Oscillations in pulse frequency modulated control systems

Warren Joseph Guy

New Jersey Institute of Technology

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BY

WARREN JOSEPH GUY Jr.

A DISSERTATION
PRESENTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE
OF
DOCTOR OF ENGINEERING SCIENCE
AT
NEWARK COLLEGE OF ENGINEERING

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1970
APPROVAL OF DISSERTATION

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CONTROL SYSTEMS

BY
WARREN JOSEPH GUY

FOR
DEPARTMENT OF ELECTRICAL ENGINEERING
NEWARK COLLEGE OF ENGINEERING

BY
FACULTY COMMITTEE

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ABSTRACT

This dissertation is concerned with oscillations and their stability which occur in a control system containing two nonlinearities, separated by linear elements. Specifically the nonlinearities are Integral Pulse Frequency Modulators and the linear elements are described by ordinary differential equations which are linear. The IPFM can be modelled by a quantizer with hysteresis and many other PFM laws are related to IPFM, thus the study applies to more than just IPFM alone.

Boundaries on the system parameters are identified within which free oscillation may be possible. These boundaries give sufficient conditions for stability and necessary conditions for instability. Also since initial conditions play such an important part in the free motion of this class of systems, certain initial condition zones will be identified. These zones give the initial conditions of the unforced system which will ultimately drive the linear plants to the origin (asymptotic stability).

Three types of motion are specifically identified: (1) free oscillation, (2) free periodic oscillation and (3) forced periodic oscillation. Free oscillation, not necessarily periodic, is studied by developing a compound des-
circuit function analysis. This type of analysis will be applicable to all systems of the given configuration and some generalizations may be made beyond the IPFM problem. Free periodic motion is very dependent upon the initial condition of the system with many modes of oscillation possible. The solution of this problem involves the solution of a set of transcendental equations and will be carried out using a modified simplex method.

The system parameters necessary for forced periodic motion are derived and the possible periods and modes of oscillation identified. The stability of the forced periodic motion is then investigated. The results of this investigation yields a set of matrices, conditions on which, if satisfied, will indicate stability in the small of the periodic motion.

Digital and analog computer techniques are used throughout the investigation to verify the theoretical results.
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CHAPTER 1

INTRODUCTION

Various forms of pulse modulated control systems have been studied for some time. The most popular of these have been pulse width modulation (PWM), pulse amplitude modulation (PAM) or sampled data control, pulse position modulation (PPM) and pulse frequency modulation (PFM). Of these only PFM is usually employed without a synchronizing clock. Although today's usage of PWM implies a clock a more general definition would not require this. Such an example of unsynchronized PWM would be a relay control system. This dissertation will be concerned with PFM systems only.

The laws which govern the actual pulse frequency modulator are as numerous as the numbers of researchers working on this class of control systems. The most popular forms are Integral Pulse Frequency Modulation (IPFM) and Neural (or Relaxation) Pulse Frequency Modulation
(N(R)PFM) (39, 43). The characteristics of the IPFM Modulator are considered in detail in Chapter 2 and will be the type considered throughout this dissertation.

The other types of modulators which have been characterized and studied are briefly described below.

Neural Pulse Frequency Modulation (NPFM), often referred to as Relaxation PFM, operates like a relaxation oscillator. The input to the modulator, \( e_{in}(t) \), is fed into a linear element with transfer function:

\[
\frac{E_R(s)}{E_{in}(s)} = \frac{1}{1 + sT_R}
\]

When the output of this linear element, \( e_R(t) \), first reaches a prescribed value, an impulse is emitted from the modulator and \( e_R(t) \) is reset to zero. The mathematical relationships are:

\[
e_R(t) = \text{Linear element output.} \\
T_R = \text{Linear element time constant.}
\]

Then,

\[
e_{in}(t) = e_R(t) + T_R \dot{e}_R(t) \\
= \text{Modulator input signal}
\]
\[ e_R(t) = A b_k \quad t = t^-_k \]
\[ e_R(t) = 0 \quad t = t^+_k \]

Thus \( t_k \) = Time of impulse firing.

\( b_k \) = Impulse polarity = \( \text{sgn}(e_R(t_k^-)) \).

\( A \) = Threshold for impulse emission.

\( u(t) = \text{Modulator output} \)

\[ = M \sum_{k=1}^{\infty} b_k \delta(t - t_k) \]

\( \delta(t) \) = Impulse function

\( M \) = Strength of the output impulse

This PFM scheme is the second most popular and has received extensive study (43, 45, 51).

A modified IPFM has been studied (4) which works in a manner similar to the normal IPFM (see Chapter 2). Its action is described by the following equation:

\[ \left| \int_{t_{k-1}}^{t_k} e_{in}(t) \, dt \right| = A + a(t_k - t_{k-1}) \]

\( A \) and \( a \) are constants, \( e_{in}(t) \) is the modulator input and \( t_k \) the firing times. When the relationship is satisfied, the modulator fires; i.e., an impulse is emitted and the integral is reset to zero.
The above methods of modulation have been generalized in Sigma Pulse Frequency Modulation (ΣPFM) (52). This scheme is described by the following relationships:

\[ p(t) = - \sum_{k=1}^{n-1} \text{sgn}(p(t_k^-)) + \int_{t_{n-1}}^{t_n} g(p) \, dt + \int_{t_{n-1}}^{t_n} e(t) \, dt \]

when \(|p(t)| = A\), the modulator fires, emitting either positive or negative impulses depending upon the polarity of \(p(t_k^-)\). The modulator output is then given by;

\[ u(t) = \sum_{k=1}^{\infty} \text{sgn}(p(t_k^-)) \delta(t-t_k) \]

g(p) can theoretically be any function of \(p(t)\). In particular if \(g(p)\) is taken to be an odd nondecreasing function, the resulting modulation is referred to as ΣPFM. Note that if \(g(p)\) is zero or a constant, then the modulator type reduces to IPFM or NPFM respectively.

While the methods indicated thus far have acted upon a continuously varying input signal, these are not the only types considered. Some modulators act in a discrete or sampled manner.

Sampled Pulse Frequency Modulation (SPFM) (7, 8, 34, 35) works on the input by first sampling it. When a sampling takes place a pulse is emitted from the modulator. The sampling instants are determined by the level of the input signal at the previous sampling instant. Described
mathematically:

\[ u(t) = M \sum_{k=1}^{\infty} (u_{-1}(t-t_k) - u_{-1}(t-t_k-\tau)) \text{sgn}(e(t_k)) \]

\[ u_{-1} = \text{Unit step function} \]

\[ t_{k+1} = t_k + mT \]

\[ m = f(e(t_k)) \]

Note that in this case the output has been taken to be finite pulses of width \( \tau \).

Delta modulation has been defined in various ways. Sometimes the definitions used define the emission of a pulse when the input signal goes beyond a certain threshold level \((2, 42)\). In other papers delta modulation is characterized by a clock pulse, whose polarity depends upon the input signal and some reference which varies. A combination of the above two schemes would give a pulse output at clock instants, the polarity being positive, negative or zero. This would give a quantized PFM signal \((59, 60)\).

The particular studies to date of PFM systems have incorporated only one modulator. Of particular interest is Meyer's work \((43)\) which first identified equilibria and periodic states in IPFM systems, incorporating one modulator. He derived theorems on the existence of equilibria and periodic states and shows exact methods for
their identification. IPFM system equilibria is not necessarily static but is more likely to follow some cyclic pattern. Meyer linearizes the IPFM system about the periodic motion and examines the linearized system for stability in the small. The linearized system about a periodic operation is a standard linear sampled data system to which \( Z \) transform technique can be applied to test for stability.

At the same time as Meyer was studying the IPFM system, Li (39, 40) was developing the describing function for the IPFM modulator. This describing function will be discussed and extended to a two modulator system in Chapter 3.

Pavlidis and Jury (52) examined the existence of a sustained oscillation using a quasi describing function for their \( \Sigma \)PFM. The input to the modulator is a square wave and the equivalent gain is then determined by the ratio of the output fundamental sinusoid to the input fundamental component of the square wave. This method gave the authors good agreement with analog studies, however it is to be avoided if the system has a small loop gain or the input to the modulator has a wave shape that differs drastically from the assumed square wave. If for example the linear plant contains an integrator, the quasi describing function will always predict a sustained oscillation. For gains sufficiently small however, none can exist.
Dymkov (16) in 1967 essentially repeated Li's work, however he did compare the describing function method and the quasi describing function method proposed by Jury and Pavlidis. He argued that for high order linear plants the output would resemble a sinusoid and not a square wave and thus the standard describing function is a better technique for predicting stability and oscillation.

Pavlidis continued the study of pulse frequency modulated control systems (54, 56). In (54) he developed five theorems which are based on the second method of Lyapunov to investigate stability (which he defined as having the system trajectories stay "close" to some set M. To be completely stable, the trajectory must be contained within a set M after some finite time). Instead of assigning the Lyapunov function (V) as a function of the state, Pavlidis has assigned it a value depending on the particular trajectory. He then shows the necessary conditions on V to insure stability or complete stability. His last two theorems use this same V to give conditions for sustained self oscillation. His paper (56) is essentially the same; he treats only the completely stable system and applies his result to a two modulator system - the modulators are in parallel in an arrangement similar to the type studied by Li (39).
Clark and Noges (8) examined the stability of SPFM systems which they had defined. Because of the nature of this modulator, a pulse will always be emitted even with no input, thus the equilibrium cannot be asymptotically stable. However they do develop two interesting theorems based on the second method of Lyapunov which can be useful in defining the bounds of an oscillation around the origin. Their paper also gives a recurrence formula for computing the system state at the pulse instants.

Kuntsevich and Chekhovoi (34) have also studied the stability of SPFM systems using the Lyapunov function:

$$V = x_{n} S' S\ x_{n}$$

Their system, similar to the SPFM, also includes a nonlinear dead zone element in the modulator. Thus unlike the Clark and Noges system, this one will have a static equilibrium because of the dead zone. The results of the paper give conditions on the linear plant and establish the width of the dead zone for stability. In this case stability is used in the Lyapunov sense; i.e., the state trajectory tends to some equilibrium set. In a later paper (35) Kuntsevich and Chekhovoi used the same Lyapunov function to investigate the stability in the large of systems with both pulse width and pulse frequency modulation.
The most recent published work on the stability of IPFM systems using Lyapunov techniques has been by King-Smith and Cumpston (31). They develop conditions for boundedness of motion. By assuming that the time between successive impulses has a maximum value, they justify the replacement of the modulator with a linear gain element. The results about the resulting linear system stability are obvious. In a second paper (32) they develop many theorems in state vector form about the existence and stability of equilibria and steady state operation. These results are very similar to Meyer's work (43).

Two papers have appeared using Popov's stability criterion in some way to investigate stability limits of a PFM system in the frequency domain. The first of these is by Dymkov (15). He applies the Popov criteria directly to a system using a nonlinear element with hysteresis to simulate PFM. His conclusions are a restatement of the Popov Theorem applied to this particular class of systems.

The most recent paper on NPFM stability using Popov's method was presented by Monopoli and Wylie (45). The authors changed the nonlinear hysteresis element in the system to make it more realizable and then reapplied the work of Dymkov (15). However they applied the theorem in a less restrictive form than Dymkov. Their modulator output is more neural like also.
As a small departure from the traditional studies, Jury and Blanchard (29) introduced a notation to attempt to unify all PFM systems and show that they are really non-linear discrete systems. They use their idea specifically on IPFM. They illustrate a phase plane method for studying IPFM systems. When the state trajectories intersect predetermined curves, the modulator will emit an impulsive signal and the state experiences a discontinuous change. The method is very descriptive but the threshold curve for pulse emission has to be recomputed after each firing. Finally they suggest Lagrange stability be used when determining stability in the large of IPFM systems. Asymptotic stability in the Lagrange sense implies boundedness; ie, starting at any initial condition, the trajectory will enter a set V and thereafter remain in a set U. Both U and V are bounded and closed and \( V \subset U \).

To date only a few articles have appeared on noise in integral pulse frequency modulated control systems, although noise immunity is one of the major virtues of such systems. C.C. Li (39) considered the problem of noise being superimposed on a pulse train output from the single signed IPFM (a noisy channel). He preceded his plant (the receiver) with a demodulator which emits a pulse when the signal input to it is positive above a certain threshold level and has a positive time derivative. Because he also now assumes finite
pulses, there exists the possibility of a signal pulse being shifted from its true value by some small time, being missing or an extra pulse appearing. Li concentrates on extra (or false) and missing pulses and develops the normalized average mean error and the normalized mean square error. These two quantities then give a measure of the number of error pulses from the demodulator.

Bombi and Ciscato (4) take up the problem of chatter (shifting of pulse from the noise free position) which Li mentioned. They introduce the noise at the input to the modulator only. The input signal to the modulator is taken to be constant with gaussian noise superimposed. Their results give a probability density function for the jitter times. The function is almost gaussian.

Hutchinson et al (26) attacked a similar problem. They introduced white noise only into the input of the modulator (IPFM and ΣPFM) and found the average number of output pulses per unit time. They checked their results experimentally and found close agreement.

The last and most recent paper dealing with noise or uncertainty in IPFM systems was by Bayly (3). Using parallel channels, each of which has an IPFM and linear plant, the outputs of which are summed and passed through a low pass (averaging) filter; he showed that the output would be
an improvement in the signal to distortion ratio over that
given by a single channel. This is true even if the mod­
ulators fired at different times. His theme being to show
increased reliability in parallel neural paths.

Standard optimal techniques (Maximum Principle, cal­
culus of variations, etc) cannot be used directly on PFM
systems. This is because of the impulse nature of the
modulator output or the period of uncontrollability of the
system. By this is meant that once the pulse (of finite
height and duration) is started, the system cannot be con­
trolled until the completion of the pulse and some pos­
sible refractory period.

Pavlidis (55) used heuristic arguments to find the
optimum control for a minimum time and minimum fuel problem.
Also since the minimum fuel solution is not unique, he
found the minimum time-fuel control. He justified using
bang-bang control into a PF modulator and then by means of
an example showed a method for calculating the switching
curve for a second order system.

Stoep (59) considered the optimization of a linear
system with a PF controller. His performance index was
based on final state error and the energy consumed to get
there. He attacked the problem indirectly by arguing that
continuous control will give a lower bound on the performance index and discrete pulse frequency modulation an upper bound. Stoep showed that for the problems considered the final difference between the bounds is very small. Thus he argued in favor of a discrete PFM system which he feels is easier to implement.

The most recent work is by Onyshko and Noges (48, 49). In their first paper they modified the Maximum Principle to take into account the uncontrollable period. In the second paper they used Dynamic Programming, but have had to modify the PF controller to fit the method.

IPFM and RPFM originally appeared as an analogy to the neural system (27), with the resulting theoretical work centered around a single neuron (modulator) in the system. This dissertation, admittedly motivated by the IPFM neural analogy, will attempt to extend the theoretical basis of IPFM systems to those containing two modulators. In particular, oscillations in this class of systems will be investigated. The idea for this study comes from simple neural networks found in man, but primarily in lower forms of life. Examples of such nets are found in a nine series neuron connection which controls the heart rate of a lobster (66), in the neural excitation of the muscles of the wings of a locust (65), or in the electrical discharge in
fish which use the resulting electric field for navigational purposes (66). Such oscillatory behavior is also thought to be responsible for the short term memory in humans. It need hardly be emphasized that no direct neurophysiological questions are to be answered in this dissertation.

Besides the almost obvious application to neurophysiology, IPFM has been used as a satellite control mechanism (10, 18, 64) where the impulse of the rocket thrust is considered as the output of the modulator. The PFM also has obvious applications when used in controlling the motion of a stepping motor (1, 19, 30). PFM has also been used in the feedback path of an adaptive aircraft flight control system (46). There have also been many papers written on the use of PFM for telemetry (41). IPFM has been incorporated as an A-D converter in a systems analyzer (17). However the full potential of PFM, in particular IPFM, has not been fully realized from an engineering point of view (24, 61).

Popularity for combination pulse techniques is coming into vogue. In particular PFM/PWM is a marriage which offers promise. The combination is a difficult one to analyze and to date has only been studied using the direct method of Lyapunov (35). The general theme for such a control
system is to use PFM for fine control and PWM for coarse control. Thus expending small amounts of energy when the error is small (PFM) and large amounts of energy when the error is large (PWM). Since a system hopefully operates with small error most of the time, a more detailed study of PFM certainly seems justified.
CHAPTER 2

THE DOUBLE INTEGRAL PULSE FREQUENCY MODULATED CONTROL SYSTEM

The problems of nonlinear control systems containing a single nonlinearity are extremely well documented. Indeed, work on this class of systems is still the primary emphasis in the nonlinear control theory. Less popular is the study of systems containing two nonlinearities. In this dissertation a system with two nonlinearities will be studied. In the present chapter the nonlinear elements and the total system are defined and the basic problem to be considered is stated. Then methods to find the total response to arbitrary inputs and initial conditions are indicated and finally the stability of this type of control system is discussed.

2.1 The Integral Pulse Frequency Modulator (IPFM)

The nonlinear element to be used is a modulator which was first introduced by C. C. Li (39) and A. U. Meyer (43). It consists of an integrator which integrates the input to the modulator (and thus its name) and an impulse emitter. When a certain prescribed magnitude, $A$, of the modulator
integrator is reached, the emitter produces an impulse and the integrator is reset to zero, ready to start integrating again. The polarity of the emitted impulse is positive if the integrator was +A at the time of emission and negative if the integrator was at -A; the impulse strength does not change and is constant at the value M. Thus control and/or information is carried by the pulse separation. Described mathematically the governing equations are:

\[
A = \int_{t_{k-1}}^{t_k} e(t) \, dt \\
\tag{2-1}
\]

\[
b_k = \pm 1 = \text{sgn} \int_{t_{k-1}}^{t_k} e(t) \, dt \\
\]

\[
u(t) = M \sum_{k=1}^{\infty} b_k \delta(t-t_k)
\]

where

- \(A\) = Integral threshold value (value of the integral for impulse emission).
- \(t_k\) = Time of impulse emission and integrator being reset to zero.
- \(e(t)\) = Input signal to modulator.
- \(b_k\) = Polarity of the emitted impulse at \(t_k\).
- \(u(t)\) = Modulator output signal which is a series of impulses.
- \(M\) = Strength of the emitted impulses.
- \(\delta(t)\) = Impulse function.
An illustration of the integrating and impulse emitting action is shown in figure 2.1. Also shown in the symbol for IPFM as adopted by Li and Meyer and used here without change.

Note also that the firing of the modulator takes place at the minimum time which satisfies the threshold conditions. It is possible for the modulator integral to satisfy the threshold conditions many times in some time interval, but only the first time it reaches "A" will be used to determine the impulse emission time.

Meyer (43) showed that such a modulator can be modelled by an integrator followed by a quantizer with hysteresis and then a differentiator. Thus the analysis and ideas developed for this class of PFM systems has a wider application than just IPFM control mechanisms.

2.2 The Two IPFM System

The system configuration to be studied in this dissertation is shown in figure 2.2. The Integral Pulse Frequency Modulators 1 and 2 may have different thresholds and impulse strengths. Their operation is discussed in the previous section.

The linear elements will be characterized by linear ordinary differential equations with constant coefficients.
Figure 2.1: Illustration of Integral Pulse Frequency Modulation.
Figure 2.2  Double Integral Pulse Frequency Modulated Control System.
Thus when these plants \((G_1(s)\) and \(G_2(s)\)) are cast into the transfer function form, the result will be a ratio of polynomials in "s" with real coefficients. The normal additional requirement that the order of the polynomial of the numerator be at least one less than that of the denominator also holds for each plant. Further restrictions will be noted as the need arises. Then \(G_1(s)\) and \(G_2(s)\) can be written with the following general transfer function:

\[
G_i(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}
\]  

(2-2)

Also note that the following standard form will be used:

\[
g(t) = L^{-1}(G(s))
\]

\[
t > 0
\]

= Impulse response of the linear element.

\[
t < 0
\]

= 0

Because of the type of system considered it will be necessary to deal with two different actions occurring at the same time. Thus those phenomena relating to IPFM 1 and \(G_1(s)\) will be characterized with a subscript "1", while those relating to IPFM 2 and \(G_2(s)\) will have the subscript "2". For example:

\[
G_1(s), \ G_2(s) = \text{Transfer function of the forward path and feedback path linear plants.}
\]
\( g_1(t), g_2(t) = \) Impulse response of \( G_1(s) \) and \( G_2(s) \) respectively.

\( y_1(t), y_2(t) = \) Outputs from \( G_1(s) \) and \( G_2(s) \) respectively.

\( A_1, A_2 = \) Threshold values for IPFM 1 and IPFM 2 respectively.

\( M_1, M_2 = \) Impulse strength of IPFM 1 and IPFM 2 emitted impulses.

\( u_1(t), u_2(t) = \) Output (impulse train) from IPFM 1 and IPFM 2.

Also terms will appear with double subscripts. Thus with the first subscript indicating the respective element:

\( t_{1k} = \) Time of the kth impulse from IPFM 1.

\( b_{1k} = \) Polarity of the kth impulse from IPFM 1.

\( t_{2k} = \) Time of the kth impulse from IPFM 2.

\( b_{2k} = \) Polarity of the kth impulse from IPFM 2.

Note that \( t_{1k} \neq t_{2k} \) or \( b_{1k} \neq b_{2k} \) necessarily; ie, the kth pulse emitted from modulator one need not correspond in time or polarity with the kth pulse emitted from modulator two.

The main theme in this dissertation will be the identification of conditions for oscillation in the defined system incorporating two IPFM modulators. The state of oscillation will be identified with the pulse patterns emitted from the two modulators. Thus for a fundamental
period of oscillation, $T$, the modulators will emit a certain number of impulses. This will be defined as the $D_{mn}$ condition when "m" is the number of impulses emitted from IPFM 1 and "n" is the number emitted from IPFM 2 in period $T$. In the unforced case for example, where the average signal must be zero over the fundamental period, $D_{42}$ would indicate 4 pulses from modulator 1, 2 positive and 2 negative. Modulator 2 would emit 2 impulses, one positive and one negative. As an example of this pattern see the modulator signals shown in figure 2.3a. Under forced conditions, the pattern can take on any number of positive and negative impulses. An example of forced oscillation is shown in figure 2.3b.

After examining the response of the system in general and its stability, attention will be focused on the conditions for oscillation in the unforced and forced cases.

2.3 Solution of the Total Response of the IPFM System

Total solution of the control system response can be carried out by exact hand computation, simulation on the analog or hybrid computer, or by approximate means on the digital computer. All methods have been tried, each with their respective advantages and disadvantages. Hand calculation is the most tedious of course; but for
Figure 2.3 Examples of Dmn operation in the Double IPFM Control System.
low order linear plants is satisfactory, especially if computer facilities are not readily available. The digital computer, while usually the most readily available, requires considerable computing time because of the two integrators in the modulators and the necessity to constantly compare their values with some threshold level. Also note that it is necessary to run two systems simultaneously; i.e., $G_1(s)$ and $G_2(s)$. The analog or hybrid computers offer a good compromise, but are accompanied by scaling problems and availability.

Some emphasis is given to the calculation of the response of the system since it is felt that by this means greater understanding of the system operating characteristics can be obtained.

