{t} \exp(-\pi k\tau^2) d\tau \right] \]

(291)
First, as before, consider bringing the complex term into the exponential. Then

\[-1 = e^{-i\pi/2}\]  \hspace{1cm} (292)

and

\[\sqrt{-1} = (e^{-i\pi/2})^{1/2} = e^{-i\pi/4}\]  \hspace{1cm} (293)

and

\[\frac{1}{\sqrt{-1}} = e^{+i\pi/4}\]  \hspace{1cm} (294)

making the appropriate substitution gives

\[e'(t) = \frac{\exp\left(2\pi i \left[ f_0 t + 1/8 \right]\right)}{\Delta \sqrt{1/k}} \left[ - \exp(+\pi i \Delta t) \int_{t-T}^{t+T} \exp(-\pi k i \tau^2) d\tau + 2 \cos \pi \Delta t \int_{t-T}^{t} \exp(-\pi k i \tau^2) d\tau \right]\]

\hspace{1cm} (295)

Consider making a substitution as follows
\[ k \tau^2 = \frac{a^2}{r} \]  

(296)

or

\[ a^2 = 2k \tau^2 \]  

(297)

then,

\[ \alpha = \sqrt{2k} \tau \]  

(298)

and

\[ da = \sqrt{2k} \, dr. \]  

(299)

Making the necessary change in equation (295) yields

\[
\epsilon'(t) = \frac{\exp\left(2\pi i\left[\int_{t'-T'}^{t'+T'} \exp\left(-\pi \alpha^2/2\right) \, d\alpha \right]\right)}{\Delta \sqrt{1/k} \, 2k} \left[- \exp(+\pi \Delta t) \int_{t'-T'}^{t'+T'} \exp(-\pi \alpha^2/2) \, d\alpha + 2 \cos \pi \Delta t \int_{t'-T'}^{t'} \exp(-\pi \alpha^2/2) \, d\alpha \right] \]

(300)

where

\[ t' = \sqrt{2k} \, t \]  

(301)

and

\[ T' = \sqrt{2k} \, T \]  

(302)
The square root of $k$ can be cancelled in the denominator to give

$$
\varepsilon'(t) = \frac{\exp\left(2\pi i \left[f_0 t + l/8\right]\right)}{\sqrt{2} \Delta} \left[ \exp(+\pi \Delta t) \int_{t'-T'}^{t'+T'} \exp(-\pi \alpha^2/2) d\alpha \right]
$$

$$
+ 2 \cos \pi \Delta t \int_{t'-T'}^{t'} \exp(-\pi \alpha^2/2) d\alpha
$$

(303)

The integral terms in equation (303) are in the Fresnel form and can be written as follows,

$$
\varepsilon'(t) = \frac{\exp\left(2\pi i \left[f_0 t + l/8\right]\right)}{\sqrt{2} \Delta} \left[ - \exp(+\pi \Delta t) \left( Z^*(t'+T') - Z^*(t'-T') \right) \right]
$$

$$
+ 2 \cos \pi \Delta t \left( Z^*(t') - Z^*(t'-T') \right)
$$

(304)

where the following substitutions have been made

$$
Z^*(t'+T') = \int_{0}^{t'+T'} \exp(-\pi \alpha^2/2) d\alpha
$$

(305)
\[ Z^\#(t') = \int_{0}^{t'} \exp(-\pi \alpha^2/2) \, d\alpha \quad (306) \]

and

\[ Z^\#(t'-T') = \int_{0}^{t'-T'} \exp(-\pi \alpha^2/2) \, d\alpha \quad (307) \]

\( \varepsilon' \) represents the sum of two terms that are inverse Fourier Transforms of frequency functions which are derivatives of frequency functions. Therefore, using equation (16) again gives

\[
\varepsilon(t) = \frac{i \exp(2\pi i[f_0 t + 1/8])}{2\pi \sqrt{2} \Delta t} \left[ - \exp(+\pi \Delta t) Z^\#(M) + 2 \cos \pi \Delta t \, Z^\#(N) \right] \quad (308)
\]

where

\[ Z^\#(M) = Z^\#(t' + T') - Z^\#(t'-T') \quad (309) \]

and

\[ Z^\#(N) = Z^\#(t') - Z^\#(t'-T') \quad (310) \]
The term "1" can be put into the exponent of the exponential as follows

\[ i = e^{1 \pi/2} \]  

(311)

therefore,

\[
\varepsilon(t) = \frac{\exp\left(2\pi i [f_0 t + 3/8]\right)}{2 \sqrt{2} \pi \Delta t} \left[ - Z^*(M) \exp(+\pi \Delta t) + 2 \cos \pi \Delta t \ Z^*(N) \right] \]

(312)

Combining the remaining terms \( \hat{\varepsilon}_{o2}(-t) \) and \( \hat{\varepsilon}_{o3}(-t) \) is all that remain.

Their values are given in equations (257) and (274).