2.3.1 Calculation of the total response. Calculation by hand requires that the initial conditions of each linear element and the input to the system be given. The procedure for the solution is shown in the chart in figure 2.4 and basically forms the flow chart for the digital computer solution also. The hand calculation will now be illustrated by means of a simple example. The example and the resulting solution are given in figure 2.5.
Figure 2.4 Flow chart for the calculation of the total response of the double IPFM control system.
Figure 2.5 Example of double IPFM system response.
Figure 2.5 Continued
For 0 < t < t_{i1}, i = 1, 2, 
\[ e(t) = r(t) - y_{20}(t) = 1 - e^{-2t} \]
\[ y_1(t) = y_{10}(t) = 10 e^{-t} \]

where \( y_{10}(t) \) and \( y_{20}(t) \) are respectively the initial conditions responses of the linear plants \( G_1(s) \) and \( G_2(s) \). Calculation of the IPFM integrals will then determine which modulator fires first; i.e., reaches its threshold first.

\[ A_1 = \left| \int_0^{t_{11}} e(t) dt \right| = \int_0^{t_{11}} (1 - e^{-2t}) dt \]
giving \( t_{11} = 1.374 \) and \( b_{11} = +1 \).
\[ A_2 = \left| \int_0^{t_{21}} y_1(t) dt \right| = \int_0^{t_{21}} (10e^{-t}) dt \]
giving \( t_{21} = 0.110 \) and \( b_{21} = +1 \).

Since \( t_{21} \) occurs first, the \( t_{11} \) calculated above is not true since the input to IPFM 1 has changed by the impulse response of \( g_2(t) \) at \( t_{21} \). At \( t_{21} \) the IPFM 2 modulator integrator is reset to zero, but the IPFM 1 modulator still has a residue value which must first be determined.

\[ \left| \int_0^{t_{21}} e(t) dt \right| = \int_0^{0.11} (1 - e^{-2t}) dt = 0.027 \]
This value must be retained during the next time phase.

Thus from $t_{21}$ forward to the next impulse:

$$
A_1 = \left| 0.027 + \int_0^{t_{11}} (1 - e^{-2t} - 3e^{-2(t-0.11)}) \, dt \right| = 1.0
$$
giving $t_{11} = 3.00$ and $b_{11} = +1$.

$$
A_2 = \left| \int_{0.11}^{t_{22}} 10 \, e^{-t} \, dt \right|
$$
giving $t_{22} = 0.230$ and $b_{22} = +1$. Note again that $t_{22}$ occurs before $t_{11}$, thus IPFM 2 fires again and the IPFM 1 integral must be carried on to the next time span. Calculation continues in this manner until IPFM 1 fires, at which time the IPFM 2 integral must be calculated and retained into the next time span with subsequent changes in the input to IPFM 2 caused by the addition of the impulse response $g_1(t)$ at $t_{11}$. Note that it must be true that the firing of the modulators must be the minimum time at which the modulator integral satisfies the threshold conditions. This requirement may present some computational difficulty if multiple roots of the IPFM integral equation exist.

If one is willing to accept more approximate results, the following scheme is useful especially if the firing time determinations are dependent upon the solution of
a difficult transcendental equation. The method is as follows:

Pick the total response time of interest and some incremental time smaller than the smallest time constant, natural or forced oscillatory period of the linear elements. Calculate the integral of the impulse response of each linear element for the accumulating time (in steps of the incremental time) to the total running time. After a certain time, this value should reach a constant so that the subsequent calculations will not be necessary. Also do the same for the initial condition response and the input, \( r(t) \). Next set up a chart with time incremented vertically as in figure 2.6 which has been set up for the same example as in figure 2.5. Calculation now proceeds by adding horizontally the values of the modulator integrals at the time increments. When the integral of IPFM 1 or IPFM 2 reaches a multiple of the threshold for the first time, an impulse is fired and an additional column is added to the input to the other modulator. For many linear plants with reasonable damping, after a few time constants, their integral values do not change and thus the calculation is made somewhat easier. The method does not give exact results, the accuracy depending only upon the incremental time selected.
### Figure 2.6 Time incremented calculation for the total response of example 2.1. * indicates a pulse emission.

<table>
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<tr>
<th>Time</th>
<th>[\int r_{dt} - \int y_{20} , dt - M_2 b_{21} \int g_2 , dt ]</th>
<th>Summation</th>
</tr>
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<td>0</td>
<td>0</td>
</tr>
<tr>
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<tr>
<td>0.2</td>
<td>0.2 -0.17</td>
<td>-0.27</td>
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<tr>
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<td>0.4 -0.28</td>
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</tr>
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-0.24
-0.70*
-1.33
-1.83*
-2.21
-2.51
-2.90*
-3.20
-3.70
-3.93*
-4.10
-4.22
-4.45
-4.36
-4.37
-4.34
-4.31
-4.28
-4.24
-3.88
-3.53
-3.37
-3.14
-2.95*
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<tr>
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Figure 2.6 Continued
As the time increment chosen approaches zero, the results become more exact, but the work more tedious.

The final method of calculation by hand can be done by the classical graphical method of adding squares from the graphs of the initial condition, impulse and input signals. This method will not be discussed since its procedure parallels the others, only differing in the way in which the modulator integrals are evaluated.

2.3.2 Digital Computer calculation of the total response. For the calculation of the total response by the digital computer, the state variable technique was used. This now requires some new definitions of terms for the compound IPFM system.

Each of the linear dynamic plants will be described by the differential equation of the form:

\[ y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y^{(1)} + a_0y = b_mu^{(m)} + \ldots + b_1u^{(1)} + b_0u \]  

where \( y^{(k)} \) and \( u^{(k)} \) are the kth derivatives of the output and input of the linear plant respectively. Also \( m<n \). Note that there are two plants and thus two equations will have a subscript indicating which linear element it represents. By well known techniques (see Appendix A) the above equation can be reduced to a collection of n
first order differential equations and cast into the matrix form:

\[
\begin{align*}
\dot{x}(t) &= A \cdot x(t) + b \cdot u(t) \\
y(t) &= c^T \cdot x(t)
\end{align*}
\]

(2-4)

where

\[
\begin{align*}
x(t) &= \text{nxl state vector} \\
A &= \text{nxn constant matrix} \\
b &= \text{nxl constant input vector} \\
u(t) &= \text{Input to the linear element} \\
y(t) &= \text{Linear element output} \\
c^T &= \text{Transpose of an nxl output vector.}
\end{align*}
\]

The solution of the above equations then may be written in the following form (see Appendix A),

\[
x(t) = e^{-A(t-t_0)} \cdot x(t_0) + e^{-A(t-t_0)} \int_{t_0}^{t} e^{-A(t-q)} \cdot b \cdot u(q) dq
\]

(2-5)

\[
e^{At} = \text{nxn state transition matrix}
\]

and

\[
y(t) = c^T \cdot x(t)
\]

For the particular case where \( u(t) \) is always a train of impulses, \( x(t) \) may then be written in the following form (see Appendix A):

\[
x(t) = e^{-A(t-t_0)} \cdot x(t_0) + \sum_{k=1}^{\infty} e^{-A(t-t_k)} \cdot b_k
\]

(2-6)

\[
y(t) = c^T \cdot x(t)
\]

\[
t > t_0
\]
$t_k$ and $b_k$ are the impulse times and polarities. Instead of calculating the summation of many matrix multiplications in the above equation, at each impulse firing a new initial condition is determined and the calculation proceeds as though it were a new problem. Now only the homogeneous solution is required with changing initial conditions of the linear plants and an additional initial condition on the modulator integrals. The equations then take on the form:

$$x(t) = e^{At} x_0 \quad t_k < t < t_{k+1}$$

where

$$x_0 = x(t_k^-) + M b_k b$$

Also the output still has the same form as in (2-6). The initial condition on the modulator which did not fire at $t_k$ will be:

$$x_i = \int_{t_{i(last fire)}}^{t_{ik}} (\text{Modulator Input}) dt$$

with

$$i = 1 \text{ for modulator IPFM 2 firing}$$
$$i = 2 \text{ for modulator IPFM 1 firing}$$

The flow chart for the digital computer program for these calculations is given in Appendix B. The above results must be extended to both modulators and plants and thus the program becomes somewhat more complex than has been indicated above. Two examples are now given; one is the same as has been previously calculated by hand and the
other is a system of higher order linear plants.

Example 2.1 The computer input data for the solution to this example is:

\[
\begin{align*}
A_1 e^{-t_1} &= e^{-t_1} & A_2 e^{-2t_2} &= e^{-2t_2} \\
b_1 &= 1 & b_2 &= 1 \\
c_1 &= 1 & c_2 &= 1 \\
A_1 &= 1 & A_2 &= 1 \\
M_1 &= 2 & M_2 &= 3 \\
x_{10} &= 10 & x_{20} &= 1 \\
r(t) &= 1.0
\end{align*}
\]

The output is shown in figure 2.7.

Example 2.2 The system configuration and results are shown in figure 2.8. The input data is:

\[
\begin{align*}
A_1 e^{-t_1} &= \begin{bmatrix} -t_1 & -0.5t_1 & -t_1 & -0.5t_1 \\ -e^{-t_1+2e} & 2e^{-t_1+2e} \\ -e^{-t_1-0.5t_1} & -e^{-t_1-0.5t_1} \\ 2e^{-t_1-e} & 2e^{-t_1-e} \end{bmatrix} \\
A_2 e^{-2t_2} &= \begin{bmatrix} -0.8t_2 & -0.7t_2 & -0.8t_2 & -0.7t_2 \\ -7e^{-0.8t_2+8e} & -10e^{-0.8t_2+10e} \\ -0.8t_2 & -0.7t_2 & -0.8t_2 & -0.7t_2 \\ 5.6e^{-0.8t_2-5.6e} & 8e^{-0.8t_2-7e} \end{bmatrix} \\
b_1 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} & b_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
c_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} & c_2 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{align*}
\]
Figure 2.7 Results for example 2.1 from the digital computer solution.
a) System Configuration

b) System response

Figure 2.8  Example 2.2 solution
2.3.3 Analog computer calculation of the total response.

The analog computer solution of an example is shown in figure 2.9. The circuit arrangements for the use of this type of computer is to be found in Appendix C.

Correlation between the results from the various methods of calculation is good. The means by which the response is determined is a matter of convenience and availability of equipment.

2.4 Stability Considerations for the Double IPFM System

In this section boundaries are obtained on the parameters of the double IPFM system for system stability (see definition 2.1 below). System stability will loosely mean that impulses cease to be emitted from both the modulators; while instability will be taken as a condition where the pulse time intervals become shorter and shorter. Stated more formally:

**Definition 2.1** If there exists a time $T$, such that for $t>T$, both modulators have zero output; ie,

$$u_i(t) = M \sum_{k=1}^{K} b_{ik} \delta(t-t_{ik}) \quad t>T$$
Figure 2.9a Analog computer output for example described above. Input = 4.0.
Figure 2.9b Analog computer output for example described above. Input = 0.1.
\[ u_i(t) = 0 \quad t > T \]

\[ i = 1, 2 \]

then the two IPFM system is said to be system stable.

Note that this definition allows for small oscillations to exist at the output of the linear elements (ripple) provided that,

\[ \left| \int_T^t (\text{Modulator input})_k dt \right| < A_i \]

\[ i = 1, k = 2 \]

or \[ i = 2, k = 1 \]

for all time \( T < t < \infty \)

This definition is motivated by the neural analogy and is consistent with neural phenomena where the impulse is of prime interest and other action in the system may tend to lower or increase the threshold of firing (modulator integral threshold in the IPFM case).

**Definition 2.2** If there exists some time \( T \), such that for \( t > T \), the impulse time intervals for both modulators approaches zero; ie,

\[ \Delta t_k = t_k - t_{k-1} \]

\[ \Delta t_k \rightarrow 0 \quad \text{for} \ t > T \]

then the double IPFM system is said to be system unstable.

Li (39) and Meyer (43) showed that when the input to the modulator became large and the pulse rate subsequently
also becomes large, the modulator may be replaced with an equivalent linear gain, \( K = M/A \). By replacing both modulators with their respective equivalent large signal gains, \( K_1 = M_1/A_1 \) and \( K_2 = M_2/A_2 \), the system becomes linear, see figure 2.10. For large inputs to the modulators and high pulse rates, the approximate analysis may be carried out in this linear mode.

If the system parameters are such as to increase the modulator inputs and pulse rates; ie, tend toward system instability, then there will exist a time \( T \) such that the linear equivalent gain is a good approximation of the system. If the modulators are then treated as linear gains, further increase in the linear dynamic outputs will make the approximation better by increasing the signal levels and pulse rates. Now if the linear equivalent system is

![Figure 2.10 The linear equivalent IPFM system.](image-url)
unstable the increasing signal levels and pulse rates will continue. Thus if the system stability boundary of the linear system is found, gains greater than this will be sufficient to cause the two IPFM system to be system unstable. For gains less than the boundary value, the system may either oscillate or be system stable. Finding the linear equivalent system gain for instability is a simple matter since any of the common linear techniques can be used (Routh-Horowitz, Root Locus, etc.). The above observations lead to the following theorem:

**Theorem 2.1** The necessary conditions for system instability for the double IPFM system correspond to conditions for instability in the linear equivalent system obtained by replacing the IPFM modulators with their large signal linear equivalent gain:

\[
\text{IPFM } 1: \quad K_1 = \frac{M_1}{A_1} \\
\text{IPFM } 2: \quad K_2 = \frac{M_2}{A_2}.
\]

This phenomena is illustrated by the following examples continued from the previous sections.

**Example 2.3** This is example 2.1 continued. The total loop transfer function with the modulators replaced by their equivalent gains is:

\[
G(s) = \frac{K_1 K_2}{(s + 1)(s + 2)}
\]
The system then has the characteristic equation:

\[ s^2 + 3s + (2 + K_1K_2) = 0 \]

By the Routh-Horowitz method it is found that this linear system is always stable and thus the two IPFM system will always be system stable or oscillate, but never will it be unstable.

Example 2.4 This is example 2.2 continued. The total loop transfer function with the modulators replaced by their respective equivalent gains is:

\[
G(s) = \frac{K_1K_2}{(s + 1)(s + 0.5)(s + 0.7)(s + 0.8)}
\]

The characteristic equation for the system is:

\[ s^4 + 3s^3 + 3.31s^2 + 1.59s + (K_1K_2 + 0.28) = 0 \]

Again using the Routh-Horowitz approach, the equivalent linear system has a marginal gain of \( K = K_1K_2 = 1.19 \).

For gains greater than this the system may be unstable and for gains less than 1.19 the system will oscillate or be system stable.

The above theorem 2.1 gives an upper bound on \( K_1K_2 \) (the modulator constants) or the linear system parameters. It is also appropriate to seek a lower boundary which will guarantee stability. The range of values, \( K_1K_2 \) for example, between the boundaries thus determined may cause the system
to be stable or oscillate. To determine the sufficient conditions for system stability for one class of systems (close to the physiological type) the following theorem is stated, the derivation may be found in the Appendix D. This theorem is based on an unpublished theorem by Meyer (43) on the single IPFM system.¹

**Theorem 2.2** If the absolute values of the initial condition responses and the impulse responses of both linear elements have finite integrals for all time; ie,

\[
y_{10}(t), y_{20}(t), g_1(t), g_2(t) \in L_1(0,\infty)
\]

where \( y_{i0}(t) \) is the initial condition response of the \( i \)th linear element, then for arbitrary initial conditions, if

\[
\int_0^\infty |g_1(t)| \, dt \int_0^\infty |g_2(t)| \, dt < \frac{A_1 A_2}{M_1 M_2}
\]

the unforced system will be system stable in terms of the definition 2.1.

**Corollary 2.2.1** If the system has all the poles of each linear element in the left half plane and Theorem 2.2 is obeyed, then all motion must go to the origin.

**Corollary 2.2.2** If \( g_1(t) \) and \( g_2(t) \) are single signed, then for stability it is sufficient that:

---

¹ As of the typing of this dissertation, generalized stability theorems for Pulse Modulated systems with one modulator have been published (68). This generalized version includes Meyer's result as a special case.
\[
\int_0^\infty g_1(t)dt < \frac{A_2}{M_1}
\]

and

\[
\int_0^\infty g_2(t)dt < \frac{A_1}{M_2}
\]

The proof of the first corollary is merely the application of the definition of system stability to a special case which has physiological interest. Proof of the second corollary is to be found in Appendix D.

The following examples continue to extend those first introduced in section 2.3.

Example 2.5  Example 2.1 continued.

\[
\begin{align*}
L_{G_1} &= \int_0^\infty \left|e^{-t}\right|dt = 1 \\
L_{G_2} &= \int_0^\infty \left|e^{-2t}\right|dt = \frac{1}{2}
\end{align*}
\]

Thus \(L_{G_1}L_{G_2} = \frac{1}{2} < \frac{A_1A_2}{M_1M_2}\)

or for system stability,

\[M_1M_2/A_1A_2 = K_1K_2 < 2\]

For free oscillatory behavior then \(K_1K_2\) must be greater than 2.

Example 2.6  Example 2.2 continued.

\[
L_{G_1} = \int_0^\infty \left|-2e^{-t} + 2e^{-0.5t}\right|dt = 2
\]
\[ L_{G_2} = \int_{0}^{\infty} (-10 e^{-0.8t} + 10 e^{-0.7t}) dt \]
\[ = 1.79 \]

Thus \( L_{G_1} L_{G_2} = 3.57 < A_1 A_2 / M_1 M_2 \)

or for system stability,
\[ M_1 M_2 / A_1 A_2 = K_1 K_2 < 0.28 \]

From the previous results for this example the range for oscillatory behavior should fall between the boundaries of system stability and instability; ie,
\[ 0.28 < M_1 M_2 / A_1 A_2 < 1.19 \]

Note that these boundaries do not guarantee oscillation if the parameters are within them, but do give a starting point in which to investigate self oscillation.

2.5 Conclusions

In this chapter the problem to be considered was defined along with the general specifications of each component within the system. First methods of calculating the total response were given so that subsequent theoretical work can be checked and a feeling for the system operation obtained. Finally the stability and instability of the system for a wide class of linear plants was discussed and theorems presented that defines a sufficient condition relationship between parameters for system
stability and a necessary condition for system instability. Having looked at the system in a "gross" way in this chapter, attention will now be turned to the special question of oscillation in the two IPFM system.
CHAPTER 3

DESCRIBING FUNCTION ANALYSIS

OF THE DOUBLE IPFM SYSTEM

The stability of and oscillations in a nonlinear control system are most practically studied with the use of the describing function, or harmonic balance. Most analysis to date has been with a single nonlinearity in the system, but some work has been published on the multiple nonlinear problem (12, 22, 28). With the exception of Davison's work (12) the techniques require a trial and error approach or are concerned with a special system configuration. In the following analysis the specialized system containing two IPFM modulators (quantizers with hysteresis) will be analyzed using the describing function approach. A new general method will be introduced for dealing with any system with two nonlinear elements separated by linear plants.

3.1 Basic Considerations

The harmonic balance technique generally considers only the first harmonic of the signals within the system. Justification for this is by assuming that the linear
portion of the system is a sufficiently good low pass filter that all higher harmonics are negligible. The analysis then proceeds by taking a sinusoidal signal at one point in the system and determining what gain and phase requirements must be met by the elements, both linear and nonlinear, around the loop to produce the original sinusoid. The nonlinear element is replaced by a gain, sensitive to input amplitude, frequency and/or phase as the case may be. This variable gain is defined as the describing function.

Conceptually there is no reason why this technique can not apply to any problem configuration. The assumptions that the input to the nonlinearities is approximately a sinusoid is critical and must be met at the input of each nonlinear element. The system to be studied has the general configuration shown in figure 3.1. By replacing the nonlinear elements with their respective describing functions and writing the system characteristic equation:

\[ 1 + N_1D(E_1, \omega, \beta_1)G_1(j\omega)N_2D(E_2, \omega, \beta_2)G_2(j\omega) = 0 \quad (3-1) \]

the stability and oscillatory behavior of the system can be studied. It is assumed that the linear plants \( G_1(j\omega) \) and \( G_2(j\omega) \) are of a low pass nature so that only the fundamental frequency is of interest (consistent with the describing function approach). Then the critical values
a) Nonlinear system configuration. N and G symbolize the nonlinear and linear elements.

b) Nonlinear system with nonlinear elements replaced by their describing function. * indicates only the first harmonic of the signal. E, ω, β are the amplitude, frequency and phase input to the nonlinear element.

Figure 3.1 Nonlinear system configuration to be studied.
are obtained by finding the intersection of the two curves \(-1/N_{1D}N_{2D}\) and \(G_1(j\omega)G_2(j\omega)\) where \(N_{1D}\) and \(N_{2D}\) are the describing functions of the two nonlinearities. The difficulty arises in the computation of \(N_{2D}\). The amplitude and phase of the sinusoid input to \(N_2\), \(E_2\sin(\omega t + \beta_2)\), is dependent upon the first nonlinearity, \(N_1\), and \(G_1(j\omega)\) at the frequency in question. Thus if
\[
e^*(t) = E_1\sin(\omega t + \beta_1)
\]
(3-2)
is the input to the first nonlinearity, the first harmonic output will be:
\[
u^*(t) = |N_{1D}(E_1,\omega,\beta_1)|E_1\sin(\omega t + \beta_1 + \alpha_1)
\]
(3-3)
where \(\alpha_1\) is the phase introduced by the first nonlinearity (if any). The output then of the first linear element is:
\[
y^*(t) = |N_{1D}(E_1,\omega,\beta_1)|E_1|G_1(j\omega)|\sin(\omega t + \beta_1 + \alpha_1 + \theta_1)
\]
(3-4)
where \(\theta_1\) is the phase introduced by the linear element \(G_1(j\omega)\). But \(y^*(t)\) is the input to the second nonlinear element \(N_2\). Therefore,
\[
E_2 = E_1|N_{1D}(E_1,\omega,\beta_1)||G_1(j\omega)|
\]
(3.5)
and the phase of the input is,
\[
\beta_2 = \beta_1 + \alpha_1 + \theta_1
\]
(3.6)
Continuing around the loop will give the following equations:
\[
u_2^*(t) = E_2|N_{2D}(E_2,\omega,\beta_2)|\sin(\omega t + \beta_2 + \alpha_2)
\]
(3-7)
with \(\alpha_2\) = angle of \(N_{2D}(E_2,\omega,\beta_2)\). Then
\[ y_2^*(t) = E_2 |N_{2D}(E_2, \omega, \beta_2)| |G_2(j\omega)| \sin(\omega t + \beta_2 + \alpha_2 + \theta_2) \]  
(3-8)

with \( \theta_2 \) the phase introduced by the \( G_2(j\omega) \) plant. Thus the critical conditions are:

\[ E_1 = E_2 |N_{2D}(E_2, \omega, \beta_2)| |G_2(j\omega)| \]  
(3-9)

\[ \beta_1 = \beta_2 + \alpha_2 + \theta_2 + 180^\circ \]  
(3-10)

where the \( 180^\circ \) has been added because of the negative feedback. Substituting the relationship for \( E_2 \) from equation (3-5) gives:

\[ E_1 = E_1 |N_{1D}(E_1, \omega, \beta_1)| |G_1(j\omega)| |N_{2D}(E_2, \omega, \beta_2)| |G_2(j\omega)| \]

or

\[ |G_1(j\omega)| |G_2(j\omega)| = \frac{1}{|N_{1D}(E_1, \omega, \beta_1)| |N_{2D}(E_2, \omega, \beta_2)|} \]  
(3-11)

and for the phase relationship use equation (3-6) in equation (3-9) to obtain:

\[ \beta_1 = 180^\circ + (\beta_1 + \alpha_1 + \theta_1) + \beta_2 + \theta_2 \]  
(3-12)

or

\[ \theta_1 + \theta_2 = -180^\circ - \alpha_1 - \alpha_2 \]

Conditions (3-11) and (3-12) are precisely the intersection conditions mentioned at the beginning of the development.

The difficulty in using the describing function approach arises in equation (3-11) and (3-12) where the gain and phase of \( G_1(j\omega) \) must be known a priori. This requirement restricts the usefulness of the method because
it introduces another parameter. If the describing functions are both real the obstacle can be overcome by plotting a family of curves for the various values of $|G_1(j\omega)|$. When one of the nonlinearities contains hysteresis the problem is more complex but can still be handled using a family of curves.

In the next section a technique for handling the system with two nonlinearities in which both their describing functions are real will be described (23). This will be necessary for a better understanding of the method used for the two IPFM system. When the two nonlinearities are sensitive to amplitude, phase and frequency the describing function method does not seem as attractive; however, in the case at hand certain information may be deleted without loss of the analytic power of the describing function method.

3.2 Construction and Use of the Compound Describing Function for a Two Nonlinearity System

For the type of nonlinearities considered in this section, the describing functions will always be real; ie, have zero imaginary part. Thus when $-1/N_D$ is plotted, it will always be on the $-180^\circ$ phase line of the Nichols chart. Likewise if both nonlinearities are as defined, the composite describing function, $N_1D N_2D$, will be real
and \(-1/N_{1D}N_{2D}\) will also be found on the -180° phase line. In the two nonlinearity system it is necessary to take into consideration \(|G_1(j\omega)|\). This is done by plotting \(-1/N_{1D}N_{2D}\) in a shifted pattern according to the parameter \(|G_1(j\omega)| = \text{constant}|. The method of construction is shown in figure 3.2. For a given \(|G_1(j\omega)| = c_1\) the \(-1/N_{1D}N_{2D}\) curve (a straight line in this case) is constructed at the point where a horizontal line \(c_1\) meets an arbitrary line \(c_T\). The parameters along the \(1/N_{1D}N_{2D}\) curve can be either \(E_1\) or \(E_2\) depending upon the information sought. In fact as the example will point out it is only necessary to connect the equal \(E_1\) (or \(E_2\)) points for the analysis (this is indicated by the dotted lines in figure 3.2).

Next construct \(G_1(j\omega)G_2(j\omega)\) and \(G_1(j\omega)\) on the same Nichols chart. For oscillations to occur \(G_1(j\omega)G_2(j\omega)\) must cross the -180° phase line at some frequency \(\omega_0\). If it does not cross this phase line the system will not oscillate. Assume that \(\omega_0\) is the crossing frequency and locate this frequency on the \(G_1(j\omega)\) curve. Construct a horizontal line from \(G_1(j\omega_0)\) to the arbitrary line, from the intersection drop a vertical line until it meets another horizontal line from \(G_1(j\omega_0)G_2(j\omega_0)\), indicated by "P" in figure 3.2.
Figure 3.2 Double nonlinearity describing function construction.
Point "P" determines the magnitude of oscillation, \( E_1 \) or \( E_2 \), and \( \omega_0 \) is the approximate frequency of oscillation. The stability of the oscillation can be determined in the usual manner for this type of plot.

The advantage of the method described above is that all elements of the system are shown on a single graph in such a way that their interactions are easily determined. Thus for example it is seen to what extent the magnitude of oscillation can be controlled by either \( G_1(j\omega) \) or \( G_2(j\omega) \) gains, variation in frequency by reshaping the \( G_1(j\omega)G_2(j\omega) \) locus, or oscillation magnitude change by reshaping the describing function curves. An example follows which illustrates the technique.

**Example 3.1** The example system, its response and the describing function graphs are shown in figures 3.3 and 3.4.

Graphical methods predict an oscillation of \( \omega_0 = 0.55 \) rad/sec and \( E_1 = 2.4 \) and \( E_2 = 5.1 \). Simulation of the system gives \( \omega_0 = 0.56, E_1 = 2.5 \) and \( E_2 = 5.0 \). The correlation can be seen to be quite good in this example.

**3.3 The Describing Function of the IPFM Modulator**

The describing function for the single IPFM modulator must first be developed. This was done by Li (39) and his results are summarized below. Consider the input to
Figure 3.3 Example of double nonlinear system and output.
Figure 3.2 Describing function method for example 3.1.
the modulator will be:

\[ e(t) = E_s \sin(\omega t + \beta) \]  

(3-13)

The output will be a train of impulses at time \( t_k \), polarity \( b_k \) and strength \( M \). Since the input is known, the times and polarities of the impulses can be computed for a half cycle as:

\[ \cos(\omega t_k) = \cos(\beta) - \frac{k}{(E_s/A\omega)} \]  

(3-14)

\[ k = 1, 2, \ldots N/2 \]

where \( N \) is the total number of pulses in the period \( T=2\pi/\omega \). Note that the pulses will have asymmetry in their position because of the symmetric sinusoidal input. Li showed that the input phase has a certain effective range. Values of phase above some critical angle can be transformed into the range from 0 to \( \beta \) (critical). Thus it is necessary only to investigate values of \( \beta \) from 0 to \( \beta \) (critical). Where,

\[ 0 < \beta \) (critical) = \beta_c \]

\[ = \cos^{-1} \left( 1 - \frac{A\omega}{E_s} \right) < \frac{\pi}{2} \]

The final pattern will depend upon \( E_s, \omega, A, \) and \( \beta \). During the first half of the cycle there will be \( N/2 \) positive impulses and \( N/2 \) negative impulses during the second half of the cycle. Thus the output over one cycle after appropriate phase shifting may be written:
Since the output is impulsive, the Fourier coefficients are easily found. If the output is written in a standard Fourier series form:

\[ u(t) = a_0 + a_1 \cos(\omega t) + a_2 \cos(2\omega t) + \ldots \]

\[ + b_1 \sin(\omega t) + b_2 \sin(2\omega t) + \ldots \]

then the coefficients may be written as:

\[ a_n = \frac{2\pi}{\omega} \int_0^{\frac{2\pi}{\omega}} M\left\{ \sum_{k=1}^{N/2} \delta(t-t_k) - \sum_{k=N+1}^{N} \delta(t-t_k) \right\} \cos(n\omega t) \, dt \quad (3-16) \]

\[ b_n = \frac{2\pi}{\omega} \int_0^{\frac{2\pi}{\omega}} M\left\{ \sum_{k=1}^{N/2} \delta(t-t_k) - \sum_{k=N+1}^{N} \delta(t-t_k) \right\} \sin(n\omega t) \, dt \quad (3-17) \]

Note that \( a_0 = 0 \) because of the pattern symmetry. At this point Li evaluated the first harmonic only and obtained the following relationships:

\[ a_1 = \frac{\omega M}{\pi} \frac{N/2}{(E_s/A\omega)} \quad (3-18) \]

\[ b_1 = \frac{\omega M}{\pi} \left\{ 2 \sum_{k=1}^{N-1} \left[ 1 - \frac{\cos\beta}{k} \right]^2 \frac{1}{k} \right\} \frac{1}{(E_s/A\omega)} + \frac{\sin(\beta)}{(E_s/A\omega)} \]

\[ + \left( 1 - \frac{\cos\beta}{N/2} \right)^2 \frac{1}{(E_s/A\omega)} \quad (3-19) \]
From which the first harmonic magnitude and phase of the output may be evaluated as:

\[ |u(\text{first harmonic})| = (a_1^2 + b_1^2)^{\frac{1}{2}} \]

and

\[ \text{Angle } u(\text{first harmonic}) = \tan^{-1}(a_1/b_1) \]

The describing function is then defined as the ratio of the output first harmonic to the input:

\[ N_D(E_s, \omega, \beta, A, M) = \frac{u(\text{first harmonic})}{E_s} \]

Thus:

\[
\begin{align*}
|N_D| &= \frac{M}{\pi A (E_s/A_\omega)} \left[ \frac{N/2}{(E_s/A_\omega)} \right] + \left( \frac{2}{N-1} \sum_{k=1}^{N-1} \left( 1 - \left( \frac{\cos \beta - k/N}{E_s/A_\omega} \right)^2 \right)^{\frac{1}{2}} \right) \\
& \quad + \left( \frac{1 - \left( \frac{\cos \beta - N/2}{E_s/A_\omega} \right)^2}{E_s/A_\omega} \right) + \sin \beta \right) \right) \right]^{\frac{1}{2}} \right. \\
\text{Angle } N_D &= \tan^{-1} \left( \frac{N/2}{E_s/A_\omega} \right) \\
& \quad + \left( \frac{2}{N-1} \sum_{k=1}^{N-1} \left( 1 - \left( \frac{\cos \beta - k/N}{E_s/A_\omega} \right)^2 \right)^{\frac{1}{2}} + \sin \beta \right) \\
& \quad + \left( \frac{1 - \left( \frac{\cos \beta - N/2}{E_s/A_\omega} \right)^2}{E_s/A_\omega} \right)^{\frac{1}{2}} \\
\end{align*}
\]

By taking \( E_s/A_\omega \) as the normalized input, the describing function may be made frequency independent, depending then only upon the normalized value, \( V \), and the phase, \( \beta \). This
gives a somewhat simpler form:

\[ |N_D| = M/A \left\{ \frac{1}{\pi V} \left[ \left( \frac{N/2}{V} \right)^2 + \left( \sum_{k=1}^{N-1} \frac{(1-(\cos \beta-k)^2)^{\frac{1}{2}} + \sin \beta}{V} \right)^2 \right]^{\frac{1}{2}} \right\} \]

Angle \( N_D = \tan^{-1} \left\{ \frac{N/2}{V} \frac{1}{\sum_{k=1}^{N-1} \left( \frac{1-(\cos \beta-k)^2)^{\frac{1}{2}} + \sin \beta}{V} \right)^2} \right\} \)

(3-21)

The constant multiplying factor \( M/A \) may be considered as a linear gain and therefore associated with the linear element. This allows the developed describing function to be universal for all IPFM modulators. The computation of the describing function is most easily accomplished on the computer, the program to do this is given in Appendix E and the results are shown graphically in figures 3.5 and 3.6. Li did not present the results in this manner, but it will be necessary to use this form for the subsequent sections. The maximum and minimum gain and phase for the describing function for each pulse pattern is tabulated in Table 3.1. This will be found to be useful in dealing with the double modulator system.
Figure 3.5 Magnitude of the describing function for the IPFM modulator.
Figure 3.6 Phase of the describing function for the IPFM modulator.
Table 3.1

Maximum and Minimum Values of the Magnitude and Phase for the Integral Pulse Frequency Modulator Describing Function

<table>
<thead>
<tr>
<th>Pulses Period</th>
<th>Maximum Magnitude</th>
<th>Minimum Magnitude</th>
<th>Maximum Phase</th>
<th>Minimum Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.273</td>
<td>0.425</td>
<td>-90°</td>
<td>-19°</td>
</tr>
<tr>
<td>4</td>
<td>0.970</td>
<td>0.615</td>
<td>-45°</td>
<td>-15°</td>
</tr>
<tr>
<td>6</td>
<td>0.971</td>
<td>0.719</td>
<td>-28°</td>
<td>-12°</td>
</tr>
<tr>
<td>8</td>
<td>0.972</td>
<td>0.783</td>
<td>-20°</td>
<td>-10°</td>
</tr>
<tr>
<td>10</td>
<td>0.975</td>
<td>0.826</td>
<td>-16°</td>
<td>-9°</td>
</tr>
</tbody>
</table>

Because of the rich harmonic content of the impulse, it is necessary to determine the harmonic content of the nonlinearity output before intelligently using the describing function analysis. If the harmonic content is low then there will not be such a severe low pass requirement on the linear elements following the IPFM modulators. Conversely, if the harmonic content is high then the linear element must reduce them to a comparatively low level if the describing function analysis is to be worthwhile.

Then if one starts with equations (3-16) and (3-17), the harmonic content describing function can be found (see Appendix F). These results are indicated below for the general harmonic term and in particular the second and third harmonic contents.
\[ a_n = \frac{\omega M}{\pi} \sum_{j=0}^{J} (-1)^j \binom{n}{2j} \left[ \left( \frac{\cos \beta - N/2}{V} \right)^{n-2j}(1 - \left( \frac{\cos \beta - N/2}{V} \right)^2)^j \right. \]
\[ \left. - \cos^{n-2j} (\beta) \sin^{2j} (\beta) \right] \quad (3-22) \]

\[ b_n = \frac{\omega M}{\pi} \left[ \sum_{k=1}^{N-1} \binom{n}{2j+1} \left[ \frac{\cos \beta - k}{V} \right]^{n-2j-1}(1 - \left( \frac{\cos \beta - k}{V} \right)^2)^{j+\frac{1}{2}} \right. \]
\[ + \sum_{j=0}^{J} (-1)^j \binom{n}{2j+1} \left[ \frac{\cos \beta - N/2}{V} \right]^{n-2j-1}(1 - \left( \frac{\cos \beta - N/2}{V} \right)^2)^{j+\frac{1}{2}} \]
\[ + \sum_{j=0}^{J} (-1)^j \binom{n}{2j+1} \cos^{n-2j-1} (\beta) \sin^{2j+1} (\beta) \right] \quad (3-23) \]

For the second harmonic,

\[ a_2 = \frac{\omega M}{\pi} \left[ \frac{N/2}{V} \right] \left[ \frac{N}{V} - 4 \cos \beta \right] \quad (3-24) \]

\[ b_2 = \frac{\omega M}{\pi} \left[ \sum_{k=1}^{N-1} \binom{2j}{2j+1} \left[ \frac{2(\cos \beta - k)}{V} \left(1 - \left( \frac{\cos \beta - k}{V} \right)^2 \right)^{\frac{1}{2}} \right] \right. \]
\[ + 2 \left( \frac{\cos \beta - N/2}{V} \right) \left(1 - \left( \frac{\cos \beta - N/2}{V} \right)^2 \right)^{\frac{1}{2}} \]
\[ + 2 \sin(\beta) \cos(\beta) \right] \]

And for the third harmonic,

\[ a_3 = \frac{\omega M}{\pi} \left[ 4 \left( \frac{\cos \beta - N/2}{V} \right)^3 - 3 \left( \frac{\cos \beta - N/2}{V} \right) - \cos(2\cos \beta + 1) \right] \quad (3-25) \]
The describing function for the general nth harmonic is:

\[
|N_D(nth)| = \frac{1}{\pi V} \left[ \left( \frac{a_n}{(\omega M/\pi)} \right)^2 + \left( \frac{b_n}{(\omega M/\pi)} \right)^2 \right]^{\frac{1}{2}} 
\]

(3-26)

\[
\text{Angle } N_D(nth) = \tan^{-1} \frac{a_n}{b_n}
\]

The computation of the describing function for the second and third harmonics can be carried out using the same program as that used for the fundamental describing function. The results of the computation for the second and third harmonic describing functions are shown in figures 3.7 and 3.8 in comparison with the fundamental variation.

As was expected in this case, the harmonic content of low pulse pattern operation, in particular D2, is very high. The greatest magnitude for D2 second harmonic is greater than 0.8, while the fundamental greatest magnitude is about 1.27. Thus if the describing function method is to be used to predict low numbered pattern operation, it will be essential that each linear element attenuate
Figure 3.7 Second harmonic describing function for the IPFM modulator.
Figure 3.8 Third harmonic describing function for the IPFM modulator.
rapidly above the predicted operating frequency. The re-
requirement for this rapid attenuation is less critical as
the pulse pattern number increases. The third harmonic of
the modulator output is also large, in fact for D2 oper-
ation it is a large as the fundamental itself. Again
rapid attenuation will be demanded; but since the linear
plants already must meet this criteria because of the
second harmonic, stating it as a requirement is redundant.

3.4 Describing Function Analysis of a Control System Con-
taining Two IPFM Modulators

Following the basic ideas developed in section 3.1,
the composite describing function for the two modulators
can be calculated using the magnitude of \( G_1(j\omega) \) as a par-
parameter. The basic equations used for the computation are:

\[
|N_{1D}(V_1, \beta_1)| = \frac{M_1}{A_1(\pi V_1)} \left[ \frac{N_1/2}{V_1} \right]^{2} + \sum_{k=1}^{N_1/2} \left[ 1 - \left( \cos \beta_1 - \frac{k}{V_1} \right)^2 \right]^{1/2} \]

\[
+ (1 - \left( \cos \beta_1 - \frac{N_1/2}{V_1} \right)^2)^{1/2} \tan^{-1} \left( \frac{N_1/2}{V_1} \right) \]

\[
\text{Ang } N_{1D}(V_1, \beta_1) = \tan^{-1} \left( \frac{N_1/2}{V_1} \right) \]

\[
\left. \sum_{k=1}^{N_1/2} \left( 1 - \left( \cos \beta_1 - \frac{k}{V_1} \right)^2 \right)^{1/2} + \sin \beta_1 \right) \]
\[ V_2 = |G_1(j\omega)| \frac{V_1}{|N_{1D}(V_1,\beta_1)|} \]  

(3-28)

\[ |N_{2D}(V_2,\beta_2)| = \frac{M_2}{A_2} \left( \frac{1}{V_2} \right)^{2} \left[ \frac{N_2}{2} \right]^{2} + \left\{ 2 \sum_{k=1}^{\frac{N_2}{2}-1} \left( 1 - \left( \frac{\cos \beta_k}{V_2} \right)^2 \right)^{\frac{1}{2}} + \left( 1 - \left( \frac{\cos \beta_k}{V_2} \right)^{2} \right)^{\frac{1}{2}} \sin \beta_k \right\}^{\frac{1}{2}} \]  

(3-29)

\[ \text{Ang } N_{2D}(V_2,\beta_2) = \tan^{-1} \left\{ \frac{-N_2/2}{V_2} \right\} \]  

The value of \( \frac{M_1}{A_2} \) is a constant or design parameter and can be associated with the gain of \( G_1(j\omega) \) without loss of generality. The same is true for \( \frac{M_2}{A_1} \) which is associated with \( G_2(j\omega) \). The computer program for the computation of the above equations is to be found in Appendix F.

For a given \( V \), \( \beta \) can vary between zero and some critical value \( \beta_c \). The resultant curves of \( N_D(V,\beta) \) may be thought of as a family of curves with \( \beta \) as the parameter. These curves define a zone and were shown in figure 3.5. An important point to note here is that for a given pulse pattern to exist, \( V_1 \) and \( V_2 \) must fall within certain
limits; ie,

\[
\frac{N_1}{4} \leq V_1 \leq \frac{N_1 + 4}{4} \tag{3-30}
\]

\[
\frac{N_2}{4} \leq V_2 \leq \frac{N_2 + 4}{4}
\]

where \( N_1 \) and \( N_2 \) are respectively the number of pulses emitted from the modulators "1" and "2" in a period of oscillation, \( T \). With this as a criteria one can obtain the range of \( |G_1(j\omega)| \) for a given pulse pattern.

**Example 3.2** Find the possible range of \( |G_1(j\omega)| \) for the pulse pattern D64, note that:

\[
1.5 \leq V_1 \leq 2.5
\]

\[
1.0 \leq V_2 \leq 2.0
\]

and from equation (3.28),

\[
|G_1(j\omega)| = \frac{V_2}{V_1 |N_{1D}(V_1, \beta_1)|}
\]

using the maximum and minimum values of \( N_{1D} \) and the corresponding values of \( V_1 \) yields:

\[
\frac{V_2_{\text{min}}}{V_1_{\text{max}} |N_{1D}(V_{1_{\text{max}}})|_{\text{max}}} \leq |G_1(j\omega)| \leq \frac{V_2_{\text{max}}}{V_1_{\text{min}} |N_{1D}(V_{1_{\text{min}}})|_{\text{min}}}
\]  

(3-31)

In this case the maximum or minimum values of \( N_{1D} \) may be obtained from figure 3.5 or table 3.1, giving:

\[
0.56 \leq |G_1(j\omega)| \leq 1.47
\]
The results from this example are extended to other patterns, tabulated in Table 3.2 and plotted in figure 3.9. These boundaries for $|G_1(j\omega)|$ are necessary for the final describing function analysis.

For large pulse numbers (greater than 20 for example) the value of $N_D(V,\beta)$ approaches 1.0 and the following estimate can be used:

$$\frac{N_2}{N_1 + 4} \leq |G_1(j\omega)| \leq \frac{N_2 + 4}{N_1}$$

(3-32)

Also for a given $V_1$ and $|G_1(j\omega)|$ not all the values of $V_2$ for a given pulse pattern are possible. Thus for the given pattern in the example, $V_2$ could range from 1.0 to 2.0, but equation (3.28) predicts a tighter boundary. For $|G_1(j\omega)| = 1.0$,

$$V_2 = |G_1(j\omega)| \frac{V_1}{N_1D(V_1,\beta_1)}$$

$$= 1 \times (1.5,2.5) \times (0.91,0.71)$$

$$= (1.36,1.77)$$

Thus although $V_2$ could fall between 1.0 and 2.0, only that portion between 1.36 and 1.77 is valid in this case.

Having looked at the ranges for $V_1$ and $V_2$ for various pulse patterns, next consider the magnitude and phase of the composite describing function. If an attempt is made to plot all the information, $V_1$, $V_2$, $\beta_1$, $\beta_2$, $|G_1(j\omega)|$ and
TABLE 3.2  

$|G_1(j\omega)|$ for the Various Pulse Patterns

<table>
<thead>
<tr>
<th>Pulse Pattern</th>
<th>Maximum Magnitude</th>
<th>Minimum Magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>D22</td>
<td>2.36</td>
<td>0.78</td>
</tr>
<tr>
<td>D42</td>
<td>1.67</td>
<td>0.40</td>
</tr>
<tr>
<td>D44</td>
<td>2.20</td>
<td>0.80</td>
</tr>
<tr>
<td>D62</td>
<td>1.10</td>
<td>0.28</td>
</tr>
<tr>
<td>D64</td>
<td>1.47</td>
<td>0.56</td>
</tr>
<tr>
<td>D66</td>
<td>1.84</td>
<td>0.83</td>
</tr>
<tr>
<td>D82</td>
<td>0.82</td>
<td>0.22</td>
</tr>
<tr>
<td>D84</td>
<td>1.10</td>
<td>0.43</td>
</tr>
<tr>
<td>D86</td>
<td>1.36</td>
<td>0.65</td>
</tr>
<tr>
<td>D88</td>
<td>1.63</td>
<td>0.87</td>
</tr>
</tbody>
</table>

| D24            | 3.15             | 1.57             |
| D26            | 3.94             | 2.35             |
| D28            | 4.72             | 3.14             |
| D46            | 2.75             | 1.20             |
| D48            | 3.33             | 1.60             |
| D68            | 2.20             | 1.10             |
Figure 3.9 Boundaries for $G_1(j\omega)$ for the various pulse patterns.
Dmn, the resultant graphs are not useful. Thus it is essential to eliminate some information, retaining only that which will ultimately be needed in the analysis or design of a system. Fortunately very good results are obtained if only Dmn and \(|G_1(j\omega)|\) are considered in the final describing function graph.