Combining gives

\[
\hat{\varepsilon}(t) = \frac{\exp\left(-2\pi i \left[f_0 + \Delta/2\right] t - 1/8\right)}{2\pi k T \sqrt{1/k t}} \left[ \int_t^{t+T} \exp(\pi k i \tau^2) d\tau \right] 
\]

\[
+ \frac{\exp\left(-2\pi i \left[f_0 - \Delta/2\right] t - 1/8\right)}{2\pi k T \sqrt{1/k t}} \left[ \int_t^{t-T} \exp(\pi k i \tau^2) d\tau \right] 
\]

(313)

Factoring yields
\[
\hat{e}(t) = \frac{\exp\left(-2\pi i [f_0 t - 1/8]\right)}{2\pi k T \sqrt{1/k t}} \left[ \exp(-\pi i\Delta t) \int_t^{t+T} \exp\left(\pi k i^2 \right) d\tau + \exp(\pi i\Delta t) \int_t^{t-T} \exp\left(\pi k i^2 \right) d\tau \right]
\]

As before, consider the following substitution,

\[\alpha^2/2 = k\tau^2 \quad (315)\]

or

\[\alpha = \sqrt{2k} \tau \quad (316)\]

and

\[d\alpha = \sqrt{2k} \ d\tau \quad (317)\]

Making these substitutions in equation (314) permits the equation to be rewritten in the following manner

\[
\hat{e}(t) = \frac{\exp\left(-2\pi i [f_0 t - 1/8]\right)}{2\pi \Delta \sqrt{2} t} \left[ \exp(-\pi i\Delta t) \int_{t'}^{t'+T} \exp\left(\pi \alpha^2/2\right) d\alpha + \exp(\pi i\Delta t) \int_{t'}^{t'-T} \exp\left(\pi \alpha^2/2\right) d\alpha \right]
\]
where $A = kT$ and

$$t' = \sqrt{2k} \ t$$  \hspace{1cm} (319)$$

and

$$T' = \sqrt{2k} \ T.$$  \hspace{1cm} (320)$$

It is possible to rewrite the first integral in equation (318) as follows

$$\int_{t'}^{t'+T'} \exp\left(\frac{i\alpha^2}{2}\right) d\alpha = \int_{0}^{t'+T'} \exp\left(\frac{i\alpha^2}{2}\right) d\alpha - \int_{0}^{t'} \exp\left(\frac{i\alpha^2}{2}\right) d\alpha$$  \hspace{1cm} (321)$$

The first integral on the right-hand side of equation (321) can also be written as

$$\int_{0}^{t'+T'} \exp\left(\frac{i\alpha^2}{2}\right) d\alpha = \int_{0}^{t'-T'} \exp\left(\frac{i\alpha^2}{2}\right) d\alpha + \int_{t'-T'}^{t'+T'} \exp\left(\frac{i\alpha^2}{2}\right) d\alpha$$  \hspace{1cm} (322)$$

Equation (322) allows equation (321) to be rewritten in the following manner,
Equation (323) can be substituted into equation (318) to yield the following equation

\[
\hat{e}(t) = \frac{\exp\left(-2\pi if_0 (t-1/8)\right)}{2\pi f \Delta \sqrt{2} t} \left[ \int_{t'-T'}^{t'+T'} \exp(-\pi t \Delta t) \exp\left(\pi \alpha^2/2\right) d\alpha - \int_{0}^{t'} \exp\left(\pi \alpha^2/2\right) d\alpha - \int_{t'-T'}^{0} \exp(-\pi t \Delta t) \exp\left(\pi \alpha^2/2\right) d\alpha - \int_{t'-T'}^{t'} \exp(\pi t \Delta t) \exp\left(\pi \alpha^2/2\right) d\alpha \right]
\]

(324)
One notes that the last two integrals in equation (324) can also be combined to yield,

\[ \hat{e}(t) = \frac{\exp\left(-2\pi i [f_0 t - l/8] / (2\Delta \sqrt{2} t)\right)}{2\Delta \sqrt{2} t} \left[ \exp(-\pi i t) \int_{t'-T'}^{t'+T'} \exp\left(\pi i \alpha^2/2\right) d\alpha \right. \\
- 2 \cos \pi \Delta t \int_{t'-T'}^{t'} \exp\left(\pi i \alpha^2/2\right) d\alpha \left. \right] \\
\]  

(325)

Using the definition of the Fresnel Integral, the two integrals in equation (325) may be expanded in the following manner,

\[ \int_{t'-T'}^{t'+T'} \exp\left(\pi i \alpha^2/2\right) d\alpha = \int_{0}^{t'+T'} \exp\left(\pi i \alpha^2/2\right) d\alpha - \int_{0}^{t'-T'} \exp\left(\pi i \alpha^2/2\right) d\alpha \\
= Z(t'+T') - Z(t'-T') \\
= Z(M) \]  

(326)

and

\[ \int_{t'-T'}^{t'} \exp\left(\pi i \alpha^2/2\right) d\alpha = \int_{0}^{t'} \exp\left(\pi i \alpha^2/2\right) d\alpha - \int_{0}^{t'-T'} \exp\left(\pi i \alpha^2/2\right) d\alpha \\
= Z(t') - Z(t'-T') \\
= Z(N). \]  