The first step is to obtain \(-1/N_{1D}N_{2D}\) for a given value of \(|G_1(j\omega)|\) and pulse pattern Dmn. These results are shown in figures 3.10 and 3.11. Note the way in which \(V_1\), \(\beta_1\), and \(V_2\), \(\beta_2\) vary. In figure 3.10 the variation in \(V_1\) and \(\beta_1\) is given for a particular pulse pattern and \(|G_1(j\omega)|\). For increasing \(V_1\) the D44 pattern moves from the lower right to the upper left, with phase angle \(\beta_1\) determining near vertical lines within each \(V_1\) sector. In figure 3.11 the variation in \(V_2\) and \(\beta_2\) has been plotted for the same pulse pattern and \(|G_1(j\omega)|\). In this figure the \(V_2\) sectors move more from left to right, with \(\beta_2\) phase lines again describing an almost vertical line. These observations may be useful in determining the magnitude of the oscillations from a compound modulator system. Intersection near the top of the constant \(|G_1(j\omega)|\) magnitude sector will indicate larger \(V_1\) and \(V_2\) than those that may intersect near the bottom of the same sector. These observations will also be used to determine the stability of the oscillations.
Figure 3.10 Variation in a Dmn sector of $V_1$ and $\beta_1$. Shown for D44, $|G_1(j\omega)| = 1.0$. 
Figure 3.11 Variation in a Dmn sector of $V_2$ and $\beta_2$. Shown for D44, $|G_1(j\omega)| = 1.0$. 
Now for a given pulse pattern, various values of $|G_1(j\omega)|$ are possible. The resulting curves form an envelope for the critical region for the defined pulse pattern. These results are shown in figure 3.12 for the D22 pattern. Note that the $|G_1(j\omega)|$ sectors move from lower right to upper left for increasing values of $|G_1(j\omega)|$. This observation will be useful when analyzing a control system using the composite describing function. The values shown on the edge of the envelope represent the approximate boundary for the values of $|G_1(j\omega)|$. Those on the lower edge represent the bottom of the respective $|G_1(j\omega)|$ sector and those on the upper edge represent the top of the $|G_1(j\omega)|$ sector. This is more clearly seen in the insert to figure 3.12.

For curves passing through a given pulse pattern envelope the probability that they actually fall within the values shown on the envelope depends upon their slope. For various lines, see figure 3.13, the range of values given by the envelope, $\Delta R$, divided by the range of values actually intersected, $\Delta r$, seems like a good measure of the reliability of the $|G_1(j\omega)|$ intersection. If $\Delta R/\Delta r$ is 1.0 the correspondence is exact. The results of this test are given in figure 3.13 for the D22 pattern. For practical low pass filtering the curve passing through the sector
Figure 3.12 Generation of the Dmn (D22) region by constant $|G_1(j\omega)|$ sectors.
Figure 3.13 Reliability of $|G_1(j\omega)|$ as determined by the boundary values.
would fall between a 45 degree and vertical line. Since this is consistent with the basic describing function assumption, the envelope values were picked for lines with 45 degree slopes.

The envelopes for various pulse patterns with $|G_1(j\omega)|$ values on the boundary are now superimposed on a single Nichols chart. The results are presented in figure 3.14. Patterns considered are limited to those containing up to 8 pulses from each modulator in a given fundamental time period, $T$. Higher pulse numbers crowd the diagram. Methods for handling these high pulse numbered patterns can be handled by an expanded scale on the chart and the program given in Appendix F. Other methods will be considered later. Note that as the pulse number becomes larger, the systems tend to their linear equivalent.

For a given value of $|G_1(j\omega)|$, the various pulse pattern sectors have been drawn in figure 3.15. Such a representation is not useful in the analysis of compound systems since it is unlikely that $G_1(j\omega)$ would maintain a constant magnitude throughout a frequency range of interest. It may be constant enough in some instances to use this type of representation for approximate determinations of pulse patterns. However the pattern of development of the constant $|G_1(j\omega)|$ sectors is interesting
Figure 3.14 Describing function curves for the 2 IPFM system.
Figure 3.15  Dmn patterns for constant $|G_1(j\omega)|=1$. 

Gain $(-1/N_1D N_2D)$
and could be useful if such a system were to be designed
with restrictions of $|G_1(j\omega)|$.

The use of the describing function for predicting
oscillations in control systems can be carried out using
the two charts given in figures 3.9 and 3.14. Two con-
ditions or intersections must exist; one intersection must
exist by the $G_1(j\omega)G_2(j\omega)$ curve through a given pulse pat-
tern envelope, and the other must be by the $G_1(j\omega)$ curve
passing through the appropriate magnitude and pulse pat-
tern sector. Both conditions must be met by the system
for oscillations to be possible.

When testing for stability of the oscillations, if
the direction of the $V_1$ variation is noted the standard
method may be used. This is discussed later. When test-
ing for stability, if either of the two intersections do
not occur, then this would be a sufficient condition for
the system to be stable. This is of course provided the
large signal equivalent system is also stable as discus-
sed in section 2.4. Stability may also be discussed using
the theorem developed in Chapter 2 (Theorem 2.2).

The best procedure for determining the possible
oscillatory conditions is to construct a table as shown
in Table 3.3 (Example 3.3). The analysis then follows
the following steps:

1. Determine the total frequency range of the intersection by the $G_1(j\omega)G_2(j\omega)$ curve with the describing function sectors for the various pulse patterns. For the case in question, see figure 3.18, this would be $\omega = (1, 2.8)$.

2. Check which pulse patterns are passed through by the $G_1(j\omega)$ plot for this same frequency range, see figure 3.17. At the same time make sure that $G_1(j\omega)G_2(j\omega)$ also pass through the same patterns. Only those in which there is correspondence need be recorded in column one of the table.

3. In column two, record the frequency range of $G_1(j\omega)G_2(j\omega)$ in each of the intersected pulse pattern regions recorded in step two.

4. Next for the above frequency range (step 3) record the magnitude of $G_1(j\omega)$. In the example given two indications are given, one is a numerical range and the other is a qualitative "low, medium, high."

5. On the total plot, $G_1(j\omega)G_2(j\omega)$, find the same magnitude range of values as in step 4, both numerical and qualitative are indicated in the example. Record them in column 4 of the table.

6. Compare the values (either numerical, which is better, or qualitative) in columns three and four. If
there is correspondence in the values and the assumptions of the describing function are valid, oscillations will occur.

It is possible for the system to oscillate in two or more modes. This is due to constant phase shifting ($\beta$) in the input signals to the modulators. This phenomena is caused by the value of the modulator integrators not exactly following the pattern requirements. Thus for example at the end of a period, $T$, an integrator should have discharged to zero, but instead may have a residual value left over. This value is carried over to the next period, causing it to be different from the previous one. This effect can accumulate causing the patterns to shift. The underlying cause of this effect is due to the fact that the signals are really not sinusoidal, and thus the time symmetric pulses assumed will not exactly hold.

The examples which follow illustrate the use of the describing function for the two IPFM modulator system and some of the peculiarities of the method.

Example 3.3 This is a continuation of example 2.1. Consider the system configuration given in figure 3.15a. The linear elements are not the best low pass filters, being only first order plants. From this follows that the outputs of the linear elements will hardly be approximate
a) System configuration

b) Digital computer simulation

Figure 3.16 Configuration and output for example 3.3.
sinusoids. This is clearly seen in figure 3.16b, the element outputs. It is expected that for such a system the describing function analysis may not yield reliable results. Because it is important to illustrate the limitations of a method as well as its good points, this example is included.

The system has the following vital data:

\[
A_1 = 0.25 \quad A_2 = 0.40 \\
M_1 = 1.00 \quad M_2 = 1.00
\]

Thus plotting \( \frac{M_1}{A_2}G_1(j\omega) \) in figure 3.17 and \( \frac{M_1M_2}{A_1A_2}xG_1(j\omega)G_2(j\omega) \) in figure 3.18 will give the superimposed curves shown. The table below (Table 3.3) follows the pattern of solution discussed above and from it, it is easily seen that there is a strong possibility of D22 operation. The basic frequencies of operation predicted will be around \( \omega = 1.5 \text{ rad/sec (T = 4.18 sec).} \) With this oscillation, estimation of the amplitude of \( E_1 \) and \( E_2 \) can be predicted as follows:

For D22 operation \( V_1 \) and \( V_2 \) must lie between 0.5 and 1.5. Therefore,

\[
V_1 = (0.5,1.5) = \frac{E_1}{A_1\omega} = \frac{y_2}{A_1\omega_{\text{peak}}}
\]

and

\[
V_2 = (0.5,1.5) = \frac{E_2}{A_2\omega} = \frac{y_1}{A_2\omega_{\text{peak}}}
\]

giving for \( E_1 \) and \( E_2 \),
Figure 3.17 \((M_1/A_2)G_1(j\omega)\) plot for example 3.3.
Figure 3.18 \((M_1M_2/A_1A_2)G_1(j\omega)G_2(j\omega)\) plot for example 3.3.
TABLE 3.3

Solution Table for Example 3.3

Total frequency range = (1, 2.7)

| Pattern | Frequency Range | $|G_1(j\omega)|$ | $G_1(j\omega)G_2(j\omega)$ Intersection |
|---------|-----------------|-----------------|------------------------------------------|
| D22     | 1-2             | 1.8-1. (M)      | 0.9-2.5 (M)                              |
| D42     | 1.5-2.5         | 1.5-1.0 (H)     | 0.5-1.0 (M)                              |
| D44     | 2.1-2.6         | 1.1-0.9 (L)     | 0.9-1.4 (L)                              |
| D64     | 2.5-2.7         | 1.0-0.9 (M)     | 0.6-0.7 (L)                              |

TABLE 3.4

Comparison of Describing Function Results and Actual Simulation

<table>
<thead>
<tr>
<th>Period, T</th>
<th>$y_1(\text{peak})$</th>
<th>$y_2(\text{peak})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulation</td>
<td>3.5</td>
<td>0.9</td>
</tr>
<tr>
<td>Describing Function</td>
<td>3.9</td>
<td>(0.3, 0.96)</td>
</tr>
</tbody>
</table>
\[ Y_{1 \text{peak}} = V_2 A_2 \omega = (0.5, 1.5)0.4(1.5) = (0.3, 0.9) \]

\[ Y_{2 \text{peak}} = V_1 A_1 \omega = (0.5, 1.5)0.25(1.5) = (0.19, 0.56) \]

Closer correlation can be obtained between the observed pattern and the predicted pattern if the specific D22 pattern plot is used. This has been done in figure 3.19 on which \( (M_1 M_2/A_1 A_2) G_1(j\omega)G_2(j\omega) \) has been plotted over the D22 pattern with \( |G_1(j\omega)| \) as a parameter shown in more detail. Using this graph and figure 3.17, the results given below may be found:

<table>
<thead>
<tr>
<th>( G_1(j\omega) ) from fig. 3.19</th>
<th>Frequency Range</th>
<th>( G_1(j\omega) ) from fig. 3.17</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9</td>
<td>( \sqrt{2} )</td>
<td>1.5</td>
</tr>
<tr>
<td>1.0</td>
<td>( \sqrt{2}-1.8 )</td>
<td>1.5-1.3</td>
</tr>
<tr>
<td>1.2</td>
<td>1.4-1.9</td>
<td>1.5-1.2</td>
</tr>
<tr>
<td>1.4</td>
<td>1.2-2.0</td>
<td>1.6-1.1</td>
</tr>
<tr>
<td>1.6</td>
<td>1.1-2.0</td>
<td>1.7-1.1</td>
</tr>
<tr>
<td>1.8</td>
<td>1.1-2.0</td>
<td>1.7-1.1</td>
</tr>
<tr>
<td>2.0</td>
<td>1.2-1.9</td>
<td>1.6-1.2</td>
</tr>
<tr>
<td>2.2</td>
<td>1.2-1.8</td>
<td>1.6-1.3</td>
</tr>
<tr>
<td>2.4</td>
<td>1.3-1.8</td>
<td>1.6-1.3</td>
</tr>
</tbody>
</table>

The frequency predicted by this closer examination is very close to that originally assumed; perhaps \( \omega = 1.6 \) (\( T = 3.92 \)) may be a better estimate. At this frequency \( |G_1(j\omega)| \) is about 1.4. The results predicted by the describing function are compared to those obtained by simulation of the system, this comparison is given in Table 3.4.
Figure 3.19 \( (M_1 M_2 / A_1 A_2) G_1(j\omega) G_2(j\omega) \) plot on the D22 pattern for example 3.3.
It is not surprising that the peak values are actually close to 1.0 since $M_1 = M_2 = 1.0$ and the impulse response change of these plants is $M_1$ or $M_2$ at the instant of the impulse. Also $y_2(\text{peak})$ may be expected to be in error since at the frequency in question ($\omega = 1.6$) the second harmonic content ($\omega = 3.2$) is down only 3.36 db from the fundamental.

**Example 3.4** This is a continuation of example 2.2. The system, output and frequency plots are shown in figures 3.20, 3.21 and 3.22. The solution table and comparison of the results are given in Tables 3.5 and 3.6.

The solution table would predict possible pulse patterns D22, D44, D66 and D88. The reason for the emitted pulses being the same from each modulator is because of the similarity of the linear plants and modulators. The predicted D22 pattern should not be an expected oscillating condition, except for very special initial conditions. This is because of the D22 oscillation frequency being so low ($\omega = 0.3$) and the corner frequencies of the linear elements; ie, 0.5, 1.0 and 0.7, 0.8. It would not appear that either linear element would filter the second or third harmonic of the D22 pattern sufficiently for a reliable prediction.
a) System configuration

b) Modulator outputs

Figure 3.20 System and modulator output for example 3.4.
Figure 3.21 \((M_1/A_2)G_1(j\omega)\) plot for example 3.4.
Figure 3.22 \( M_2 \) plot for example 3.4.
TABLE 3.5

Solution Table for Example 3.4

Total frequency range = (0.23,0.65)

| Pulse Pattern | Frequency Range | $|G_1(j\omega)|$ | $G_1(j\omega)G_2(j\omega)$ Intersection |
|---------------|----------------|----------------|---------------------------------------|
| D22           | 0.23-0.40      | 1.45-1.80      | 0.90-2.50                             |
| D42           | 0.33-0.45      | 1.35-1.60      | 0.60-1.20                             |
| D44           | 0.42-0.49      | 1.34-1.40      | 0.90-1.60                             |
| D64           | 0.46-0.51      | 1.20-1.35      | 0.70-0.90                             |
| D66           | 0.49-0.55      | 1.30-1.20      | 0.90-1.70                             |
| D86           | 0.59-0.53      | 1.15-1.20      | 0.80-1.10                             |
| D88           | 0.53-0.60      | 1.15-1.20      | 0.90-1.60                             |

TABLE 3.6

Comparison of Describing Function Results and Actual Simulation

<table>
<thead>
<tr>
<th>Period, T</th>
<th>$y_1$(peak)</th>
<th>$y_2$(peak)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulation</td>
<td>10.8</td>
<td>1.11</td>
</tr>
<tr>
<td>Describing Function</td>
<td>12.1</td>
<td>(0.78,1.30)</td>
</tr>
</tbody>
</table>
Figure 3.25 $\left( M_1 M_2 / A_1 A_2 \right) G_1(j\omega)G_2(j\omega)$ plot for example 3.5.
For large numbers in the pulse pattern designation; ie, D20/2, D2/20, D16/16 etc., the defining sectors become too small and crowded for convenient illustration. In this case approximate boundaries can be obtained by using maximum and minimum values of the separate describing functions. The areas thus obtained are rectangular in shape, but since they will be narrow compared to the other lower numbered patterns, this does not seem a bad way to indicate them. If the $G_1(j\omega)G_2(j\omega)$ curve passes near the origin and large numbered patterns are suspected, these rectangles may be constructed using the following for their apexes:

Apex 1: $-1/N_1D_{max} N_2D_{max}$

Apex 2: $-1/N_1D_{min} N_2D_{max}$

Apex 3: $-1/N_1D_{max} N_2D_{min}$

Apex 4: $-1/N_1D_{min} N_2D_{min}$

Note that these rectangles need only be drawn for those patterns which fall near the frequency ranges indicated by the $G_1(j\omega)$ curve. These techniques are illustrated in the following example.

**Example 3.5** Consider the system shown in figure 3.23a. The $G_1(j\omega)$ and $G_1(j\omega)G_2(j\omega)$ curves are shown in figures 3.24 and 3.25 respectively. In the frequency range of
Figure 3.23 System and modulator output for example 3.5.
Figure 3.24 \((M_1/A_2)G_1(j\omega)\) plot for example 3.5.
about 0.25 to 0.32 rad/sec there is a strong possibility for high pulse numbered patterns. This is indicated by an example of D10/6 pattern in figure 3.23b. The rectangle for the D10/6 pattern is shown in figure 3.25 and the $|G_1(j\omega)|$ range is given in figure 3.24. That this is a possible mode of oscillation is verified in figure 3.23b.

It is possible to continue listing possible patterns of operation in the way indicated above; however all patterns will have the same approximate frequency range with varying amplitudes. The amplitudes will be larger for the numbered patterns. As to which actual mode the system will operate in is very dependent upon the initial conditions. In fact as has been previously pointed out it is possible to operate in more than one mode. This is indicated in this system by the D10/6 and D12/6 pattern appearing in the output.

If the systems are stable when the linear equivalent gain of the modulators are used, then if oscillations are predicted they will be stable. For a specific pattern the stability of oscillation may be determined by noting that the normalized input to the modulators, $V = E/A\omega$, will increase for increasing numbered patterns. Thus in figure 3.26 increasing $V_1$ and $V_2$ is indicated by an
Figure 3.26 Stability of oscillation determination.
arrow (see figures 3.10 and 3.11), $A$ will be constant and $\omega$ will increase according to the $G_1(j\omega)G_2(j\omega)$ curve. If the amplitude of the oscillation, $E$, tends to increase then $V$ will also increase. Thus the $G_1(j\omega)G_2(j\omega)$ curve falls below the new point for increased amplitude. This corresponds to an attenuation and thus a reduction in the amplitude. The reverse situation occurs for decreasing amplitudes. The oscillations are basically stable for the examples given except that they may shift between various patterns. This is called neutral stability by Li (39) in his consideration of the single modulator system.

Thus far the pulse patterns considered, $D_{mn}$, have assumed $m>n$. The condition for $n>m$ may also occur if $|G_1(j\omega)|$ or $M_1/A_2$ is sufficiently large enough to increase the level of the input signal to IPFM 2. The analysis of these patterns can easily be considered by reversing $G_1(j\omega)$ and IPFM 1 with $G_2(j\omega)$ and IPFM 2 respectively. Since it is only the loop condition which is to be satisfied this arrangement will not effect the analysis. The first indication that such a pattern may occur is given by $G_1(j\omega)$ having high gain in the frequency range of intersection of the $G_1(j\omega)G_2(j\omega)$ curve. Note that with this method, the describing function curves and $G_1(j\omega)G_2(j\omega)$ are the same for both possible patterns ($D_{26}$ and $D_{62}$). It is
only necessary to draw \((M_2/A_1)G_2(j\omega)\) on the \(|G_1(j\omega)|\) boundary graph. These techniques are illustrated in the example given below.

**Example 3.6** Consider the system shown in figure 3.27a. By the methods of chapter 2 the maximum \(M_1M_2/A_1A_2\) must be less than 0.925 to guarantee stable oscillations. No lower bound can be found since the system does not meet the criteria for Theorem 2.2. The Nichols chart and \(G_1(j\omega)\) graphs are shown in figures 3.28 and 3.29. There are numerous possible patterns of oscillation all with a frequency range around \(\omega = 0.2\) (\(T = 31.4\)). Because of the high gain of \(G_1(j\omega)\) in this frequency range, high pulse numbered patterns are suggested. Thus instead of plotting \((M_1/A_2)xG_1(j\omega)\), plot \((M_2/A_1)G_2(j\omega)\). When this is done the solution proceeds as before yielding many possible pulse patterns.

**3.5 Conclusions**

In this chapter the conditions for describing function analysis of the two modulator problem were determined. The composite describing function was then calculated and applied to various examples. The second and third harmonic content of the single IPFM describing function was calculated and shown to contain high magnitudes at the lower numbered pulse patterns. Thus the method should
Figure 3.27 System and output for example 3.6.
Figure 3.28 \( \frac{M_1 M_2}{A_1 A_2} G_1(j\omega) G_2(j\omega) \) plot for example 3.6.
Figure 3.29 \((M_2/A_1)G_2(j\omega)\) plot for example 3.6.
not be used when low numbered patterns are predicted unless ample attenuation of the harmonics is provided by the linear elements; ie, good low pass filtering.

It is to be noted that the describing function predicts oscillations that may occur, but does not guarantee their existence under all initial conditions. Thus in the system under consideration it must be kept in mind that the actual oscillating patterns is dependent upon the initial conditions. Indeed predicted oscillations need not even occur under some initial conditions. In the next chapter conditions for exact periodic oscillations will be considered.
CHAPTER 4

STABILITY ZONES AND

PERIODIC SELF OSCILLATION

In the previous chapters boundaries on the system parameters were established within which self oscillation could occur and the describing function method was developed to identify these oscillations. In this chapter computational techniques are developed to identify the initial conditions for certain classes of linear plants which will guarantee system stability. Initial conditions not in these zones will yield periodic or cyclic motion; by identifying those periodic trajectories, the entire state space may be mapped according to the resulting motion. Unfortunately in general there will be a large number of zones and oscillating conditions, thus complete mapping is impractical. Attention will be focused on the D22 periodic condition for two reasons: (1) it is the fundamental mode and (2) the prediction of this mode is less reliable when the describing function method is used. Fundamental to the problem of periodic self oscillation is the solution of systems of transcendental equations.
4.1 Initial Condition Stability Zones for the Double IPFM System

The computation of the initial conditions which will result in an asymptotically stable trajectory basically employs a backward mapping technique. For the single modulator system this results in a relatively simple algorithm (see Appendix H) because the effect of the discontinuity of the impulse occurs simultaneously around the loop. However this is not true for the double modulator system; i.e., \( G_1 \) can experience an impulse input while \( G_2 \) is operating continuously.

Consider the double IPFM system with both linear elements having all their poles in the left half of the complex s plane. Now define a capture zone, \( S_{00'} \), in which the initial condition response of the system will produce no impulses from either modulator. This zone will be in a composite state space, \( \mathbb{R}^{n_1 \times n_2} \), where \( n_1 \) and \( n_2 \) are the orders of the linear elements \( G_1 \) and \( G_2 \) respectively. The initial conditions in this zone must obey the following inequalities:

\[
A_1 > \max_0^t \left| \int_0^t c_2^T e^{A_2^\tau} x_{20} \, d\tau \right| \\
\text{and} \quad A_2 > \max_0^t \left| \int_0^t c_1^T e^{A_1^\tau} x_{10} \, d\tau \right| \\
0 \leq t < \infty
\]

(4-1)
where $x_{10}$ and $x_{20}$ are the initial condition vectors for linear elements $G_1$ and $G_2$. Thus if the system initial conditions are:

$$x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \in S_{00}$$

the system will be asymptotically stable because of the assumptions of the class of linear plants. This capture zone is the starting point for subsequent computation.

Note that such a zone will exist for all systems if the assumptions above are satisfied. This is true even if the necessary conditions of Theorem 2.1 (Chapter 2) for instability are satisfied. Thus with the above linear elements it will never be possible to establish sufficient conditions for instability for all initial conditions. This observation is true not only for the $S_{00}$ zone but will hold for other initial condition zones to be defined.

Now let it be desired to locate the initial conditions whose response will produce $m$ pulses from IPFM 1 and $n$ pulses from IPFM 2. These $m-n$ pulses will drive the system to $S'_{00}$ after which the trajectory will tend to the origin, never again leaving $S'_{00}$.

$S'_{00}$ is given by the $x_1, x_2$ states satisfying (4-1) except $2A_1$ and $2A_2$ are used in the inequality. The reason
for this is to include the possibility of the trajectory entering $S'_{00}$ with a modulator just about to fire. Thus the accumulated integral value $A$ can be cancelled by $-A$ for example and still the integral could accumulate up to $-A$ again. Thus the boundary should permit $2A$ as the constraint.

Designate the zone which gives the $m-n$ pulse condition as $S_{mn}$. There will be $m+n$ total pulses emitted from the modulators. If the actual pulse pattern sequence is not designated a priori, the total number of possibilities will be:

$$\frac{(m+n)!}{m! \ n!} \ 2^{(m+n)}$$

corresponding to $b_1=+1$ or $b_2=+1$ in each of the $m$ and $n$ positions. Thus for large $m$ and/or $n$ the number of permutations possible is very large.

To determine the initial condition zone corresponding to a given sequence, arbitrarily select a point in $S'_{00}$, say $x_0=(x_{10}, x_{20})^T$. Associated with this point are in general four numbers corresponding to the maximum and minimum values the modulator integrals would accumulate if $x_0$ where the last switching point, see figure 4.1. These numbers are given by:
Figure 4.1 Illustration of backward mapping to find initial conditions which guarantee stability.
\begin{align*}
a_{11} &= \max\left\{ \int_{0}^{t} c_{2}^{\tau} e^{\frac{A_{2}^{\tau}}{t}} x_{20} d\tau \right\} \\
\frac{a_{21}}{a_{21}} &= \max\left\{ \int_{0}^{t} c_{1}^{\tau} e^{\frac{A_{1}^{\tau}}{t}} x_{10} d\tau \right\} \\
a_{12} &= \min\left\{ \int_{0}^{t} c_{2}^{\tau} e^{\frac{A_{2}^{\tau}}{t}} x_{20} d\tau \right\} \\
\frac{a_{22}}{a_{22}} &= \min\left\{ \int_{0}^{t} c_{1}^{\tau} e^{\frac{A_{1}^{\tau}}{t}} x_{10} d\tau \right\} 
\end{align*}

These numbers should be stored temporarily. Since the impulse sequence is known, the last impulse is used to determine the position of the state, \( x_{i} \), just prior to this last firing. Thus:

\begin{align*}
x_{i}^\wedge(t_{1}^-) &= (x_{i})_{0} - M_{i} b_{il} b_{i} \\
x_{i}(t_{1}^-) &= (x_{i})_{0} \quad \text{if } i=1, \hat{i}=2 \\
&\quad \text{or } i=2, \hat{i}=1
\end{align*}

where \( t_{1}^- \) is the last firing time and \( b_{il} \) is the last impulse emitted before entering \( S'_{00} \), see figure 4.1.

A constraint may be placed on \( x_{i}^\wedge(t_{1}^-) \); ie,

\[ \text{sgn } y_{i}^\wedge(t_{1}^-) = +1 = b_{i} \]

+ if \( i=2 \) and - if \( i=1 \). This must be so since the argument of the modulator integral must be the same sign (polarity)
as the output at $t_1^-$ for the linear systems studied. If the linear elements are written in the standard normal form (see Appendix A) then:

$$\text{sgn} \ x_i^\wedge(t_1^-) = + b_i$$

Since the pulse pattern must be known a priori, first consider the simplest case; $S_{10}$ (or $S_{01}$). With $x_0'$ located map backwards along the trajectory passing through $x_0'$. This backwards motion is continued until the integral for IPPM $i$ is satisfied or a violation noted. These are now discussed.

Whenever a modulator is fired, those temporarily stored numbers ($a_{1j}$'s) are set equal to zero since they can not affect the modulator in the backward mapping anymore. Thus at point $x_0^\prime$ backward motion can continue until,

$$b_{1i} A_i = \pm \int_0^{t_1} c_i T e^{\hat{A}_{1i}^\wedge \tau} x_i^\wedge(0) d\tau \quad (4-4)$$

with $+$ when $\hat{i}=1$ and $-$ when $\hat{i}=2$. These signs will not be carried on; it will be assumed that the proper sign will be associated with $c_i^\wedge$. The integral must be evaluated along the curve satisfying:

$$x_i^\wedge(t_1^-) = e^{-\hat{A}_{1i}^\wedge t_1} x_i^\wedge(0) \quad (4-5)$$
or from (4-4)

\[ b_{i1}A_i = c_i^TA_i^{\top} \{ e^{-A_i^{\top}t_1} - I \} x_i^\wedge(0) \]

then using (4-5) gives:

\[ b_{i1}A_i = c_i^TA_i^{\top} \{ x_i^\wedge(t_1^-) - x_i^\wedge(0) \} \quad (4-6) \]

The last equation represents a hyperplane (straight line in two dimensions). The intersection of the trajectory with this plane is the \( x_{q0} \), the desired initial condition. However modulator IPFM \( \hat{i} \) can never fire. This places requirements on the interval between \( x_0^i \) and \( x_{q0} \). For IPFM \( \hat{i} \) the \( a_{ij} \)'s are retained and added to the modulator integral. Then the requirement,

\[ A_i^\wedge > a_{i1} + \int_0^{t_1} c_i^T e^{A_i^{\top}} x_i(0) \, d\tau \]

\[ -A_i^\wedge < a_{i2} + \int_0^{t_1} c_i^T e^{A_i^{\top}} x_i(0) \, d\tau \quad (4-7) \]

must also be satisfied. There is the possibility of (4-7) being satisfied, but IPFM \( \hat{i} \) firing before reaching \( x_0^i \), especially in oscillatory systems (complex poles). Thus it must also be true that:

\[ A_i > \max_t \left| \int_0^t c_i^T e^{A_i^{\top}} x_i(0) \, d\tau \right| \quad (4-8) \]
Then in summary the backward mapping from $x'_0$ follows the trajectory,

$$
\begin{bmatrix}
A_1 t_1 & x_1 \\
A_2 t_1 & x_2
\end{bmatrix}
$$

until intersection with (4-6). This will determine $x_{00}$. Then (4-7) is checked. If (4-7) is valid then (4-8) must be verified for all time $0 \leq t \leq t_1$. This last step need not be used if $y_1(t)$ is single signed throughout $0 \leq t \leq t_1$; then only the final value of the integral need be checked (equation (4-5) without the $a_{ij}$'s). If all conditions are satisfied then $x_{00}$ is a valid point in the initial condition zone. If however there is any violation, a new $x_0$ must be selected and the process repeated until a good $x_{00}$ is found. Once one $x_{00}$ is located in $S_{10}$ or $S_{01}$, then a search around this point (time moving forward) should yield the boundaries of the zone.

Figure 4.1 shows two trajectories, a and b. Trajectory b is valid, yielding $x_{00}$. Trajectory a however is not valid, since if the initial conditions, $A_1$, satisfying (4-6) are used, (4-8) is not satisfied. The point at which IPFM $i$ would fire is indicated by $A_2$.

**Example 4.1** For the system of example 2.1 with $A_1=A_2=1$ and $M_1=M_2=1.5$ find all the initial condition zones $S_{10}$ and $S_{01}$. 

The capture zone $S_{00}$ is found first:

$$1 > \left| \int_{0}^{\infty} e^{-t} x_1 \, dt \right|$$

$$1 > |x_1|$$

and

$$1 > \left| \int_{0}^{\infty} e^{-2t} x_2 \, dt \right|$$

$$2 > |x_2|$$

see figure 4.2. There are two $S_{10}$ and $S_{01}$ zones. Examine $S_{10}$ ($b_{11} = -1$) first. Select $x'_0 = (-0.85, 1.25)^T \in S'_{00}$ or $S'_{01}$. Then,

$$a_{11} = 0.625 \quad a_{12} = 0$$

$$a_{21} = -0.850 \quad a_{22} = 0$$

$x'_0 = (0.65, 1.25)^T$, thus $x_2(t_1) = 1.25$ which can be used to calculate the intersection from (4-6);

$$0.5 (1.25 - x_2(0)) = -1$$

$$x_2(0) = 3.25$$

Then $x_1(0) = 1.25$. $x_1$ is single signed; then checking the integrals indicates that $x_{00} = (1.25, 3.25)^T$ is a valid point for $S_{10}$. From this point a search is made to determinethe boundary of $S_{10}$. Because the resultant $R^{n1} x R^{n2}$ is two dimensional, graphical techniques can be used (see Appendix I). The other zones are similarly determined.
Figure 4.2 Determination of $S_{01}$ and $S_{10}$ zones for example 4.1.
Note that symmetric pulse patterns will produce symmetrically located zones through the origin.

The zone determination for a general $S_{mn}$ case is considerably more difficult. To illustrate the method consider the $S_{21}$ zone with pattern given in figure 4.3a. The construction is shown in figure 4.3b. As in the basic case an $x_0$ is selected and mapped backwards according to (4-3) and the sign of $y_2(t_1)$ checked. The next to the last impulse can occur at any time, switching the state to the trajectory passing through $x_0'$. However there is a limit to this switching point; i.e., that corresponding to the point satisfying:

$$2A_1 > \min_{t} \left| \int_{0}^{t} e^{\frac{A_2}{2} \tau} x_2 \, d\tau \right|$$

or

$$2A_2 > \min_{t} \left| \int_{0}^{t} e^{\frac{A_1}{2} \tau} x_1 \, d\tau \right|$$

where $x_1$ and $x_2$ are the points on the trajectory passing through $x_0'$.

These limits correspond to entering the trajectory with IPFM 1 or 2 just about to fire, but then not reaching threshold because of a change in signal polarity. Translated geometrically this means that the first backward time intersection of the trajectory through $x_0'$ with either hyperplane:
a) Pulse pattern

b) $S_{21}$ zone determination

Figure 4.3 Illustration for higher $S_{mn}$ zone determination.
A point on the trajectory, \( x \), is chosen between \( x_0 \) and the boundary point determined by (4-9). The next to the last impulse is assumed to switch the state to this point; in this case, \( b_2(1-1) = b_{2l} = +1 \). Thus the state just prior to switch must be:

\[
\begin{align*}
    x(t_{1-1}^-) : & \quad x_1(t_{1-1}^-) = x_1 \\
    & \quad x_2(t_{1-1}^-) = x_2 - M_2 b_2(1-1) b_2
\end{align*}
\] (4-10)

Check the sign of \( y_1(t_{1-1}^-) \) for proper polarity. Expression (4-10) has the same form as (4-3) except with regards to the state being switched. If \( j \) is the impulse number counted from the last impulse and \( k \) is the IPFM modulator firing, \( \hat{k} \) will be the modulator not firing. Then (4-3) or (4-10) may be written in the standard form:

\[
\begin{align*}
    x(t_j^-); & \quad x_k(t_j^-) = x_k - M_k b_{kj} b_k \\
    & \quad x_k(t_j^-) = (x_k^\circ)(t_j^-) \\
    & \quad j = 1, 1-1, 1-2, \ldots, 1 \\
    & \quad k = 1, 2 \text{ and } k=2, 1
\end{align*}
\] (4-11)

The point \( x(t_{1-1}^-) \) defines a trajectory passing through it. This trajectory will determine the points to which a switch can occur (the last in this case). Again there is
a limit to the backward time which is possible; and again the results are exactly like (4-9) except for the subscript modification, thus in general (4-9) becomes:

$$+2A_k = C_k^T A_k^{-1} (x_k^-(t_j) - x_k^+)$$  \hspace{1cm} (4-12)$$

where \( k \) now designates the IPFM modulator about to pulse in the backward mapping. (4-12) determines the intersection conditions with the trajectory passing through \( x(t_{1-1}) \) (or \( x(t_j^-) \) in general).

Select a point on this trajectory between \( x(t_{1-1}) \) (in general \( x(t_j^-) \)) and the limit point \( x_1 \) (\( x_k \) in general). Map this point backwards according to (4-11); ie,

\[
\begin{align*}
  x(t_1^-): & \quad x_2(t_1^-) = x_2 \\
  x_1(t_1^-) &= x_1 - M_1 b_{11} b_1
\end{align*}
\]

Since there are no other impulses, conditions for \( x_{00} \), the starting point will be given by:

$$A_1 = -C_2^T A_2^{-1} (x_2(t_1^-) - x_{20})$$

in general this can be written:

$$A_k = C_k^T A_k^{-1} (x_k^-(t_1^-) - x_k^+)$$  \hspace{1cm} (4-13)$$

Thus \( x_2 \) (or \( x_k^+ \)) is uniquely determined.
Since the points at which the switching took place were initially approximations, they must now be checked. Starting at $x_{00}$ the condition, for $0 \leq t < t_1$

$$A_2 > \max_t \left| \int_0^t c_1 e^{A_1 \tau} x_{10} \, d\tau \right| \quad (4-14)$$

must be satisfied. Let

$$a_2 = \int_0^{t_1} c_1 e^{A_1 \tau} x_{10} \, d\tau$$

on expanding,

$$a_2 = c_1 T A_1^{-1} (x_1(t_1^-) - x_{10})$$

Now after the $b_{11}$ impulse, modulator IPFM 2 fires. Thus:

$$A_2 b_{21} = a_2 + \int_0^t c_1 e^{A_1 \tau} x_1(t_1^+) \, d\tau$$

or

$$A_2 b_{21} = a_2 + c_1 T A_1^{-1} (x(t_2^-) - x_1(t_1^+)) \quad (4-15)$$

During the interval $t_1^- < t < t_2^-$ modulator IPFM 1 may not fire, thus:

$$A_1 > \max_t \left| \int_0^t c_2 e^{A_2 \tau} x_2(t_1^+) \, d\tau \right| \quad (4-16)$$

$$0 \leq t \leq t_2 - t_1$$

let

$$a_1 = \int_0^{t_2-t_1} c_2 e^{-A_2 \tau} x_2(t_1^+) \, d\tau$$
Then for IPFM 1 to fire \((b_{12})\)

\[
A_1 b_{13} = a_1 + \int_0^{t_3-t_2} C_2^T e^{-A_2 \tau} x_2(t_2^+) \, d\tau (4-17)
\]

\[
= a_1 + C_2^T A_2^{-1} (x_2(t_3^-) - x_2(t_2^+))
\]

Finally

\[
A_2 > \max_t \left| \int_0^t C_1^T e^{-A_1 \tau} x_1(t_2^+) \, d\tau \right| (4-18)
\]

letting

\[
a_2 = C_1^T A_1^{-1} (x_1(t_3^-) - x_1(t_2^+))
\]

then

\[
A_2 > \max_t |a_2 + \int_0^t C_1^T e^{-A_1 \tau} x_1(t_3^+) \, d\tau| (4-19)
\]

If the outputs from the linear elements considered in \((4-14)\), \((4-16)\), \((4-18)\) and \((4-19)\) are single signed throughout the time interval in question, then only the end point need be considered. Thus in this special case,

\[
|a_{k,j}^\wedge| < |A_k^\wedge| (4-20)
\]

\[
A_k^\wedge > |a_{k,j}^\wedge| = |C_{k-1}^T A_{k-1}^{-1} (x_k(t_j^-) - x_k(t_j-1^-))|
\]

If any of the conditions \((4-14-19)\) are not satisfied, then \(x_{00}\) determined by \((4.13)\) is not valid. In this case the original \(x_0\) may still be satisfactory, but most likely one of the switch choices was incorrect. Thus a systematic
iteration of the first impulse switching is made. If this fails to produce a valid $x_{00}$, the second switching point is varied. This continues until all switching points for $x_0$ are considered; if no valid $x_{00}$ is found, a new $x_0$ is selected and the process repeated until one valid $x_{00}$ is found. Once a good $x_{00}$ is located, a search around this point will locate the $S_{mn}$ boundaries.

A generalized flow chart for these calculations is shown in figure 4.4 and an example follows.

Example 4.2 Find the $S_{20}$ and $S_{11}$ stability zones corresponding to the system example 4.1 and the pulse patterns shown in figure 4.5.

The zone determined by the above method is shown in figure 4.5b. Note that $S_{20}$ can not exist for this system.

The complete mapping (all $S_{mn}$) is not possible for even the simplest system considered here. Other practical considerations, such as maximum physically possible initial conditions, can be added as constraints to the zone determination. If a special initial condition region is to be investigated, a mapping by computation of response is probably the most logical approach (see Chapter 2, section 2.3).
Read pulse sequence and polarity

Select $x_0 \in S'_{\infty}$

Pulse 1 (last)

$(4-11) \rightarrow x_1(L,M)\quad x_2(L,M)$

Check sign

[If pulse 1-1 from same modulator as pulse 1, use:]

$$A_k = C_k^T A_k^{-1} (x_k(L-1,P) - x_k(L,M))$$

to determine $x(L-1,P)$

Increment backward time $\Delta t$ seconds. Determine $x(L-1,P)$ from the eqn.

$$x(L,M) = A_1^t e^{-1} x_1(L-1,P)\quad A_2^t e^{-1} x_2(L-1,P)$$

Check (4-12)

Pulse 1-1 (second from last)

Same pattern as for pulse 1 above.
Violation

Increment $\Delta t$

Check on integrals between pulses:

In each time interval:

$$\int_{t_{k-1}}^{t_k} \frac{c-T-1}{A_j} \left( x_j(x_{k-1}, M) - x_j(x_k, M) \right) dt = 0$$

Set $l_a = 0$

or

$$\sum_{j=1}^{n} a_j \left( x_j(x_{k-1}, M) - x_j(x_k, M) \right) = 0$$

and

$$\sum_{j=1}^{n} b_j \left( x_j(x_{k-1}, M) - x_j(x_k, M) \right) = 0$$

Check $sgn$ (4-11) + $x_{2L}(1, M)$

Check $sgn$ (4-13) + $x_{00}$

Pulse 1

Pulse 2

$X_{00}$ good
Search around $x_{00}$ and store boundary points.

$x_j(K,M) = \text{State of linear element } j \text{ at } t_k^-$

$x_j(K,P) = \text{State of linear element } j \text{ at } t_k^+$

Interval $k-1$ to $k$, $j$ modulator fires

$\hat{j}$ modulator does not fire

Figure 4.4 Flow chart to determine $S_{mn}$ stability zone.
Figure 4.5 Patterns and stability zone for example 4.2.
4.2 Fundamental Self Periodic Oscillation

The fundamental mode is characterized by the minimum number of impulses from each modulator in a period $T$. For free motion this will in general be the D22 mode. There will be one positive and one negative impulse from each modulator in an oscillatory period, $T$. This must be true so that the average signal is zero (see Theorem 5.1). This pulse pattern is shown below where the following definitions apply:

$$T = \text{Period of oscillation, } T>0$$
$$\alpha = \text{Fractional part of period to the first impulse from IPFM 1, } 0<\alpha<1$$
$$\beta_1, \beta_2 = \text{Fractional part of period to the first and second impulses from IPFM 2 respectively, } 0<\beta_1<\beta_2<1$$

![Diagram of D22 oscillatory pattern definitions](image)

Figure 4.5 D22 oscillatory pattern definitions.
The starting time \(t=0\) for the oscillation period has been assumed to be just after the positive IPFM 1 impulse. This is arbitrary, but makes the analysis easier. Then at \(t=0^+\), \(x_{10}\) and \(x_{20}\) will be respectively the state of \(G_1\) and \(G_2\).

For periodic motion the state of \(G_1\) and \(G_2\) must again be at \(x_{10}\) and \(x_{20}\) at \(t=T^+\). This condition for periodic oscillation can be used to derive necessary and sufficient conditions on the system for the D22 pattern. At \(t=0^+\),

\[
x_{10} = x_1(0)
\]

Through the interval \(0<t<\alpha t\) the state equation becomes,

\[
x_1(t) = e^{A_1t}x_{10}
\]

At \(t=\alpha T\) the state is changed by an impulse input (see Appendix A), giving at \(t=\alpha T^+\):

\[
x_1(\alpha T^+) = e^{A_1\alpha T}x_{10} - M_1b_1
\]

During the next time period, \(\alpha T<t<T\),

\[
x_1(t) = e^{A_1t}x_{10} - M_1e^{A_1(t-\alpha T)}b_1
\]

and finally at \(t=T^+\), after the positive impulse:

\[
x_1(T^+) = e^{A_1T}x_{10} - M_1e^{A_1(1-\alpha)T}b_1 + M_1b_1
\]
But the state must be the original $x_{10}$ at $t=T^+$; therefore,

$$x_{10} = e^{A_1 T} x_{10} + M_1 (I - e^{A_1 (1-\alpha) T}) b_1$$

on rearranging,

$$x_{10} = (I - e^{A_1 T})^{-1} M_1 (I - e^{A_1 (1-\alpha) T}) b_1$$

(4-21)

Thus $x_{10}, \alpha$ and $T$ must satisfy (4-21) for D22 oscillation to be possible.

The same development may be carried out for the $G_2$ plant. This derivation is outlined below:

$$x_{20} = x_2(0)$$

$$x_2(\beta_1 T^+) = e^{A_2 \beta_1 T} x_{20} + b_{21} M_2 b_2$$

$$x_2(\beta_2 T^+) = e^{A_2 \beta_2 T} x_{20} + b_{21} M_2 e^{A_2 (1-\beta_1) T} b_2 + b_{22} M_2 b_2$$

$$x_2(T^+) = e^{A_2 T} x_{20} + b_{21} M_2 e^{A_2 (1-\beta_1) T} b_2 + b_{22} M_2 e^{A_2 (1-\beta_2) T} b_2$$

Then since $x_2(T^+) = x_{20}$ for D22 motion,

$$x_{20} = M_2 (I - e^{A_2 T})^{-1} (b_{21} e^{A_2 (1-\beta_1) T} + b_{22} e^{A_2 (1-\beta_2) T}) b_2$$

(4-22)

This second equation (4-22) is another necessary condition that must be met for D22 oscillation.
The next set of equations to be derived will force the modulators to fire at the times given in figure 4.5. If these next equations along with \((4-21)\) and \((4-22)\) are satisfied then periodic motion will occur provided that the solution gives the minimum firing times of the modulators (see Section 2.1). Then for the period \(0 < t < T\) the IPFM 1 modulator must integrate its input and reach the threshold, \(A_1\), at \(t = T\). Thus:

\[
-A_1 = \int_0^T -y_2(t) \, dt
\]

or

\[
A_1 = \int_0^T c_2 x_2(t) \, dt
\]

\[
= \left[ \frac{\beta_1 T}{c_2} x_2 \right]_0^T = \frac{\beta_1 T}{c_2} x_2(0)
\]

\[
+ \int_{\beta_1 T}^{\alpha T} c_2 e^{A_2 t} x_2 e^{A_2 (t-\beta_1 T)} e^{-e^{A_2 (t-\beta_1 T)}} b_2\, dt
\]

Carrying out the integration and noting that for the linear elements considered \(A_1^{-1}\) and \(A_2^{-1}\) exist, yields,

\[
\frac{T}{c_2 a_2} e^{A_2 T} \frac{x_2(0)}{-c_2 a_2} + \frac{c_2 A_2^{-1} x_2(0) + c_2 A_2^{-1} M_2 b_2}{c_2 A_2^{-1} M_2 b_2} = A_1
\]

rearranging to a more convenient form gives

\[
\frac{T}{c_2 a_2} e^{A_2 T} \frac{x_2(0)}{-I} + \frac{A_2 (\alpha-\beta_1) T}{c_2 a_2} e^{A_2 (\alpha-\beta_1) T} b_2 = A_1
\]

\[
\frac{T}{c_2 a_2} \left\{ (e^{A_2 T} - I) x_2(0) + M_2 b_2 \right\} = A_1
\]

\(4-23\)
The second time interval for IPFM 1, \(\alpha T < t < T\), is integrated in the same manner. The result is:

\[
\begin{align*}
\int_{-c_2A_2}^{A_2} \{(e^{\frac{-A_2T}{e}} - e^{\frac{-A_2\alpha T}{e}})x_{20} + M_2(b_{21}e^{\frac{A_2(1-\beta_1)T}{e}} - b_{21}e^{\frac{A_2(\alpha-\beta_1)T}{e}}) + b_{22}e^{\frac{A_2(1-\beta_2)T}{e}} \} = A_1
\end{align*}
\] (4-24)

Equation (4-24) is not independent of (4-21) and (4-23); ie, by adding (4-23) and (4-24) equation (4-21) is obtained.

Thus one of the three can be eliminated from further consideration. (4-24) is eliminated because it seems to be the most difficult to manipulate.