(328)
If the results of equations (306) and (307) are substituted into equation (304), the following result:

$$
\hat{e}(t) = \exp\left(-\frac{2\pi i [f_0 t - 1/8]}{2\pi \sqrt{2} \Delta t}\right) \left[ \exp(-\pi \Delta t) Z(M) - 2 \cos \pi \Delta t Z(N) \right]
$$

(328)

In order to be able to combine $e(t)$ in equation (312) with $\hat{e}(t)$ in equation (328), consider expanding the a part of the exponent of the exponential terms. Namely, from equation (312)

$$
\exp\left(2\pi i \left(\frac{3}{8}\right)\right) = \exp\left(3/4\pi i\right) = \cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi
$$

$$
= -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}
$$

(329)

From equation (328)

$$
\exp\left(-2\pi i \left(-\frac{1}{8}\right)\right) = \exp\left(2\pi i \left(\frac{1}{8}\right)\right) = \exp(\pi i/4)
$$

$$
= \cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi
$$

$$
= \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}
$$

(330)

Now, consider the first term of equation (312)
\[
\varepsilon_A(t) = - \frac{\exp(2\pi f_0 t)\exp(+\pi i\Delta t)Z^*(M)}{2 \sqrt{2} \pi \Delta t} \left( -\frac{\sqrt{2}}{2} + 1 \frac{\sqrt{2}}{2} \right) \\
= \frac{Z^*(M)\exp\left(\frac{\pi i[f_0 + \Delta/2]t}{2}\right)}{2 \pi \Delta t} (+1 - 1) 
\]

(331)

Similarly, the first term of equation (328) is given by

\[
\hat{\varepsilon}_A(t) = \frac{\exp(-2\pi f_0 t)\exp(-\pi i\Delta t)Z(M)}{2 \sqrt{2} \pi \Delta t} \left( \frac{\sqrt{2}}{2} + 1 \frac{\sqrt{2}}{2} \right) \\
= \frac{Z(M)\exp\left(-2\pi i[f_0 + \Delta/2]t\right)}{4 \pi \Delta t} (1 + 1) 
\]

(332)

The expressions in equations (331) and (332) will now be combined to yield the following expression

\[
\varepsilon_A(t) + \hat{\varepsilon}_A(t) = \frac{1}{4 \pi \Delta t} \left[ \exp\left(2\pi i[f_0 + \Delta/2]t\right)(1-1)(R-ix) \\
+ \exp\left(-2\pi i[f_0 + \Delta/2]t\right)(1+1)(R+ix) \right] 
\]

(333)

where the following expression has been substituted for \(Z(M)\)

\[
Z(M) = R + ix 
\]

(334)
In order to combine the terms, the two complex terms in equation (333) will be expanded first. That is

\[(l-i)(R-ix) = R + i^2 x - iR - ix\]

\[= (R-x) - i(R+x) \quad (335)\]

and

\[(l+i)(R+ix) = R + i^2 x + iR + ix\]

\[= (R-x) + i(R+x) \quad (336)\]

Thus, making these substitutions into equation (333) yields the following equation

\[e_A(t) = \frac{1}{4\pi\Delta t} \left[ (R-x) \left( \exp(2\pi i[f_o + \Delta/2]t) + \exp(-2\pi i[f_o + \Delta/2]t) \right) \right.\]

\[\left. - i(R+x) \left( \exp(2\pi i[f_o + \Delta/2]t) - \exp(-2\pi i[f_o + \Delta/2]t) \right) \right] \quad (337)\]

where \(e_A(t)\) has been used as the sum of \(e_A(t)\) and \(\hat{e}_A(t)\).

It should be noted that the exponential expressions inside the brackets of equation (337) have a very special form which permits them to be combined to produce a cosine and sine term, respectively. Performing the combination on equation (337) yields the result,
It is instructive to expand the cosine and sine functions in order to put the equation into a somewhat more usable form. The expansions are as follows,

\[
e^{\Delta}(t) = \frac{1}{2\pi\Delta t} \left[(R-x)\cos 2\pi[f_o + \Delta/2]t + (R+x)\sin 2\pi[f_o + \Delta/2]t\right]
\]

(338)

and for the sine term

\[
\cos 2\pi[f_o + \Delta/2]t = \cos 2\pi f_o t \cos \pi\Delta t - \sin 2\pi f_o t \sin \pi\Delta t
\]

(339)

and for the sine term

\[
\sin 2\pi[f_o + \Delta/2]t = \sin 2\pi f_o t \cos \pi\Delta t + \cos 2\pi f_o t \sin \pi\Delta t
\]

(340)

Replacing the cosine and sine terms with their respective expansions yields equation (341),

\[
e^{\Delta}(t) = \frac{1}{2} \left[(R-x)\left(\cos 2\pi f_o t \frac{\cos \pi\Delta t}{\pi\Delta t} - \sin 2\pi f_o t \frac{\sin \pi\Delta t}{\pi\Delta t}\right)\right.

+ (R+x)\left(\cos 2\pi f_o t \frac{\sin \pi\Delta t}{\pi\Delta t} + \sin 2\pi f_o t \frac{\cos \pi\Delta t}{\pi\Delta t}\right)\right]
\]

(341)

At this point, the expression for \(\sin \pi\Delta t/\pi\Delta t\) can be replaced by sinc \(\Delta t\) and \(\cos \pi\Delta t/\pi\Delta t\) can be replaced by cosc \(\Delta t\). Making these substitutions in equation (341) will produce the following result,
The next task is to combine the second term of equation (312) with the second term of equation (328). Use will be made of the results of equation (329) and (330). Hence, for the second term of equation (312) the following can be written

\[
e_A(t) = \frac{1}{2} \left[ \cos(2\pi f_ot (R-x) \csc \Delta t + (R+x) \text{sinc} \Delta t) + \sin 2\pi f_ot ((R-x) \text{sinc} \Delta t + (R+x) \csc \Delta t) \right] (342)
\]

In a similar fashion, the second term of equation (328) will yield

\[
e_B(t) = \frac{\exp(2\pi if_ot) \cos \pi \Delta t \, Z^*(N)}{\sqrt{2} \, \pi \Delta t} \left( - \frac{\sqrt{2}}{2} + 1 \frac{\sqrt{2}}{2} \right)
\]

\[
= \frac{Z^*(N) \exp(2\pi if_ot) \cos \pi \Delta t}{2\pi \Delta t} (-1 + 1) (343)
\]

\[
= \frac{\exp(-2\pi if_ot) \cos \pi \Delta t \, Z(N)}{\sqrt{2} \, \pi \Delta t} \left( \frac{\sqrt{2}}{2} + 1 \frac{\sqrt{2}}{2} \right)
\]

\[
= \frac{Z(N) \exp(-2\pi if_ot) \cos \pi \Delta t}{2\pi \Delta t} (-1 - 1) (344)
\]
Now, equations (343) and (344) should be combined, as indicated before, to give

\begin{equation}
\begin{aligned}
e_B(t) &= \frac{\cos \frac{\pi \Delta t}{2}}{2\pi \Delta t} \left[ e^{2\pi i f_0 t} (P-iQ)(-1+i) + e^{-2\pi i f_0 t} (P+iQ)(-1-i) \right] \\
&= \frac{\cos \frac{\pi \Delta t}{2}}{2\pi \Delta t} \left[ e^{2\pi i f_0 t} \right] \left[ (P-iQ)(-1+i) + (P+iQ)(-1-i) \right] \\
&= \frac{\cos \frac{\pi \Delta t}{2}}{2\pi \Delta t} \left[ (P-iQ)(-1+i) + (P+iQ)(-1-i) \right]
\end{aligned}
\end{equation}

where

\begin{equation}
e_B(t) = e_B(t) + \hat{e}_B(t) \tag{346}
\end{equation}

and

\begin{equation}
Z(N) = P + iQ \tag{347}
\end{equation}

Before combining the terms, it will be necessary to expand the two complex expressions. Namely,

\begin{equation}
(P-iQ)(-1-i) = -P - i^2 Q + iP + iQ \tag{348}
\end{equation}

\begin{equation}
= (-P+Q) + i(P+Q)
\end{equation}

and

\begin{equation}
(P+iQ)(-1-i) = -P - i^2 Q - iP - iQ \tag{349}
\end{equation}

\begin{equation}
= (-P+Q) - i(P+Q)
\end{equation}
Replacing these expressions by their expansions in equation (345) will allow the following to be written,

\[ e_B(t) = \frac{\cos \pi \Delta t}{2\pi \Delta t} \left[ \exp(2\pi f_0 t)(-P+Q) + i(P+Q) \right] \]  

(350)

\[ + \exp(-2\pi f_0 t)(-P+Q) - i(P+Q) \]

Expanding the equation a little more produces the result given below in equation (351)

\[ e_B(t) = \frac{\cos \pi \Delta t}{2\pi \Delta t} \left[ (-P+Q)\left(\exp(2\pi f_0 t) + \exp(-2\pi f_0 t)\right) \right. \]

(351)

\[ + i(P+Q)\left(\exp(2\pi f_0 t) - \exp(-2\pi f_0 t)\right) \]

One notes that the two exponentials can be combined to give a cosine and sine term, respectively. Performing these indicated combinations yields the following result,

\[ e_B(t) = \frac{\cos \pi \Delta t}{2\pi \Delta t} \left[ 2(-P+Q)\cos 2\pi f_0 t - 2(P+Q)\sin 2\pi f_0 t \right] \]  

(352)

Now combining equations (342) and (352) will produce the final term necessary to complete the picture. This work gives
\[ e_{o2}(t) = e_A(t) + e_B(t) \]  

\[ = \frac{1}{2} \left[ \cos 2\pi f_0 t \left( \csc \Delta t \left( R - x - (P-Q)\alpha \right) + \text{sinc} \Delta t[R+x] \right) 
+ \sin 2\pi f_0 t \left( \csc \Delta t \left( R + x - (P+Q)\alpha \right) - \text{sinc} \Delta t(R-x) \right) \right] \]  