The same type of result is obtained for IPFM 2 modulator; ie, for IPFM 2 to fire at \(t = \beta_1 T\):

\[
\begin{align*}
\int_{0}^{\beta_1T} y_1(t)dt + \int_{\beta_2T}^{T} y_1(t)dt = b_{21}A_2
\end{align*}
\]

yielding:

\[
\begin{align*}
\int_{c_1A_1}^{A_1} \{(e^{\frac{-A_1\beta_1T}{e}} - e^{\frac{-A_1T}{e}} - e^{\frac{-A_1\beta_2T}{e}})x_{10} + M_1(-e^{\frac{A_1(1-\alpha)T}{e}} + e^{\frac{A_1(\beta_2-\alpha)T}{e}})b_{1}\} = b_{21}A_2
\end{align*}
\] (4-25)

For the second interval,

\[
\int_{\beta_1T}^{\beta_2T} y_1(t)dt = b_{22}A_2
\]

or

\[
\begin{align*}
\int_{c_1A_1}^{A_1} \{(e^{\frac{-A_1\beta_2T}{e}} - e^{\frac{-A_1\beta_1T}{e}})x_{10} - M_1(e^{\frac{A_1(\beta_2-\alpha)T}{e}} - 1)b_{1}\} = b_{22}A_2
\end{align*}
\] (4-26)
Again if (4-25) and (4-26) are added, (4-22) is obtained. Thus (4-25) is eliminated from further consideration.

Thus four equations remain, (4-21), (4-22), (4-23), and (4-26). These can be reduced to two relationships by eliminating $x_{10}$ and $x_{20}$ from (4-23) and (4-26) respectively. Using (4-21) and (4-22) to eliminate this initial condition gives:

\[
T - 1 r = 2aT - 2bT - \alpha_1^2 e_2^1 (e - 1) \beta_1 T A_2 - \alpha_2^2 (1 - e_2^1 (e - 1) \beta_1 T A_2)
\]

or

\[
T - 1 r = 2aT - 2bT - \alpha_1^2 e_2^1 (e - 1) \beta_1 T A_2 - \alpha_2^2 (1 - e_2^1 (e - 1) \beta_1 T A_2)
\]

From (4-26):

\[
T - 1 r = 2aT - 2bT - \alpha_1^2 e_2^1 (e - 1) \beta_1 T A_2 - \alpha_2^2 (1 - e_2^1 (e - 1) \beta_1 T A_2)
\]

or

\[
T - 1 r = 2aT - 2bT - \alpha_1^2 e_2^1 (e - 1) \beta_1 T A_2 - \alpha_2^2 (1 - e_2^1 (e - 1) \beta_1 T A_2)
\]

The equations (4-27) and (4-28) represent the necessary and sufficient conditions for D22 periodic oscillation,
provided the times $\alpha T$, $\beta_1 T$, $\beta_2 T$ and $T$ are the minimum times which satisfy the modulator integral conditions. These equations reduce to two transcendental relationships in four unknowns, $\alpha$, $T$, $\beta_1$, $\beta_2$. The solution, if it exists, will certainly not be unique.

4.2.1 Existence of the basic solution. Numerical techniques will be used to confirm or not the existence of an $\alpha$, $T$, $\beta_1$, $\beta_2$ which satisfy (4-27) and (4-28). The method to be used is a modified gradient-simplex method which will be described below. As a first step let:

$$f_1 = c_1 A_1^{-1} \{ (e^{\beta_2 T} - e^{-\beta_1 T}) (I - e^{-1}) (I - e^{-1}) - A_1 (1-\alpha) T$$

$$- (e^{-1} (\beta_2 - \alpha) T - I) b_1 - b_2 A_2 / M_1$$

$$f_2 = c_2 A_2^{-1} \{ (e^{\alpha T} - e^{-\beta_2 T}) (I - e^{1}) - b_1 A_2 (1-\beta_1) T$$

$$+ b_2 e^{2} + b_2 (e^{2} - 1) b_2 - A_1 / M_2$$

Now square $f_1$ and $f_2$ so that their values will always be positive, then form the squared sum: i.e,

$$f_1^2 + f_2^2 = K \quad (4-30)$$

If a value for $\alpha$, $T$, $\beta_1$, $\beta_2$ can be found such that $K=0$, these values will also satisfy (4-27) and (4-28) and thus give a point on the solution surface. But since $K=0$ is the minimum value (4-30) can obtain, the problem can be
reduced to a minimization of the squared sum expression
over possible \( \alpha, T, \beta_1, \beta_2 \).

There are numerous numerical ways in which this func-
tion can be minimized (37), the method which was finally
selected is a combination of a gradient- simplex technique.
Briefly the procedure is to select an arbitrary value of
\( \beta_1 \) and \( \beta_2 \) satisfying the inequality
\[
0 < \beta_1 < \beta_2 \leq +1
\]
Then minimize (4-30) over all possible \( \alpha, 0<\alpha<1 \), and \( T>0 \)
using the simplex search (to be described). If \( K=0 \) then
the solution is \( \beta_1, \beta_2 \) selected originally and the value
of \( \alpha \) and \( T \) which minimizes (4-30). If \( K\neq0 \) but is \( K_1 \) for
example, \( \beta_1 \) may be varied by some small \( \delta \) amount. The
minimization process is repeated (over \( \alpha \) and \( T \)) to obtain
\( K_2 \). For \( K_2=0 \), the existance of a solution is shown; if
\( K_2\neq0 \) then a new \( \beta_1 \) is chosen in such a direction so as to
reduce \( K \), this is shown in figure 4.6. Once a minimum
for the constant \( \beta_2 \) is found, then search along a constant
\( \beta_1 \) line is performed until a value for \( \alpha, T, \beta_1, \beta_2 \) is
found for which \( K=0 \). If there is no such point then \( D22 \)
oscillation cannot occur.

The method to find the minimum value of the function
(4-30) for a given \( \beta_1, \beta_2 \) is to use a simple polygon; in
Figure 4.6 Variation in $K$ as $\beta_1$ and $\beta_2$ are changed.

Figure 4.7 Simplex method for finding the minimum of a function.
this case it is a triangle (58), see figure 4.7. The method is based upon function evaluation and does not require the computation of derivatives. At each apex of the simplex (polygon) the function is evaluated. The first evaluation in the figure 4.7 is at points A, B, and C. The apex at which the function obtains its maximum value is selected. The mirror image of this vertex is then found. For the case in figure 4.7, the function has a maximum value at apex A. Point D is the mirror image of A. Vertices CBD form a new simplex.

The direction of the sequential triangles is indicated in the figure 4.7 by dashed lines. Once the triangles enclose the minimum value, they will oscillate on a common base or rotate. When this occurs, the triangle size may be reduced or the centroid used to estimate the minimum value.

4.2.2 Variation in D22 oscillation. If the existence of an $\alpha, T, \beta_1, \beta_2$ is established by the minimization technique described in section 4.2.1 which satisfy equations (4-27) and (4-28), then the Implicit Function Theorem may be used to verify that in the vicinity of the above point the functional relationships exist, $\beta_1 = g_1(\alpha, T)$ and $\beta_2 = g_2(\alpha, T)$. The resultant solution represents two
three dimensional surfaces.

The freedom of the variables in the solution makes their display difficult. The following example illustrates D22 oscillation for the neural like system.

**Example 4.3** Consider the system of example 4.2 with $A_1=0.25$, $A_2=0.40$, $M_1=M_2=1.0$. Is D22 oscillation possible?

Select $\beta_1=0.2$ and $\beta_2=0.8$ as a starting point in the simplex search. Choose the sides of the triangle to be 0.05 units in length and establish the vertices at $(\alpha,T)=$((0.5,2.0),(0.5,2.05),(0.4566,2.025)). The initial evaluation of (4-30) for this system is given in Table 4.1 along with 2 subsequent moves which are illustrated in figure 4.8. The resultant ultimately converges to $\beta_1=0.2$, $\beta_2=0.8$, $\alpha=0.50$ and $T=2.50$ where $K=0$ and thus a solution is possible.

Example 4.3 is continued by determining the solution surface for $\beta_1$, $\alpha$, $T$. Then plot the relationship between $\alpha$ and $\beta_1$ for minimum $T$ on this surface. These results are shown in figure 4.9 and 4.11 and an example illustrating the results on the time axis and phase plane in figures 4.12 and 4.13.
TABLE 4.1

Simplex Minimization for Example 4.3

\[ f_1 = \frac{-2(1-\beta_1)T -2(1-\beta_2)T -2\alpha T}{(1-e^{-2T})} + (1-e^{-2(\alpha-\beta_1)T}) \]

\[ f_2 = \frac{-\beta_1 T -\beta_2 T -(1-\alpha)T}{(1-e^{-T})} - (1-e^{-2(\beta_2-\alpha)T}) + 0.4 \]

**Simplex No. 1**

Coordinates of Apex

\begin{array}{ccc}
& T & \text{Value of } f_1^2 + f_2^2 \\
(A) & 0.5000 & 2.0000 & 0.1053 \\
(B) & 0.5000 & 2.0500 & 0.1022 \\
(C) & 0.4566 & 2.0250 & 0.1065 \\
\end{array}

(1) 1.0000 4.0500  Coordinate Sum (less discard)
(2) 1.0000 4.0500  2xAverage of (1)
(3) 0.4566 2.0250  Coordinate of discarded apex
(4) 0.5434 2.0250 (D) New coordinate (2)-(3)

**Simplex No. 2**

\begin{array}{ccc}
& T & \text{Value of } f_1^2 + f_2^2 \\
(A) & 0.5000 & 2.0000 & 0.1053 \\
(B) & 0.5000 & 2.0500 & 0.1022 \\
(D) & 0.5434 & 2.0250 & 0.1033 \\
\end{array}

(1) 1.0434 4.0750
(2) 1.0434 4.0750
(3) 0.5000 2.0000
(4) 0.5434 2.0750 (E)
Figure 4.8 Simplex method for example 4.3.
Figure 4.9 Solution surface for example 4.3.
Figure 4.10 Contours of figure 4.9 (Example 4.3) for constant $\alpha$.

Figure 4.11 Graph of example 4.3 of the minimum period, $T$, vs. $\alpha$ and corresponding $\beta_1$. 
Figure 4.12 Output of linear elements for example 4.3, $\alpha=0.5$, $\beta_1=0.2$, $\beta_2=0.8$, $T=2.5$.

Figure 4.13 Phase plane plot of example 4.3 (Figure 4.12 illustration).
4.3 The Stability of Free D22 Oscillation

The stability of the D22 oscillatory pattern is considered in section 5.2, Chapter 5, where the only difference in the resultant relationships is a constant $R$. For free systems, $R=0$.

4.4. Conclusions

The investigation of the free motion of the general compound IPFM system is very difficult because of the variety in the resultant motion. Initial condition zones scattered throughout the state space yields stable trajectories, while in other zones motion may be periodic. The exact determination of the existence and shape of these regions is difficult numerically, impossible theoretically.

The methods proposed in this chapter are computational and when used in conjunction with the describing function should yield satisfactory descriptions of possible oscillatory patterns.
CHAPTER 5

FORCED PERIODIC MOTION

Thus far attention has been focused on self oscillations; attention is now turned to periodic oscillations under forced conditions. For the class of systems considered in this chapter, a motion of some type will usually be present when the system has an input; the objective of the following discussion is to identify those conditions under which this motion will be periodic. The stability of the periodic motion in the small is then considered with particular attention on the fundamental mode, D11. It is found that the system, when linearized about the periodic oscillation, behaves like a linear discrete system. To check the stability of a general Dmn condition it will be required to know the exact periodic motion a priori. This appears to be an insurmountable drawback because in general this information is not easily obtained due to its dependance on the system initial conditions. For the fundamental mode however, certain techniques are available to check the stability.
5.1 Conditions for Periodic Operation Under constant Input R

The analysis and derivations will be carried out in state variable form; so first cast the linear elements of the system into the standard form (see Appendix A).

For the linear element $G_1(s)$:

$$\begin{align*}
\dot{x}_1(t) &= A_1 x_1(t) + b_1 u_1(t) \\
y_1(t) &= c_1^T x_1(t)
\end{align*}$$

(5-1)

and for the linear element $G_2(s)$:

$$\begin{align*}
\dot{x}_2(t) &= A_2 x_2(t) + b_2 u_2(t) \\
y_2(t) &= c_2^T x_2(t)
\end{align*}$$

(5-1)

Because the system now has an input assumed to be a constant signal, $R$, the error or input to the modulator IPFM will be:

$$e(t) = R - y_2(t)$$

(5-2)

Because the linear elements are excited by impulses from the modulators the solution of (5-1) may be written (see Appendix A):

$$\begin{align*}
x_1(t) &= e^{A_1 t} x_{10} + \sum_{k=1}^{\infty} e^{A_1 (t-t_{1k})} b_{1k} M b_1 \\
y_1(t) &= c_1^T x_1(t)
\end{align*}$$

(5-3)
and
\[ x_2(t) = e^{A_2 t} x_{20} + \sum_{k=1}^{\infty} e^{A_2 (t-t_{2k})} b_{2k} M_2 b_2 \]
\[ y_2(t) = c^T_2 x_2(t) \quad (5-3) \]

If periodic oscillation is to occur, then there must be a time after which,
\[ x_i(t) = x_i(t+T) \quad i=1,2 \]
\[ T = \text{Period of oscillation} \]

It is assumed for the present that the eigenvalues of \( A_1 \) and \( A_2 \) have only negative real parts (ie, the poles of \( G_1(s) \) and \( G_2(s) \) have \( \text{Re} \, s < 0 \)). Then if oscillation is to occur there will be a time when the effect of the initial conditions will be negligible. This does not mean that the initial conditions will not have any influence on the ultimate pattern. This time at which the initial condition response is insignificant can theoretically be considered as \( t=0 \), by letting the real time begin at \( t=-\infty \) and starting the periodic investigation at \( t=0 \). In reality this cannot be done; but for practical considerations after ten times the longest time constant of the linear elements, the approximation to starting at \( t=-\infty \) should be reasonably valid.
If \( t = -\infty \) is the assumed starting time, then at \( t = 0 \) the system will be in periodic oscillation if the conditions are correct for this type of motion. Assuming that such conditions will exist, the motion of the system during the periodic time \( T \) can be described by the equations:

\[
x_1(t) = e^{\frac{A_1}{A_2} t} x_{10} + M_1 \sum_{k=1}^{m} b_{1k} e^{\frac{A_1}{A_2} (t-t_{1k})} y_{1} \]
\[
y_1(t) = c_{11}^T x_1(t) \quad 0 < t \leq T \quad (5-4)
\]

and

\[
x_2(t) = e^{\frac{A_2}{A_2} t} x_{20} + M_2 \sum_{k=1}^{n} b_{2k} e^{\frac{A_2}{A_2} (t-t_{2k})} y_{2} \]
\[
y_2(t) = c_{22}^T x_2(t) \quad 0 < t \leq T \quad (5-4)
\]

where \( m \) is the number of impulses emitted from modulator IPFM 1 in an oscillatory period \( T \) and \( n \) is the number emitted from modulator IPFM 2 in the same period. \( x_{10} \) and \( x_{20} \) represent the initial conditions at times \( t = kT^+ \), \( k = 1, 2, 3, \ldots \). The "+" time is needed to establish the proper initial conditions if an impulse occurs at the time \( t = kT \).

If the input to any IPFM modulator is integrated over an arbitrary time period corresponding to "j" impulses, and if the integrator is reset to zero after each firing, the total integral must be:
\[ \int_0^t e_{in}(t) dt = \sum_{k=1}^{j} b_k A \]

If the motion is periodic then the actual integration boundaries over a period \( T \) is arbitrary and need not coincide with the impulses. Thus for the case of Dmn operation integration over a cycle period \( T \) of the modulator input will yield:

For modulator 1: \[ \sum_{k=1}^{m} b_{1k} A_1 \]

For modulator 2: \[ \sum_{k=1}^{n} b_{2k} A_2 \]

Because of the above arguments, it must be true that for IPFM 1:

\[ \int_0^T e(t) dt = \sum_{k=1}^{m} b_{1k} A_1 \]

\[ = \int_0^T (R - y_2(t)) dt \]

\[ = \int_0^T \left( R - \frac{T}{2} \left( e^{-2T} x_{20} + \sum_{k=1}^{n} e^{-2k(T-t_2^k)} b_{2k} M_{2b_2} \right) \right) dt \]

Since the assumption has been made that \( A_1 \) and \( A_2 \) are to have eigenvalues with negative real parts, \( A_1^{-1} \) and \( A_2^{-1} \) are guaranteed to exist. Thus the integration may be carried on further, giving:

\[ \sum_{k=1}^{m} b_{1k} A_1 = RT - \frac{T}{2} \left( e^{-2T} x_{20} + \sum_{k=1}^{n} e^{-2k(T-t_2^k)} b_{2k} M_{2b_2} \right) \]
This equation can be rearranged into a form suitable for further reduction; ie,

\[
\sum_{k=1}^{m} b_{1k}A_1 = RT - c_{2A_2}^{-1} \left( e^{A_2 T} x_{20} + M_2 \sum_{k=1}^{n} b_{2k}e^{-20} \right)
\]
\[
- x_{20} + \sum_{k=1}^{n} (M_2 b_{2k} b_2)
\]

But periodic motion is assumed so that the motion of \(x_2(t)\) at \(t=T\) will give:

\[
x_2(T) = e^{-20} x_{20} + M_2 \sum_{k=1}^{n} b_{2k}e^{-20} b_2
\]

Since the motion is periodic

\[
x_2(T) = x_2(t_0+T) = x_2(T) = x_{20}
\]

This result when combined with (5-7) and used in (5-6) yields:

\[
\sum_{k=1}^{m} b_{1k}A_1 = RT - c_{2A_2}^{-1} \left( x_{20} - x_{20} - M_2 \sum_{k=1}^{n} b_{2k} b_2 \right)
\]
\[
= RT - c_{2A_2}^{-1} \left( M_2 \sum_{k=1}^{n} b_{2k} \right)
\]

This same development can be carried out for modulator IPPM 2. Starting out with (5-4) and integrating over a period \(T\) and equating the result with (5-5) gives:

\[
\int_{0}^{T} y_1(t) dt = \sum_{k=1}^{n} b_{2k}A_2
\]
\[
\begin{align*}
\int_{0}^{T} c_{1}^T (e^{-x_{10}} + m_{1} \sum_{k=1}^{m} b_{lk} e^{A_{1}(t-t_{1k})} b_{1}) dt
\end{align*}
\]

Then
\[
\sum_{k=1}^{n} b_{2k} A_{2} = c_{1}^T A_{1}^{-1}(e^{A_{1}T} x_{10} + m_{1} \sum_{k=1}^{m} (e^{A_{1}(T-t_{1k})} b_{lk} b_{1})}
\]

(5-10)

Again rearrange into a form suitable for further reduction:
\[
x_{1}(T) = e^{A_{1}T} x_{10} + m_{1} \sum_{k=1}^{m} b_{lk} e^{A_{1}(T-t_{1k})} b_{1} = x_{10}
\]

Substituting this expression in (5-10) gives:
\[
\sum_{k=1}^{n} b_{2k} A_{2} = -c_{1}^T A_{1}^{-1} (x_{10} - x_{10} - m_{1} \sum_{k=1}^{m} b_{lk} b_{1})
\]

(5-11)

From the definition of the modulator operation in Chapter 2, \( b_{lk} = +1 \) and \( b_{2k} = +1 \) is the polarity of the emitted impulses. Thus the following definition can be made which indicates the number of excess positive impulses over negative (or visa-versa); ie,
\[
\begin{align*}
\gamma_{1} &= \sum_{k=1}^{m} b_{lk} = \text{Excess positive (or negative)} \\
&\text{impulses from IPFM 1} \\
\gamma_{2} &= \sum_{k=1}^{n} b_{2k} = \text{Excess positive (or negative)} \\
&\text{impulses from IPFM 2}
\end{align*}
\]

If these definitions are then applied to (5-9) and (5-11) they can be written in the simpler form:
\[ Y_1A_1 = RT + c_{2A_2}^{-1}b_2(M_2 \gamma_2) \]

\[ Y_2A_2 = -c_{1A_2}^{-1}b_1(M_1 \gamma_1) \]  \hspace{1cm} (5-13)

If the second equation of (5-13) is used to eliminate \( \gamma_2 \) from the first equation, the result is:

\[ Y_1A_1 = RT + c_{2A_2}^{-1}b_2(M_2)\left(\frac{-c_{1A_2}^{-1}b_1M_1\gamma_1}{A_2}\right) \]

Rearranging the above equations gives:

\[ RT = Y_1\left[A_1 + M_1M_2 \left(c_{1A_2}^{-1}b_1\right)\left(c_{2A_2}^{-1}b_2\right)\right] A_2 \]  \hspace{1cm} (5-14)

Also \[ \gamma_2 = Y_1\left[\frac{-M_1}{A_2}\left(c_{1A_2}^{-1}b_1\right)\right] \]

Equations (5-14) allow the following group of theorems to be stated:

Theorem 5.1 If both linear elements of the double IPFM system, \( G_1(s) \) and \( G_2(s) \), have all their poles in the left hand of the s plane; ie, \( \text{Re } s < 0 \), then for the unforced oscillatory case of period T, the number of positive and negative impulses from each modulator must balance; ie, \( \gamma_1 = 0 \) and \( \gamma_2 = 0 \).

Proof: From equation (5-14) if \( R = 0 \) then \( \gamma_1 = 0 \) since all other terms are assumed to be non-zero. Thus the second equation of (5-14) then gives \( \gamma_2 = 0 \).
This result has been previously assumed by a heuristic argument, but here it has been proven. Note that this theorem says nothing about the possibility of self oscillation, only that if it happens the signals must have zero average over the period \( T \).

**Theorem 5.2** If both linear elements of the double IPFM system, \( G_1(s) \) and \( G_2(s) \), have all poles in the left hand of the \( s \) plane; ie, \( \text{Re} \ s_p < 0 \), then for the forced oscillatory case, \( R = \text{constant} \), the possible periods of oscillation are integer multiples of:

\[
T = \frac{1}{R} \left[ A_1 + \frac{M_1 M_2}{A_2} \left( \frac{c_1 A_1^{-1} b_1}{A_1} \right) \left( \frac{c_2 A_2^{-1} b_2}{A_2} \right) \right]
\]

**Proof:** This result follows directly from equation (5-14) with \( \gamma_1 \) representing the integer multiplier. The actual period of oscillation, \( T \), and the resulting pattern will depend upon the signal \( R \) and the system initial conditions. The possibilities are so numerous that the only practical method of determination of the actual pattern is by an impulse by impulse calculation as given in Chapter 2.

**Theorem 5.3** If both linear elements of the double IPFM system, \( G_1(s) \) and \( G_2(s) \), have all their poles in the left hand of the \( s \) plane; ie, \( \text{Re} \ s_p < 0 \), then for a constant
input \( R \), periodic motion of period \( T \) given by Theorem 5.2 will occur if

\[
\frac{\gamma_2}{\gamma_1} = -\frac{M_1}{A_2} \left( c_1 A_1^{-1} b_1 \right)
\]

where \( \gamma_1 \) and \( \gamma_2 \) are given by expression (5-12) and characterize the possible oscillatory modes.

**Proof:** From (5-14) this result automatically follows.

This theorem requires some explanation. From the analytic point of view the right hand side of the expression represents the mathematical model of the system in question. It is recognized that this model is an approximation of the real physical system and thus \( \gamma_2/\gamma_1 \) oscillations may not be observed. For example the predicted mode may be 5/2 with a period \( T=3 \text{sec.} \). The physical system may however show a mode 5001/2002 with a period \( T=3003 \text{ sec.} \). Thus observation of the system for a time span of about 30-300 seconds would probably note a 5/2 mode. From a synthesis point of view if a given mode is to be obtained, the right hand side of the expression in Theorem 5.3 must be achieved as closely as possible with the realization that perfect correlation is impossible.