Finally, the output will be given by the summation of equations (285) and (353). Denoting this as \( e(t) \), the following equations results,

\[ e(t) = \sqrt{2} A \text{sinc} \Delta t \cos 2\pi \left( f_0 t - kt^2/2 + a \right) \]  

\[ + \cos 2\pi f_0 t \left[ \csc \Delta t(S-V') + \text{sinc} \Delta t(S) \right] \]  

\[ + \sin 2\pi f_0 t \left[ \csc \Delta t(S'-V) - \text{sinc} \Delta t(S) \right] \]  

where the following substitutions have been made.

\[ S = \frac{1}{2} (R-X) \]  

\[ S' = \frac{1}{2} (R+X) \]  

\[ V = P + Q \]  

\[ V' = (P - Q) \]
At this point, it will be useful to reiterate all of the definitions of the previous terms so that they will be all in one place.

\[ P + iQ = Z(N) = Z(t') - Z(t'-T') \]

\[ = \int_{0}^{t'} \exp(i\pi \alpha^2/2) d\alpha - \int_{0}^{t'-T'} \exp(i\pi \alpha^2/2) d\alpha \]

\[ = \int_{t'-T'}^{t'} \exp(i\pi \alpha^2/2) d\alpha \quad (359) \]

and

\[ P + iX = Z(M) = Z(t'+T') - Z(t'-T') \]

\[ = \int_{0}^{t'+T'} \exp(i\pi \alpha^2/2) d\alpha - \int_{0}^{t'-T'} \exp(i\pi \alpha^2/2) d\alpha \]

\[ = \int_{t'-T'}^{t'+T'} \exp(i\pi \alpha^2/2) d\alpha \quad (360) \]
and

\[ t' = \sqrt{2k} t \]  \hspace{1cm} (361)

and

\[ T' = \sqrt{2k} T \]  \hspace{1cm} (362)

also,

\[ Ae^{-ic} = Z^*(\sqrt{2D}) \]  \hspace{1cm} (363)

and

\[ a = 1/8 - c' \text{ i.e. } c' = c/2\pi \]  \hspace{1cm} (364)

It should be pointed out that these equations are completely
general and can be applied to any chirp radar system as long as
the values for "D" and "T" are given.

Since the results shown in reference 1 are for the special
of an infinite gate width, one should be able to obtain the
same results from the generalized results of equation (354)
if the gate width is permitted to become infinite. Hence,
if \( T \to \infty \), \( D \to \infty \) and

\[ \lim_{D \to \infty} Z^* (\sqrt{2D}) = \frac{\sqrt{2}}{2} e^{i\pi/4} \]  \hspace{1cm} (365)
Therefore,

\[ A = \frac{\sqrt{2}}{2} \]  

and

\[ a = 1 - c' = 0 \]  

because

\[ c' = \frac{c}{2\pi} = \frac{1}{8} \]

In addition, since \( T \to \infty \)

\[ k = \frac{\Delta}{T} \to 0 \]  

Therefore, the 2nd and 3rd terms in equation (354) approach zero because the upper and lower limits on the integral are the same. Now, by substituting the results in equations (366) and (367) into the remaining term in equation (354), one obtains the following result

\[ e_d(t) = \text{sinc} \Delta t \cdot \cos(2\pi f_0 t) \]  

This equation is exactly the same result shown in reference 1.
This appendix describes the computer program used to evaluate the de-chirp output waveform given in equation (36). The program, shown below, is a representative sample of the computer programs used for the various sampling rates.

For $D = 30$, $Z^* \left(\sqrt{2D}\right) = \frac{1}{2} - i\frac{1}{2} = \frac{1}{\sqrt{2}} e^{-i\pi/4}$. . ., $A = \frac{1}{\sqrt{2}}$, $C = \pi/4$

and $a = \frac{1}{8} - C' = \frac{1}{8} - \frac{1}{8} = 0$ since $c = c/\sqrt{2} = \frac{1}{8}$

1. PROGRAM VOLT
2. DIMENSION ZC1(2048), ZC2(2048), ZC3(2048), ZS1(2048), ZS2(2048), ZS3(2048), X(2048), Y(2048), Z(2048), U(2048), W(2048), V(2048),
3. V1(2048), V2(2048), S1(2048), S2(2048), E(2048), E0(2048), AM(2048), AN(2048), AO(2048), AP(2048)
4. PK = ((20./3.)*.5)*1.0E + 06
5. T = 3.0 E = 06
6. DO 10 I = 1, 2048
7. A = I
8. AR1 = A* .15625 E-08*PK
9. CALL FRICS (AR1, ZC1(I), AS1(I))
10. 10 CONTINUE
11. DO 11 I = 1, 2048
12. A = I
13. AR2 = (.15625E-08*A-T)*PK
14. CALL FRICS(AR2, ZC2(I), ZS2(I))
15. 11 CONTINUE
17. DO 12  I = 1,2048
18. A = I
19. AR3 = (.15625-08*A+T)*PK
20. CALL FRICS (AR3, ZC3(I), ZS3(I))
21. 12  CONTINUE
22. DO 13  I = 1, 2048
23. A = I
24  AT = (.15625-08)*A
25. F0 = 30, E+06
26. W(I) = 3.14*1.0E+07*AT
27. U(I) = 2.*3.14*AT*F0
28. AM(I) = ZC2(I)-ZC2(I)
29. AN(I) = ZS2(I)-ZS3(I)
30. AO(I) M ZC1(I) - ZC2(I)
31. AP*(I) = ZS1(I)-ZS2(I)
32. S2(I) = .5*(AM(I)+AN(I))
33. S1(I) = .5*(AM(I)-AN(I))
34. V1(I) = AO+AP
35. V2(I) = AO-AP
36. X(I) = COSF((2.*3.14)*(F0*AT-.25*(P**2)*(AT**2)))
37. Y(I) = A1(I)*COSF(U(I))
38. Z(I) = B(I)*SINF(U(I))
39. V(I) = (SINF(W(I))/W(I))*(Q(I)-Y(I)+Z(I))
40. E(I) = COSF(W(I))/W(I)*((C(I)+A1(I))*SINF(U(I))+(D(I)+B(I))*COSF(U
41. 1(I)))
42. EO(I) = V(I)-E(I)
Since only 2048 storage locations have been requested for the numerous variables in statements 1-5, the program only produces values of the time signal which are valid for t > 0. The program is then repeated for t < 0 and an additional 2048 samples are produced by the program. This process yields upon completion 4096 samples of the waveform.

\[ P_k = \sqrt{2k} = \sqrt{2\Delta/T} = \left(\sqrt{20/3}\right) \times 10^6 \]  \hspace{1cm} (B-1)

Statements 8-12 evaluate the following equation

\[ \int_{0}^{\sqrt{2k}t} \exp(i\pi\alpha^2/2) \, d\alpha = ZC1(\sqrt{2k}t) + iZS1(\sqrt{2k}t) \]  \hspace{1cm} (B-2)

Those from 13-17 determine

\[ \int_{0}^{\sqrt{2k}(t-T)} \exp(i\pi\alpha^2/2) \, d\alpha = ZC2(\sqrt{2k}(t-T)) + iZS2(\sqrt{2k}(t-T)) \]  \hspace{1cm} (B-3)

And from 18-22

\[ \int_{0}^{\sqrt{2k}(t+T)} \exp(i\pi\alpha^2/2) \, d\alpha = ZC3(\sqrt{2k}(t+T)) + iZS3(\sqrt{2k}(t+T)) \]  \hspace{1cm} (B-4)
\( W(I) = \pi \Delta t \cdot \text{AT} \) \( \text{i.e., } (\pi \Delta t) \). 
\( U(I) = 2\pi \cdot \text{AT} \cdot \text{FO} \) \( \text{i.e., } 2\pi f_o t \).

\( \text{AM}(I) = ZC2(I) - ZC3(I) \text{ i.e., } R = -R_e [Z(t_1 + T) - Z(t_1 - T)] \) (B-7)

\( \text{AN}(I) = S2(I) - S3(I) \text{ i.e., } X = -I_m [Z(t_1 + T) - Z(t_1 - T)] \) (B-8)

\( \text{AO}(I) = ZC1(I) - ZC2(I) \text{ i.e., } P = R_e [Z(t_1) - Z(t_1 - T)] \) (B-9)

\( \text{AP}(I) = ZS1(I) - ZS2(I) \text{ i.e., } Q = I_m [Z(t_1) - Z(t_1 - T)] \) (B-10)

\( S2(I) = (0.5)(\text{AM}(I) + \text{AN}(I)) \text{ i.e., } S_2 = \frac{1}{2}(R+X) \) (B-11)

\( S1(I) = (0.5)(\text{AM}(I) - \text{AN}(I)) \text{ i.e., } S_1 = \frac{1}{2}(R-X) \) (B-12)

\( V1(I) = R(I) + X(I) \text{ i.e., } V_1 = P+Q \) (B-13)

\( V2(I) = R(I) - X(I) \text{ i.e., } V_2 = P-Q \) (B-14)

\( X(I) = \cos(2\pi \cdot [\text{FO} \cdot \text{AT} - \frac{1}{4} \cdot P^2 \cdot \text{AT}^2]) \)
\text{i.e., } \cos 2\pi [f_o t - \frac{1}{2} kt^2] \) (B-15)

\( Y(I) = A1(I) \cdot \cos(U(I)) \text{ i.e., } = S_2 \cos 2\pi f_o t \) (B-16)

\( Z(I) = B(I) \cdot \sin(U(I)) \text{ i.e., } = S_1 \sin 2\pi f_o t \) (B-17)

\( V(I) = \frac{\text{SINW}(I)}{W(I)} \cdot (Q(I) - Y(I) + Z(I)) \)
\text{i.e., } \frac{\sin \pi \Delta t}{\pi \Delta t} [\cos 2\pi [f_o t - \frac{1}{2} kt^2] - S_2 \cos 2\pi f_o t + S_1 \sin 2\pi f_o t] \) (B-18)
\[ E(I) = \frac{\cos(w(I))}{w(I)} \cdot [(C(I) + A(I)) \cdot \sin(u(I) + (D(I) + B(I)) \cdot \cos(u(I))] \]

i.e., \[ = \frac{\cos \Delta t}{\pi \Delta t} \cdot [(V_1 + S_2) \sin 2\pi f_o t + (V_2 + S_1) \cos 2\pi f_o t] \] (B-19)

\[ E_0(I) = V(I) - E(I) \quad i.e., = e_0(t) \] (B-20)

The values of \( E_0 \) have been punched onto cards in order to plot them on a separate device because the plot routine was available only in FORTRAN IV and the Fresnel Integral in FORTRAN II. FRICS (A,B,C) is the subroutine used for the evaluation of the Fresnel Integral.
This appendix describes the method used to evaluate and plot the frequency response of the de-chirp output waveform.