**Example 5.1** With \( G_1(s)=1/s+1 \) and \( G_2(s)=1/s+2 \) and \( M_1=M_2=1 \) and \( A_1=0.25 \) and \( A_2=0.4 \), Theorem 5.3 gives:

\[
\frac{\gamma_2}{\gamma_1} = -(1/0.4)(-1) = 5/2
\]

From Theorem 5.2 the period of oscillation will be

\[
T=(\gamma_1/R)\left\{0.25 + (-1)(-1/2)(1/0.4)\right\}
\]

\[ = 1.5 \left( \gamma_1/R \right) \]
Since \( \gamma_1 \) and \( \gamma_2 \) must be integers, the possible patterns are limited by this last expression; i.e., \( \gamma_1 \neq 1 \) for example.

The stability of the forced oscillation for the above case will be discussed in section 5.2.

Next consider the case where \( G_1(s) \) has a simple pole at the origin. This means that \( A_1 \) will be singular and the previous derivation will have to be modified to take this into account. This modification can be made by first casting the \( G_1(s) \) equations into the canonical form (25). By doing this it is possible to choose one of the states of the system to be constant between impulses since no other inputs to the linear elements occur. Let \( x_{11}(t) \) be this variable. Now further assume that \( A_2 \) is nonsingular. Then for periodic motion all states must satisfy (5-4), thus for \( x_{11}(t) \)

\[
x_{11}(t_0) = x_{11}(t_0 + T)
\] (5-15)

Because this state is constant between impulses, it will consist of step functions and will have the form:

\[
x_{11}(t) = x_{11}(t_0) + M_1 \sum_{k=1}^{m} b_{1k} b_{11} u_1(t-t_{1k})
\]

\[t_0 \leq t < T\]

where \( b_{11} \) is the first element in the \( b_1 \) vector. This constant relates the input signal (impulses) with the state
of the system; there is no loss in generality by selecting it as the first element of the \( b_1 \) vector. At time \( t=T \),

\[
x_{11}(t_0+T) = x_{11}(t_0) + M_1 \sum_{k=1}^{m} b_{1k} \hat{b}_{11}
\]

Thus

\[
x_{11}(t_0+T) - x_{11}(t_0) = M_1 \sum_{k=1}^{m} b_{1k} \hat{b}_{11}
\]

from equation (5-15)

\[
0 = M_1 \hat{b}_{11} \gamma_1
\]

This means that since \( M_1 \) and \( \hat{b}_{11} \) cannot physically be zero, then \( \gamma_1 \) must be zero. If this is so then the number of positive and negative impulses from modulator IPFM 1 must be equal. Therefore over a period \( T \), the average of the input signal to the modulator must be zero. If this average were not zero the integrator of the IPFM 1 would for example accumulate an additional amount during each period \( T \), causing the pulses to shift from one period to the next and eventually producing an extra impulse. Such a condition would not allow \( \gamma_1 = 0 \). If the average input to IPFM 1 is to be zero, then it must be true that:

\[
\frac{1}{T} \int_{0}^{T} e(t) \, dt = 0
\]

\[
\frac{1}{T} \int_{0}^{T} (R - c_2 x_2(t)) \, dt = 0
\]
or
\[ R = \frac{1}{T} \int_{0}^{T} c_{2} x_{2}(t) \, dt \quad (5-16) \]
since in a time period \( T \)
\[ x_{2}(t) = e^{A_{2} t} x_{20} + \sum_{k=1}^{n} b_{2k} e^{A_{2}(t-t_{2k})} b_{2} \]
0 \( \leq t < T \)

Then (5-16) may be written:
\[ R = \frac{1}{T} \int_{0}^{T} c_{2} \{ e^{A_{2} t} x_{20} + \sum_{k=1}^{n} b_{2k} e^{A_{2}(t-t_{2k})} b_{2} \} \, dt \]
\( A_{2}^{-1} \) will exist because of the assumptions made about \( G_{2}(s) \)
and therefore the integration can be carried out:
\[ R = \frac{1}{T} c_{2} A_{2}^{-1} \{ (e^{A_{2} T} - I) x_{20} + \sum_{k=1}^{n} b_{2k} (e^{A_{2}(T-t_{2k})} - I) b_{2} \} \]
again invoke the periodicity condition:
\[ x_{2}(T) = x_{20} = e^{A_{2} T} x_{20} + \sum_{k=1}^{n} b_{2k} e^{A_{2}(T-t_{2k})} b_{2} \]
to reduce (5-17) to,
\[ R = \frac{1}{T} c_{2} A_{2}^{-1} \{ - \sum_{k=1}^{n} b_{2k} M_{2} b_{2} \} \quad (5-18) \]
Using definition (5-12), equation (5-18) reduces to the simpler form:
\[ R = \frac{-M_{2} Y_{2}}{T} \{ c_{2} A_{2}^{-1} b_{2} \} \quad (5-19) \]
This result leads to the following theorem:
Theorem 5.4  For a double IPFM system with linear element $G_1(s)$ containing only poles in the negative half of the $s$ plane with one simple pole at the origin and $G_2(s)$ containing all poles in the left half of the $s$ plane, then for a constant input, $R$, the period of oscillation, $T$, which may occur is an integer multiple of:

$$T = \frac{M_2}{R} (-\frac{T}{C_2A_2^2}b_2)$$

Proof:  This result follows directly from (5-19) by simple algebraic manipulation.

The exact multiple of $T$ depends on $\gamma_2$, the number of excess positive impulses from IPFM 2. The final pattern and thus the value of $\gamma_2$, is dependent upon the initial conditions and/or how the input signal $r(t)$ became constant. The stability of this motion will be discussed in section 5.2.

Theorem 5.5  For a double IPFM system with linear element $G_1(s)$ containing only poles in the negative half of the $s$ plane with a simple pole at the origin and $G_2(s)$ containing all poles in the left hand of the $s$ plane, then for zero input, $r(t)=0$, the number of positive and negative impulses from each modulator must balance; ie $\gamma_1=\gamma_2=0$ during an oscillatory period $T$. 
Proof: These results follow immediately from (5-19) and \( R = 0 \).

The last case to be considered is opposite to the previous situation; i.e., now \( G_2(s) \) will have a simple pole at the origin with all other poles in the left half of the s plane, and \( G_1(s) \) will contain only poles with negative real parts. The argument for oscillation is similar to the previous situation with one additional comment. First observe that in this situation it is possible to cast \( G_2(s) \) into the normal form, with \( x_{21}(t) \) the state variable which represents the output. The \( A_2 \) matrix will then have a form (see Appendix A);

\[
\begin{bmatrix}
0 & 1 & 0 & \ldots \\
0 & 0 & 1 & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
0 & -a_1 & -a_2 & \ldots
\end{bmatrix}
\]

Now an equilibrium condition will exist for \( R \neq 0 \) if all initial conditions of \( G_2(s) \) are zero except for \( x_{21}(t) \) which will have the value \( x_{21}(0) = R/c_{21} \). With \( c_{21} \) the first element in the \( c_2 \) vector. If this is the case,

\[
y_2(t) = c_{21}x_{21} = R
\]

and there will never be an impulse emitted by modulator IPFM 1. The equilibrium point for \( G_1(s) \) corresponding to this condition is the origin of the state space represent-
ing G. For such a condition to be achieved $G_2(s)$ must have the initial conditions stated above, but the conditions on $G_1(s)$ which must be obeyed are that:

$$\max_{t \geq 0} \left| \int_0^t c_1^T e^{-A_1 t} x_{10} \ dt \right| < A_2$$

$$0 \leq t < \infty$$

If $x_{10}$ satisfies the above, IPFM 2 will not fire and thus all motion will tend to the origin of $G_1$ space and the $G_2$ state defined above.

Periodic motion is also possible under these assumptions of the plant configurations. First cast $G_2(s)$ into the cononical form such that $x_{21}(t)$ remains constant between impulses. Then for periodic motion:

$$x_{21}(t_0 + T) = x_{21}(t_0) + M_2 b_{21} b_{21}$$

$$\gamma_2 M_2 b_{21} = 0$$

Thus $\gamma_2 = 0$. The input to modulator IPFM 2 must therefore have zero average over period $T$. But for the input to IPFM 2 (the output of $G_1(s)$) to have zero average, the input to $G_1(s)$ must have zero average since $G_1(s)$ has all poles in the left hand plane. Thus $\gamma_1 = 0$. From the above argument the following theorem can be established.

**Theorem 5.6** For a double IPFM system with linear element $G_1(s)$ containing only poles in the left half of the s
plane and \( G_2(s) \) having a simple pole at the origin but all other poles have negative real parts, the two equilibrium conditions possible for a constant input, \( R, \) are:

1. Periodic motion with \( \gamma_1 = \gamma_2 = 0 \) and

2. No motion with an equilibrium point at \( x_1^T = (0,0,\ldots) \)
   and \( x_2^T = (R/c_{21}, 0, 0,\ldots). \)

Equations for the periodic oscillation will be developed in the next section on stability where they will be used to determine the stability of the equilibrium conditions in (1), Theorem 5.6.

5.2 The Stability of Periodic Motion

The sequence of system restrictions considered in section 5.1 will be followed in this section also. Thus attention is first focused on the system with both linear elements having real part negative poles only.

Consider first the fundamental case, D11, in which only one impulse is being fired from each modulator in a period \( T_e. \) Take the state of the system to be \( x_0 \) immediately after the impulse change at time \( t=0. \) See figure 5.1. Then IPFM 2 will emit an impulse at time \( t=t_{21} \) relative to \( t=0. \) The input to the system is a constant \( R, \) which may be zero. Let a disturbance be measured immediately after the \( I \)th pulse from IPFM 1. This means that the initial conditions after the impulse is no longer \( x_0 \) as was assumed in sec-
Figure 5.1 DII operation nomenclature.

Figure 5.1, but now becomes:

\[ x(0^+) = x_0 + X(I) \]

where \( X(I) \) is the amount of the disturbance. This perturbation will now cause \( x(t) \), the state of the total system, to traverse a slightly different path, resulting in an alteration in the \( t_{21} \) time because of the now different integral of \( y_1(t) \). Altering the firing time of IPFM 2 then alters the output of \( G_2(s) \), \( y_2(t) \). Thus IPFM 1 fires not at \( T_e \), but at some different time interval \( T_1 \), with \( T_1 \neq T_e \) necessarily. Then the net effect is to vary \( x(t) \) after the \((I+1)^{st}\) impulse so that it is not necessarily equal to \( x_0 \). Then the relationship after the impulse at \( T_1 \) is:

\[ x(T_1^+) = x_0 + X(I+1) \]
where $X(I+1)$ is the variation after the $(I+1)$st impulse from the equilibrium state $x_0$. This variation at $(I+1)$ will be functionally related to the initial disturbance at $I$, thus:

$$X(I+1) = g(X(I))$$

If the perturbations $X(I)$ are small then the above equation can be expanded in a Taylor series about $X(I)=0$ and approximated by the first term of the expansion only. Thus:

$$X(I+1) = \frac{\partial g(X(I))}{\partial X(I)} \bigg|_{X(I)=0} X(I)$$  \hspace{1cm} (5-20)

where $\partial g(X(I))/\partial X(I)$ is an nxn Jacobian matrix with the $(ij)$th element equal to

$$\frac{\partial g_i(X(I))}{\partial X(I)}$$

evaluated at $X(I)=0$. Note that (5-20) can be written in the form:

$$X(I+1) = \begin{bmatrix}
\frac{\partial X_1(I+1)}{\partial X_1(I)} & \frac{\partial X_1(I+1)}{\partial X_2(I)} & \frac{\partial X_1(I+1)}{\partial \psi(I)} \\
\frac{\partial X_2(I+1)}{\partial X_1(I)} & \frac{\partial X_2(I+1)}{\partial X_2(I)} & \frac{\partial X_2(I+1)}{\partial \psi(I)} \\
\frac{\partial \psi_2(I+1)}{\partial X_1(I)} & \frac{\partial \psi_2(I+1)}{\partial X_2(I)} & \frac{\partial \psi_2(I+1)}{\partial \psi_2(I)}
\end{bmatrix}\begin{bmatrix}
X_1(I) \\
X_2(I) \\
\psi_2(I)
\end{bmatrix}_{X=0} = G \begin{bmatrix}X_1(I) \\
X_2(I) \\
\psi_2(I)\end{bmatrix}$$  \hspace{1cm} (5-21)

$$= G X(I)$$  \hspace{1cm} (5-22)
where $X^1(K) =$ Perturbation in the state of $G^1(s)$ immediately following the IPFM 1 impulse.

$X^2(K) =$ Perturbation in the state of $G^2(s)$ immediately following the IPFM 1 impulse.

$\Psi^2(K) =$ Perturbation in the IPFM 2 integral value immediately following the IPFM 1 impulse. This integral value is thus considered a state variable of the system.

This matrix form is that of a linear discrete system if $G$ is a constant matrix. Thus for $X$ to approach zero and thus $x(t)$ approach the original periodic motion after a small disturbance it will be necessary for the eigenvalues of $G$ to have an absolute value less than unity. Thus the problem is to find the $G$ matrix so that its eigenvalues may be calculated.

TABLE 5.1

<table>
<thead>
<tr>
<th>Eigenvalues of $G$</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>one value $&gt; 1$</td>
<td>Unstable</td>
</tr>
<tr>
<td>one value $= 1$</td>
<td>No information</td>
</tr>
<tr>
<td>all value $&lt; 1$</td>
<td>Stable</td>
</tr>
</tbody>
</table>

Attention is now turned to the calculation of the elements of the $G$ matrix. With the system in periodic motion, Dll, the state of $G^1(s)$ is governed by the equation:

$$x^1(t) = e^{A_1t}x^1_{10} \quad 0 < t < T_e$$
and just after the impulse:

\[ x_1(T_e^+) = e^{-A_1 T_e} x_{10} + b_{11} M_1 b_1 \]  

(5-23)

\[ = x_{10} \]

The disturbance has been assumed just after the impulse so that there is a new initial condition:

\[ x_{10}^* = x_{10} + x_1(I) \]  

(5-24)

Thus the motion during the next time interval, which may not necessarily be of duration \( T_e \), will be,

\[ x_1^*(t) = e^{-A_1 t} x_{10}^* \quad 0 < t < T_1 \]  

(5-25)

where \( T_1 \) is the new time interval between the impulses at \( I \) and \( I+1 \). Thus using (5-24) in (5-25) yields:

\[ x_1^*(t) = e^{-A_1 t} \left( x_{10} + x_1(I) \right) \quad 0 < t < T_1 \]  

(5-26)

and the state just after the \((I+1)\)st impulse will be:

\[ x_1^*(T_1^+) = e^{-A_1 T_1} \left( x_{10} + x_1(I) \right) + b_{11} M_1 b_1 \]

which can be written in the form:

\[ x_1^*(T_1^+) = x_{10} + x_1(I+1) \]

Then

\[ x_1(I+1) = x_1^*(T_1^+) - x_{10} \]  

(5-27)

\[ = e^{-A_1 T_1} \left( x_{10} + x_1(I) \right) - x_{10} + b_{11} M_1 b_1 \]
In a similar manner the perturbations in $x_2(t)$ and $y_2(t)$ at the impulse instants are:

$$x_2(I+1) = e^{A_2(T_1 - t_{21})} (x_{20} + x_2(I)) + M_2 b_{21} e^{A_2(t_{21})} - x_{20}$$  \hspace{1cm} (5-28)

and

$$y_2(I+1) = \int_{t_{21}}^{T_1} c_{11} e^{A_1 t} (x_{10} + x_1(I)) dt - y_2$$

$$= c_{11} A_1^{-1} \left( e^{A_1 T_1} - e^{A_1 t_{21}} \right) (x_{10} + x_1(I)) - y_2$$  \hspace{1cm} (5-29)

where $y_2$ is the IPFM 2 equilibrium integral value at $t=0$.

Now differentiate (5-27) with respect to $x_1(I)$ noting that $T_1$ is also of a functional relationship to $x_1(I)$.

This differentiation, when carried out yields:

$$\frac{\partial x_1(I+1)}{\partial x_1(I)} = e^{A_1 T_1} + A_1 e^{A_1 T_1} (x_{10} + x_1(I)) \left[ \frac{\partial T_1}{\partial x_1(I)} \right]^T$$

$$\left[ \begin{array}{c} \partial T_1 \\ \partial x_1(I) \end{array} \right] x(I=0)$$  \hspace{1cm} (5-30)

where $\partial T_1/\partial x_1(I)$ is an nx1 column vector with the ith element equal to $\partial T_1/\partial x_1(I)$. This vector may be obtained by noting that:

$$\frac{\partial T_1}{\partial x_1(I)} = \frac{\partial T_1}{\partial t_{21}} \frac{\partial t_{21}}{\partial x_1(I)}$$  \hspace{1cm} (5-31)

where $t_{21}$ is the time of the impulse from modulator IPFM 2 after the Ith impulse from modulator IPFM 1.
Taking the impulse instant as the time of disturbance allows the modulator equation for IPFM 2 integrator to be written as,

\[ \psi_2(I) + \int_0^{t_{21}} y_1(t) dt = b_{21}A_2 \]

where \( \psi_2(I) \) is the value on the IPFM modulator integrator at the Ith impulse time. Given equation (5-26) then,

\[ \psi_2(I) + \int_0^{t_{21}} c_1e^{A_1t} (x_{10} + x_1(I)) dt = b_{21}A_2 \]

or

\[ \psi_2(I) + c_1T^{-1}e^{A_1t_{21}-I} (x_{10} + x_1(I)) = b_{21}A_2 \]

Differentiate with respect to \( x_1(I) \) to determine \( \partial t_{21}/\partial x_1(I) \). This expression gives the variation of the impulse times from IPFM 2 with small disturbances in the state of \( G_1(s) \).

Thus:

\[ 0 + c_1T^{-1}e^{A_1t_{21}} (x_{10} + x_1(I)) \left[ \frac{\partial t_{21}}{\partial x_1(I)} \right]^T + (e^{A_1t_{21}-I}) = 0 \]

rearranging:

\[ c_1T^{-1}e^{A_1t_{21}} (x_{10} + x_1(I)) \left[ \frac{\partial t_{21}}{\partial x_1(I)} \right]^T = c_1T^{-1}(I-e^{A_1t_{21}}) \]

Note that the multiplying factor of \( \partial t_{21}/\partial x_1(I) \) is a scalar quantity since the dimensions of the matrices are respectively \((1xn)(nxn)(nx1)\). Thus it is permissible to divide both sides of the expression by this factor to give:
Now turning attention to modulator IPFM 1, the disturbance will occur after the \((I+1)\)st impulse at time \(T_1\).

Therefore
\[
\int_0^{T_1} (R - y_2(t)) dt = b_{11} A_1
\]

substituting the expression for \(y_2(t)\) then gives,
\[
\int_0^{T_1} (RT_1 - c_1 A_1^{T_1} (x_{20} + x_2(I)) + M_2 b_{21} e^{A_2(T_1 - t_21) - I}) dt = b_{11} A_1
\]

performing the integration yields:
\[
RT_1 - c_2 A_2 \{e^{T_1 - I} (x_{20} + x_2(I)) + M_2 b_{21} e^{A_2(T_1 - t_21) - I} b_2 \}
\]

\[
= b_{11} A_1
\]

Differentiate (5-34) with respect to \(t_{21}\) and obtain,
\[
R \frac{\partial T_1}{\partial t_{21}} - c_2 A_2 \{e^{T_1 - I} (x_{20} + x_2(I)) \frac{\partial A_2}{\partial t_{21}} (T_1 - t_{21}) \}
\]

\[
M_2 b_{21} (e^{A_2(T_1 - t_{21}) - I} b_2) = 0
\]

Collect all \(\frac{\partial T_1}{\partial t_{21}}\) terms,
\[
\{R - c_2 A_2 \{e^{T_1 (x_{20} + x_2(I)) + M_2 b_{21} e^{A_2(T_1 - t_{21})} b_2 \}} \frac{\partial T_1}{\partial t_{21}}
\]

\[
= -c_2 M_2 b_{21} e^{A_2(T_1 - t_{21})} b_2
\]
or
\[
\frac{\partial T_1}{\partial t_{21}} = \frac{T_2 M_2 b_{21} e^{-A_2(T_1-t_{21})}}{R-c_2^2 e^{-A_2^2 T_1} (x_{20}+x_2(I)) + M_2 b_{21} e^{-A_2(T_1-t_{21})}} \quad (5-35)
\]

\[
\frac{\partial t_{21}}{\partial X_1(I)} \text{ can now be calculated from } (5-31) \text{ using } (5-33) \text{ and } (5-35) \text{ to give:}
\]
\[
\frac{\partial T_1}{\partial t_{21}} = \frac{-M_2 b_{21} c_2 e^{-A_2^2 T_1} b_2 \{ c_1^T A_1^{-1} (I-e^{-A_1 T_{21}}) \}}{R-c_2^2 e^{-A_2^2 T_1} (x_{20}+x_2(I)) + M_2 b_{21} e^{-A_2(T_1-t_{21})}} \quad (5-35)
\]

And finally the desired element for the \(G\) matrix,
\[
\frac{\partial X_1(I+1)}{\partial X_1(I)} \bigg|_{X=0} = \begin{bmatrix} a_1 T_1 & a_1 T_1 \\ a_1 T_1 & a_1 T_1 \\ \end{bmatrix} \begin{bmatrix} \frac{\partial T_1}{\partial X_1(I)} \end{bmatrix} \bigg|_{X=0}
\]

If this matrix is evaluated at \(X(I)=0\) then \(T_1=T_e\), and \(t_{21}\) will be the equilibrium time interval between IPFM 1 and IPFM 2 impulses, see figure 5.1. Therefore,
\[
\frac{\partial X_1(I+1)}{\partial X_1(I)} \bigg|_{X(I)=0} = e^{-A_1 T_e} + A_1 e^{-A_1 T_e} X_{10} \begin{bmatrix} \frac{\partial T_1}{\partial X_1(I)} \end{bmatrix} \bigg|_{X(I)=0}
\]

Some further simplification can be made by observing that for periodic Dll motion, the linear elements must obey the relationship:
\[
x_2(T_e) = e^{-A_2 T_e} x_{20} + M_2 b_{21} e^{-A_2(T_e-t_{21})} b_2 = x_{20}
\]
Then,
\[
\frac{\partial X_1 (I+1)}{\partial X_1 (I)} = \frac{A_1 T e}{e_1} + A_1 e_1 T e X_{10} \frac{M_2 b_2 c_2 T e A_2 (T e - t_{21})}{(R - c_2 x_{20})}
\]
(5-36)

The $\frac{\partial X_1 (I+1)}{\partial X_2 (I)}$ can be found in a similar manner. Taking the derivative of (5-27) with respect to $X_2 (I)$:

\[
\frac{\partial X_1 (I+1)}{\partial X_2 (I)} = \frac{A_1 e_1 T e X_{10}}{e_1} \left[ \frac{\partial T_1 - \partial X_2 (I)}{\partial X_2 (I)} \right]_{X(I)=0}
\]
(5-37)

where $\frac{\partial T_1}{\partial X_2 (I)}$ can be found from (5-32) to be:

\[
\frac{\partial T_1}{\partial X_2 (I)} = \frac{C_2 A_2 T e A_2 T e T_{21}}{C_2 x_{20} (R - C_2 X_{20})}
\]
(5-38)

And finally $\frac{\partial X_1 (I+1)}{\partial \psi_2 (I)}$ is,

\[
\frac{\partial X_1 (I+1)}{\partial \psi_2 (I)} = \frac{A_1 e_1 T e X_{10}}{e_1} \left[ \frac{\partial T_1}{\partial \psi_2 (I)} \right]_{X(I)=0}
\]
(5-39)

with:

\[
\frac{\partial T_1}{\partial \psi_2 (I)} = \frac{\partial T_1}{\partial t_{21}} \left. \frac{-1}{T_{21} c_1 e_1 T_{21} X_{10}} \right|_{X=0} \frac{X=0}
\]
(5-40)

where $\frac{\partial T_1}{\partial t_{21}}$ is obtained from (5-35) and $\frac{\partial t_{21}}{\partial \psi_2 (I)}$ results when,
\[
\psi_2(I) + \psi_2 = b_2 A_2 \frac{t_{21}}{2} \int_0^{c_1 t_{12}} e^{A_1 t} (x_{10} + x_1(I)) dt
\]

is differentiated with respect to \( \psi_2(I) \). That is,

\[
\psi_2(I) = b_2 A_2 - c_1 t_{12}^{-1} (e^{A_1 t_{12}} - 1) (x_{10} + x_1(I))
\]

(5-41)

Variation in \( x_2(t) \) at the impulse instants is now obtained from (5-28). Taking the derivative of (5-28), after some rearranging, with respect to \( x_1(I) \) gives:

\[
\frac{\partial x_2(I+1)}{\partial x_1(I)} = A_2 e^{A_1 T_1} (x_{20} + x_2(I)) \begin{bmatrix}
\frac{\partial T_1}{\partial X_1(I)}
\end{bmatrix}^T
+ M_2 b_2 A_2 e^{A_1 (T_1 - t_{21})} \frac{T_1 - t_{21}}{b_2} \begin{bmatrix}
\frac{\partial T_1}{\partial x_1(I)} - \frac{\partial t_{21}}{\partial x_1(I)}
\end{bmatrix}^T
\]

Evaluating at \( x(I) = 0 \) and using the periodicity condition for plant \( G_2(s) \) gives the \( G_2^2 \) element:

\[
\frac{\partial x_2(I+1)}{\partial x_1(I)} \bigg|_{x=0} = A_2 \left( x_{20} \frac{\partial T_1}{\partial t_{21}} \right) - M_2 b_2 A_2 e^{A_1 (T_1 - t_{21})} \frac{T_1 - t_{21}}{b_2} \begin{bmatrix}
\frac{\partial T_1}{\partial x_1(I)} - \frac{\partial t_{21}}{\partial x_1(I)}
\end{bmatrix}^T
\]

(5-42)

Since the remaining derivatives in (5-42) are already known, (5-35) and (5-33), the matrix can now be evaluated.