1. DIMENSION EO(4096), FZ(4096), TT1(12), TT2(12), ST1(12), ST2(12), YL1(6)
2. 1, YL2(6), XL1(6), V(4096), X(4096), Y(4096), XL2(6), U(4096), TZ(4096),
3. 2, TT3(12), ST3(12), YL3(6), XL3(6)
4.
5. READ(5,1) TT1, TT2, TT3, ST1, ST2, ST3
6. READ(5,2) YL1, YL2, YL3, XL1, XL1, XL3
7. READ(5,3) (EO(I), I = 1, 4096)
8. 1 FORMAT (12A6)
9. 2 FORMAT (6A6)
10. 3 FORMAT (E15,8)
11. N2POW = 12
12. DO 12 I = 1, 4096
13. X(I) = EO(I)
14. Y(I) = 0.
15. 12 CONTINUE
16. CALL FRXFM(N2POW, X, Y)
17. DO 11 I = 1, 4096
18. IF(2048)30, 30, 31
19. 31 J = I - 2048
20. U(J) = X(I)
21. V(J) = Y(I)
22. GO TO 11
23. 30 J = I - 2048
24. U(J) = X(I)
25. V(J) = Y(I)
26 11 CONTINUE
27. DO 14 J = 1, 4096
28. FZ(J) = (U(J)**2*V(J)**2)**5
29. TZ(J) = ATAN(V(J)/U(J))
30. 14 CONTINUE
31. DO 10 I = 1, 4096
32. A = I
33. X(I) = A
34. 10 CONTINUE
35. WRITE (6, 21) (J, FZ(J), J = 1, 4096)
36. 21 FORMAT (1H1, (1H, 2HFZ, 14, 1H =, E15, 8))
37. WRITE (6, 23) (J, EZ(J), J = 1, 4096)
38. 23 FORMAT (1H1, (1H, 2HE0, 14, 1H =, E15, 8))
39. WRITE (6, 24) (J, TZ(J), J = 1, 4096)
40. 24 FORMAT (1H1, (1H, 2HTZ, 14, 1H =, E15, 8))
41.
42. CALL TPLOT(X, E0, TT1, ST1, YL1, 36, XL1, 36, 1, 1, 1, 4, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1
43. 1, 1, 1, 2, D, 1) 4096
44. CALL TPLOT(4096, X, FZ, TT2, ST2, YL2, 36, XL2, 36, 1, 1, 1, 4, 1, 1, 1, 0, 0, 0, 0, 0, 1
45. 1, 1, 1, 2, D, 1)
46. CALL TPLOT(4096, X, TZ, TT3, ST3, YL3, 36, XL3, 36, 1, 1, 1, 4, 1, 1, 1, 0, 0, 0, 0, 0, 1
47. 1, 1, 1, 2, D, 1)
The quantities TT1, TT2, TT3, ST1, ST2, ST3 are various labels which find their way onto the final plots. Similarly, YL1, YL2, YL3, XL1, XL2, XL3 are the labels for the X and Y axis. Statement 7 reads the values of EO found in the previous program. Statements 8-10 are the format statements for the previously mentioned read statements.

N2POW represents the number of samples used by the program; i.e.,

\# samples = 2^{N2POW}.

Statements 12-15 set the values of the variables used in the Fast Fourier Transform (FFT). The values of X represent the real part of the waveform and Y the complex part.

Statement 16 calculates the Fourier Transform via use of the subroutine FRXFM (FFT). This calculation yields two arrays, X and Y, which are the real and complex values, respectively, of the Fourier Transform.

Statements 17-26 re-arrange the spectral components because this particular form of the FFT does not produce spectral components in the negative frequency domain but produces the image of these components in the positive frequency domain which exists for \(2 > N/2\), \(N\) being the number of samples employed by the FFT. Hence, the components which lie between \(0 \rightarrow N/2\) are shifted to the region between
\( \frac{N}{2} \rightarrow N \) and those between \( N/2 + 1 \rightarrow N \) are shifted to \( 0 \rightarrow \frac{N}{2} \). In addition, the variables are changed to \( U \) and \( V \) instead of \( X \) and \( Y \) respectively.

Statements 27-30 ascertain the magnitude and phase spectra of the de-chirp waveform.

Statements 35-40 print the values of the variables which are plotted by the commands in statements 42-47.

It should be noted that the output waveform computed by the program in Appendix B has also been plotted by this program.
Figure 2. - (a) Ideal envelope of actual chirp signal, of $T$ seconds duration and chosen to be of unit amplitude. (b) Instantaneous frequency vs. time characteristic of chirp signal; a band of frequencies, $\Delta$, centered at $f_0$ is linearly swept during the pulse duration. (c) Schematic diagram of a signal having the properties indicated in (a) and (b).
FIG. 3 SPECTRAL AMPLITUDE OF A RECTANGULAR CHIRP SIGNAL. THE SHAPE IS SYMMETRIC ABOUT THE POINT \((f-f_0)/\Delta = 0\)
Figure 4. - Network delay vs. frequency characteristic suitable for phase equalization of the chirp signal in figure 2; ideally, this network is chosen to have a flat loss characteristic.
Figure 5. - Envelope of output response from the network in figure 4; this pulse now has a pulse width about $2/\Delta$ and an amplitude increase given by $\sqrt{D}$, where $D=T\Delta$ is called the Dispersion Factor.

$$e(t) = \sqrt{D} \text{sinc} \Delta t \cos(2\pi f_0 t)$$
FIG. 6 (a) THE RETURNS FROM TWO INDIVIDUAL TARGETS.
(b) THE RESULTANT RETURN AS SEEN IN THE RECEIVER
FIG. 8 TIME GATED CHIRP RADAR SYSTEM
FIG. 9 AMPLITUDE OF CHIP PULSE SHAPE IS SYMMETRIC ABOUT THE POINT $t/T = 0$
FIG. 10. THE DE-CHIRP FILTER OUTPUT
FIG. 11A. DE-CHIRP OUTPUT SPECTRUM.
FIG. 11 B. PHASE ANGLE OF THE DE–CHIRP OUTPUT
FIG. 12A AMPLITUDE SPECTRUM FOR THE OPTIMUM SIDE LOBE SUPPRESSION FILTER
FIG. 12B OUTPUT AFTER FIRST STAGE OF EQUALIZATION
FIG. 12C  AMPLITUDE SPECTRUM REQUIRED FOR FIRST STAGE OF EQUALIZATION
FIG. 12D SPECTRUM RESULTING FROM PULSE WIDENING STAGE
### TABLE 1

<table>
<thead>
<tr>
<th>EXPO NET</th>
<th>SIDELOBE LEVEL *</th>
<th>PULSE WIDTH *</th>
</tr>
</thead>
<tbody>
<tr>
<td>-.05</td>
<td>3.4 db</td>
<td>.970</td>
</tr>
<tr>
<td>-.07</td>
<td>1.9 db</td>
<td>.985</td>
</tr>
<tr>
<td>-.10</td>
<td>.8 db</td>
<td>.995</td>
</tr>
<tr>
<td>-.11</td>
<td>0 db</td>
<td>1.00</td>
</tr>
<tr>
<td>-.12</td>
<td>-.1 db</td>
<td>1.04</td>
</tr>
<tr>
<td>-.15</td>
<td>-.15 db</td>
<td>1.08</td>
</tr>
<tr>
<td>-.20</td>
<td>-.18 db</td>
<td>1.13</td>
</tr>
</tbody>
</table>

Normalized to values obtained with the exponent of (-.11).

The Sidelobe Level and Pulse Width of the Output Pulse as a Function of the Exponent of the Exponential Decay Function.
FIG. 12E AMPLITUDE SPECTRUM REQUIRED FOR PULSE WIDENING STAGE
FIG. 12F AMPLITUDE SPECTRUM OF THE SIDELobe SUPPRESSION FILTER. PHASE SPECTRUM EQUAL ZERO.
FIG. 14 AMP. SPECTRUM OF THE SIDELOBE SUPPRESSION FILTER FOR D = 80, Δ = .5, f₀ = 30 mc
PHASE SPECTRUM = 0
FIG. 15
AMP. SPECTRUM OF THE SIDELOBE SUPPRESSION FILTER FOR $D = 20$, $\Delta = 20$, $f_0 = 30 \text{mc}$, PHASE SPECTRUM $= 0$.
FIG. 17. SIDELOBE LEVEL (BELOW MAIN LOBE) vs DISPERSION FACTOR.

\( \Delta_1 < \Delta_2 < \Delta_3 < \Delta_4 \)

\( \Delta_1 = 5\text{MC} \quad \Delta_2 = 10\text{MC} \)

\( \Delta_3 = 15\text{MC} \quad \Delta_4 = 20\text{MC} \)
FIG. 18. PERCENT INCREASE IN PULSE WIDTH vs DISPERSION FACTOR. $\Delta_1 < \Delta_2 < \Delta_3 < \Delta_4$
FIG. 19. SNR LOSS vs DISPERSION FACTOR. Δ₁ < Δ₂ < Δ₃ < Δ₄
VITA

Stephen Honickman was born in New York, New York, on June 1, 1962. He received the B.E.E. degree from the City College of New York, New York, New York, in June, 1962 and the M.E.E. degree from New York University, New York, New York, in September, 1964.

From June, 1962 to September, 1963 he was employed by North American Aviation, Anaheim, California, as a research engineer. From September, 1964 to September, 1967 he was employed by Bell Telephone Labs, Whippany, New Jersey, as a member of the technical staff. Presently he is employed by RCA Labs, Princeton, New Jersey, as a member of the technical staff.

The research work on this paper began April, 1968 and was completed in September, 1970.

Mr. Honickman received the N.C.E. Alumni Scholarship from September, 1967 till May, 1969.