Next \( \frac{\partial x_2(I+1)}{\partial x_2(I)} \) is obtained from (5-28) as:

\[
\frac{\partial x_2(I+1)}{\partial x_2(I)} \bigg|_{x=0} = A_2 e^{A_1 T_1} x_{20} \begin{bmatrix}
\frac{\partial T_1}{\partial x_2(I)}
\end{bmatrix}^T + A_2 e^{A_1 T_1} + A_2 M_2 b_2 A_2 e^{A_1 (T_1 - t_{21})} \frac{T_1 - t_{21}}{b_2} \begin{bmatrix}
\frac{\partial T_1}{\partial x_2(I)} / \partial x_2(I)
\end{bmatrix}^T
\]

(5-43)
with the time derivative evaluated from (5-34) to be,

\[
\begin{align*}
\frac{\partial T_1}{\partial x_2(I)} & = \frac{T \cdot A_1 e^{-\frac{T}{c_2} R - c_2 x_2}}{\alpha^2} & \left(\frac{\partial T_1}{\partial x_2(I)}\right)_{x=0} = \frac{T \cdot A_1 (e^{-\frac{T}{c_2} R - c_2 x_2})}{\alpha^2} \\
\end{align*}
\]

(5-44)

The remaining results are stated without explanation, they may be obtained from (5-28), (5-29), (5-32) and (5-34).

\[
\begin{align*}
\frac{\partial x_2(I+1)}{\partial \psi_2(I)} & = A_2 e^{\frac{A_2 T e^{\frac{T}{c_2} R - c_2 x_2}}{\alpha^2}} \left(\frac{\partial T_1}{\partial \psi_2(I)}\right)_{X=0} + M_b e^{\frac{A_2 T e^{\frac{T}{c_2} R - c_2 x_2}}{\alpha^2}} b_2 \\
\frac{\partial x_2(I)}{\partial \psi_2(I)} & = \frac{\partial t_1}{\partial \psi_2(I)} - \frac{\partial t_21}{\partial \psi_2(I)} \\
\end{align*}
\]

(5-45)

\[
\begin{align*}
\frac{\partial t_21}{\partial \psi_2(I)} & = \frac{-1}{e^{\frac{T}{c_2} R - c_2 x_2} x_2} \\
\end{align*}
\]

(5-46)

\[
\begin{align*}
\frac{\partial x_2(I+1)}{\partial \psi_2(I)} & = \frac{T \cdot A_1 e^{\frac{T}{c_2} R - c_2 x_2}}{\alpha^2} + \frac{T \cdot A_1 e^{\frac{T}{c_2} R - c_2 x_2}}{\alpha^2} x_2 \left(\frac{\partial T_1}{\partial x_1(I)}\right)_{X=0} \\
\end{align*}
\]

(5-47)

\[
\begin{align*}
\frac{\partial x_2(I+1)}{\partial \psi_2(I)} & = \frac{T \cdot A_1 e^{\frac{T}{c_2} R - c_2 x_2}}{\alpha^2} x_2 \left(\frac{\partial T_1}{\partial x_2(I)}\right)_{X=0} \\
\end{align*}
\]

(5-48)

\[
\begin{align*}
\frac{\partial x_2(I+1)}{\partial \psi_2(I)} & = \frac{T \cdot A_1 e^{\frac{T}{c_2} R - c_2 x_2}}{\alpha^2} x_2 \left(\frac{\partial T_1}{\partial x_2(I)}\right)_{X=0} \\
\end{align*}
\]

(5-49)
Combining all of the above derivation will give the following theorem.

**Theorem 5.7** For a compound IPFM system with linear elements having only negative real part poles, Dll motion produced by a constant input $R$ will be stable in the small if the $G$ matrix (5-22) has all eigenvalues with absolute values less than unity.

The expressions for this matrix appear to be very complex but certain computational short cuts are possible.

From Theorem 5.3 the period $T_e$ may be found for Dll motion:

$$T_e = \frac{1}{R} \left\{ A_1 + \frac{M_1 M_2}{A_2} \left( C_{22} A_2^{-1} b_2 \right) \left( C_{11} A_1^{-1} b_1 \right) \right\}$$

Also the initial conditions can easily be determined by noting that:

$$x_1(T_e^+) = x_{10} = e^{A_1 T_e} x_{10} + M_1 b_{11} b_1$$

solving for $x_{10}$ gives,

$$x_{10} = (I - e^{A_1 T_e})^{-1} M_1 b_{11} b_1$$

Likewise for $x_{20}^+$:

$$x_{20}^+ = x_2(T_e^+) = e^{A_2 T_e} x_{20} + M_2 b_{21} e^{A_2 (T_e - t_{21})} b_2$$

$$= (I - e^{A_2 T_e})^{-1} M_2 b_{21} e^{A_2 (T_e - t_{21})} b_2$$

This gives $x_{20}$ as a function of $t_{21}$ only. If $t_{21}$ is known a priori, $x_{20}$ can be determined and the eigenvalues of $G$
checked for stability. If however $t_{21}$ is not known, this single variable can be varied between $0 < t_{21} < T_e$ to determine the elements of the $G$ matrix. This can be done on a root locus plot to insure that for $0 < t_{21} < T_e$ the roots of $|I_q - G|$ are within the unit circle in the complex $q$ plane. Note that this root locus will probably be unlike those normally plotted for polynomials if $t_{21}$ is the variable. Failure for the roots to lie within the unit circle means that the Dll oscillations are not stable, and thus can not be physically observed, or that for the particular $t_{21}$, Dll motion is not possible. The following example illustrates the above stability theorem.

**Example 5.2** Consider the system of example 5.1, except with $A_1 = A_2 = M_1 = M_2 = 1$ so that Dll motion is possible. Then the fundamental period is from Theorem 5.2,

$$T = \frac{1}{1 + 1x1(-1)(-0.5)} = 1.5 \text{ sec.}$$

$$\gamma_2 = -(-1)^{1} \frac{1}{1} \gamma_1 = \gamma_1$$

Then $x_{10}$ and $x_{20}$ may be found to be:

$$x_{10} = (1 - e^{-1.5})^{-1}$$

$$x_{20} = (1 - e^{-3})^{-1} e^{-2(1.5-t_{21})}$$

The eigen-values of the $G$ matrix have been found for $t_{21} = 1.0, t_{21} = 1.245,$ and $t_{21} = 1.4$. These values have been plotted in figure 5.2. Note that for $t_{21} = 1.0$ the Dll
Figure 5.2 Root locus of G matrix eigenvalues for $t_{21} = 1.0, 1.245, 1.4$. 

- $\triangle$ at 5.741
- $X$ for $t_{21} = 1.0$
- $\bigcirc$ for $t_{21} = 1.245$
- $\bigtriangleup$ for $t_{21} = 1.4$
motion is stable, but for $t_{21}=1.40$, Dll motion cannot physically occur. $t_{21}=1.245$ is the boundary time below which the motion will be stable.

For the general Dmn case the determination of stability is much more difficult. First assume that all motion is known for this mode of oscillation; i.e., the $t_{ik}$'s, $T_e$, $x_1(t)$ and $x_2(t)$ in the period of oscillation. Then following the same line of reasoning it will be necessary to show that all disturbances $x$ go to zero. These disturbances will again be measured just after an impulse from the IPFM 1 modulator.

Equations (5-21) and (5-22) still essentially apply, however the elements of the $G$ matrix are based on a general Dmn motion of the system. Thus they take on the form:

$$X(I+m) = G(X(I))$$

(5-50)

where $m$ is the number of impulses in a period $T_e$ from IPFM 1; $n$ is the number from IPFM 2 in the same period. Then following the same development as for the Dll case,

$$x_1(I+m) = e^{A_1 T_1}(x_{10} + x_1(I)) + M_1 \sum_{k=1}^{m} b_{1k} e^{A_1(T_1-t_{1k})} b_1 = x_{10}$$

$$x_2(I+m) = e^{A_2 T_1}(x_{20} + x_2(I)) + M_2 \sum_{k=1}^{n} b_{2k} e^{A_2(T_1-t_{2k})} b_2 = x_{20}$$

(5-51)
\[ \psi_2(I+m) = c_1 a_1 {T-1} \{ (e^{A_1 T_1} - e^{A_1 t_2 n}) (X_{10} + X_1(I)) \} \]  
\[ + M_1 \sum_{k=1}^{m-1} b_{1k} (e^{A_1 (T_{1-t_1 k})} - e^{A_1 (t_2 n-t_1 k)}) b_1 \} - \psi_2 \]

The resulting elements of the G matrix are then:

\[ \frac{\partial X_1(I+m)}{\partial X_1(I)} \bigg|_{X(I)=0} = e^{A_1 T_{1}e} + A_1 e^{A_1 T_{1}e} x_{10} \frac{\partial T_{1}}{\partial X_1(I)} \bigg|_{X(I)=0}^T \]
\[ + M_1 A_1 \sum_{k=1}^{m} b_{1k} e^{A_1 (T_{1-t_1 k})} b_1 \frac{\partial T_{1} - \partial t_{1k}}{\partial X_1(I) \partial X_1(I)} \bigg|_{X(I)=0}^T \]
\[ = e^{A_1 T_{1}e} + A_1 x_{10} \frac{\partial T_{1}}{\partial X_1(I)} \bigg|_{X(I)=0}^T \]

\[ \frac{\partial X_1(I+m)}{\partial X_2(I)} \bigg|_{X(I)=0} = A_1 x_{10} \frac{\partial T_{1}}{\partial X_2(I)} \bigg|_{X(I)=0}^T \]
\[ - A_1 M_1 \sum_{k=1}^{m} b_{1k} e^{A_1 (T_{1-t_1 k})} b_1 \frac{\partial t_{1k}}{\partial X_2(I)} \bigg|_{X(I)=0}^T \]
\[
\frac{\partial X_1(I+m)}{\partial \psi_2(I)} = \frac{A_1 x_{10} \frac{\partial T_1}{\partial \psi_2(I)}}{x(I)=0} \quad (5-54)
\]

\[
\frac{\partial X_1(I+m)}{\partial \psi_2(I)} = \frac{A_1 M_1 \sum_{k=1}^{m} b_{1k} e^{A_1(Te-t_{1k})} - A_1 M_1 \sum_{k=1}^{m} b_{1k} e^{A_1(Te-t_{1k})}}{x(I)=0} \quad (5-55)
\]

\[
\frac{\partial X_2(I+m)}{\partial \psi_2(I)} = \frac{A_2 x_{20} \frac{\partial T_1}{\partial \psi_2(I)}}{x(I)=0} \quad (5-56)
\]

\[
\frac{\partial X_2(I+m)}{\partial \psi_2(I)} = \frac{A_2 M_2 \sum_{k=1}^{n} b_{2k} e^{A_2(Te-t_{2k})} - A_2 M_2 \sum_{k=1}^{n} b_{2k} e^{A_2(Te-t_{2k})}}{x(I)=0} \quad (5-57)
\]

\[
\frac{\partial \psi_2(I+m)}{\partial X_2(I)} = \frac{T_{-1}(e^{A_1T_{-1}x_{10}} + M_1 \frac{\partial T_{-1}}{\partial X_2(I)} x_{10} - \frac{\partial t_{2n}}{\partial X_2(I)} \frac{\partial X_1(I)}{X(I)=0}}{X(I)=0} \quad (5-58)
\]

\[
\frac{\partial \psi_2(I+m)}{\partial X_1(I)} = \frac{\sum_{k=1}^{m-1} b_{1k} e^{A_1(Te-t_{1k})} - \sum_{k=1}^{m-1} b_{1k} e^{A_1(Te-t_{1k})}}{X(I)=0} \quad (5-59)
\]
The time derivatives may be obtained from the integral conditions on the modulators in much the same way as was done for the Dll case.

**IPFM 1:**
\[
\psi_2 + \int_{t_1(k-1)}^{t_{1k}} \left( R - \frac{A_t}{2} (e^{x_{20} + x_2(I)}) \right) dt = A_1 b_{1k}
\]
\[
k = 1, 2, \ldots, m
\]
\[
t_{10} = 0
\]

with \( p_2 \) the number of impulses fired from IPFM 2 in the period \( 0 < t < t_{1k} \).

**IPFM 2:**
\[
\psi_2 + \int_{t_2(k-1)}^{t_{2k}} \left( \frac{A_t}{2} (x_{10} + x_1(I)) \right) dt = A_2 b_{2k}
\]
\[
k = 1, 2, \ldots, n
\]
\[
t_{20} = 0
\]
where $\psi_2$ is the value of the IPFM 2 integral at $t=0$ and is valid for $k=1$ only. For $k\neq1$, $\psi_2$ is taken to be zero. $p_1$ is the number of impulses fired from IPFM 1 in the period $0<t<t_{2k}$.

Because of the infinite number of possible patterns in an IPFM system, evaluation beyond this point is best illustrated by means of an example. After the example, generalized forms will be given.

**Example 5.3** Consider the following example of D33 operation, see figure 5.3.

The following manipulations on the IPFM integral expressions will yield the needed time derivatives to evaluate the $G$ matrix. For IPFM 1 for the first time interval:
\[ R_{t_{11}} = c_2 A_2^{-1} \{ (e_1 t_{11} - I)(x_{20} + x_2(I)) + M_2 \sum_{k=1}^{2} (e_2 (t_{11} - t_{2k}) - I)b_2 \} = A_1 \]

Then

\[ R \frac{\partial t_{11}}{\partial X_1(I)} \bigg|_{X(I)=0} = c_2 \left( e_2 t_{11} x_{20} \frac{\partial t_{11}}{\partial X_1(I)} \right) \bigg|_{X(I)=0} + M_2 \left( e_2 (t_{11} - t_{21}) b_2 \left( \frac{\partial t_{11} - \partial t_{21}}{\partial X_1(I)} \right) \right) \bigg|_{X(I)=0} + e_2 (t_{11} - t_{22}) b_2 \left( \frac{\partial t_{11} - \partial t_{22}}{\partial X_1(I)} \right) \bigg|_{X(I)=0} \]

Rearranging,

\[ R \frac{\partial t_{11}}{\partial X_1(I)} \bigg|_{X(I)=0} = c_2 \left( e_2 t_{11} x_{20} + M_2 e_2 (t_{11} - t_{21}) b_2 \right) \frac{\partial t_{11}}{\partial X_1(I)} \bigg|_{X(I)=0} + M_2 e_2 (t_{11} - t_{22}) b_2 \frac{\partial t_{11}}{\partial X_1(I)} \bigg|_{X(I)=0} + c_2 M_2 \sum_{k=1}^{2} e_2 (t_{11} - t_{2k}) b_2 \frac{\partial t_{2k}}{\partial X_1(I)} \bigg|_{X(I)=0} = 0 \]

Using the periodicity requirements then yields:

\[ 0 = (R - c_2 x_2(t_{11})) \frac{\partial t_{11}}{\partial X_1(I)} \bigg|_{X(I)=0} \]

(5-63)
The IPFM 2 firing time derivatives may be found from (5-61):

\[ \psi_2(I) + c_1^T A_1^{-1} (e_1 t_{21} - I)(x_{10} + x_{1}(I)) = A_2 \]

Then

\[ c_1^T A_1^{-1} (e_1 t_{21} - I) + c_1^T e_1 t_{21} x_{10} \frac{\partial t_{21}}{\partial x_{1}(I)} \bigg|_{X(I)=0} = 0 \quad (5-64) \]

allows \( \partial t_{21}/\partial x_{1}(I) \) to be found. For the second time interval of IPFM 2, (5-61) gives:

\[ c_1^T A_1^{-1} (e_1 t_{22} - e_1 t_{21}) + c_1^T e_1 t_{22} x_{10} \frac{\partial t_{22}}{\partial x_{1}(I)} \bigg|_{X(I)=0} = 0 \quad (5-65) \]

allows \( \partial t_{22}/\partial x_{1}(I) \) to be found since \( \partial t_{21}/\partial x_{1}(I) \) is already known from (5-64). Using the results of (5-64) and (5-65) in (5-63) gives \( \partial t_{12}/\partial x_{1}(I) \). Next \( \partial t_{12}/\partial x_{1}(I) \) is found in the same manner from (5-61):

\[ 0 = (R - c_2^T x_2(t_{12})) \frac{\partial t_{12}}{\partial x_{1}(I)} \bigg|_{X(I)=0} = \left( R - c_2^T x_2(t_{11}) \right) \frac{\partial t_{11}}{\partial x_{1}(I)} \bigg|_{X(I)=0} \]

\[ + c_2^T M_2 \sum_{k=1}^{2} (e_2(t_{12} - t_{2k}) - e_2(t_{11} - t_{2k})) b_2 \frac{\partial t_{2k}}{\partial x_{1}(I)} \bigg|_{X(I)=0} \]

Since \( \partial t_{11}/\partial x_{1}(I) \), \( \partial t_{21}/\partial x_{1}(I) \) and \( \partial t_{22}/\partial x_{1}(I) \) have already been found, \( \partial t_{12}/\partial x_{1}(I) \) may be evaluated. Finally from (5-61) to find \( \partial T_{1}/\partial x_{1}(I) \) yields,
\[ 0 = (R - c_{22}x_{20}) \frac{\partial T_1}{\partial X_1(I)} \bigg|_{X(I)=0}^T - (R - c_{22}(t_{12})) \frac{\partial t_{12}}{\partial X_1(I)} \bigg|_{X(I)=0}^T \]

\[ + c_{22}^2 \sum_{k=1}^{3} b_{2k} (e^{A_{2}(t_2-t_{2k})} - e^{A_{2}(t_{12}-t_{2k})})b_{2} \frac{\partial t_{2k}}{\partial X_1(I)} \bigg|_{X(I)=0}^T \]

where only \( \partial t_{23}/\partial X_1(I) \) need be found to evaluate \( \partial T_1/\partial X_1(I) \).

Thus from (5-62),

\[ 0 = c_{1}A_{1}^{-1} (e^{A_{1}t_{23}} - e^{A_{1}t_{22}}) + c_{1}x_{1}(t_{23}) \frac{\partial t_{23}}{\partial X_1(I)} \bigg|_{X(I)=0}^T \]

\[ - c_{2}x_{1}(t_{22}) \frac{\partial t_{22}}{\partial X_1(I)} \bigg|_{X(I)=0}^T \]

\[ - M_{1}c_{1} \sum_{k=1}^{2} b_{1k} (e^{A_{1}(t_{23}-t_{1k})} - e^{A_{1}(t_{22}-t_{1k})})b_{1} \frac{\partial t_{1k}}{\partial X_1(I)} \bigg|_{X=0}^T \]

Only \( \partial t_{23}/\partial X_1(I) \) is unknown and may thus be solved for.

Thus the derivatives of all the impulse times, \( t_{11}, t_{12}, T_1, t_{21}, t_{22}, t_{23} \), with respect to \( X_1(I) \) have now been found.

Carrying out the same operations for the time derivatives with respect to \( X_2(I) \) and \( \Psi_2(I) \) gives:

From (5-62),

\[ \frac{\partial t_{21}}{\partial X_2(I)} \bigg|_{X(I)=0} = \frac{\partial t_{22}}{\partial X_2(I)} \bigg|_{X(I)=0} = 0 \]
\[ 0 = c^T_1 x_1(t_{23}) \frac{\partial t_{23}}{\partial x_2(I)} \left[ \begin{array}{c} T \\ x(I)=0 \end{array} \right] - \frac{2}{c^T_1 M_1 \sum_{k=1}^{2} b_{1k} (e^{(t_{23}-t_{1k})} - e^{(t_{22}-t_{1k})}) b_1 \frac{\partial t_{1k}}{\partial x_2(I)} \right] \]

From (5-61)

\[ (R - c^T_2 x_2(t_{11})) \frac{\partial t_{11}}{\partial x_2(I)} \left[ \begin{array}{c} T \\ x(I)=0 \end{array} \right] - c^T_2 A_2^{-1} (e^{A_2 t_{11}} - I) = 0 \]

which gives \( \frac{\partial t_{11}}{\partial x_2(I)} \). Then \( \frac{\partial t_{21}}{\partial x_2(I)} \) may be similarly obtained from (5-61),

\[ 0 = (R - c^T_2 x_2(t_{12})) \frac{\partial t_{12}}{\partial x_2(I)} \left[ \begin{array}{c} T \\ x(I)=0 \end{array} \right] - (R - c^T_2 x_2(t_{11})) \frac{\partial t_{11}}{\partial x_2(I)} \left[ \begin{array}{c} T \\ x(I)=0 \end{array} \right] - c^T_2 A_2^{-1} (e^{A_2 t_{12}} - e^{A_2 t_{11}}) \]

Since \( \frac{\partial t_{11}}{\partial x_2(I)} \) is known this expression gives \( \frac{\partial t_{12}}{\partial x_2(I)} \).

Then finally for \( \frac{\partial t_1}{\partial x_2(I)} \):

\[ 0 = (R - c^T_2 x_{20}) \frac{\partial t_1}{\partial x_2(I)} \left[ \begin{array}{c} T \\ x(I)=0 \end{array} \right] - (R - c^T_2 x_2(t_{12})) \frac{\partial t_{12}}{\partial x_2(I)} \left[ \begin{array}{c} T \\ x(I)=0 \end{array} \right] - \frac{c^T M_2 b_{23} e^{A_2 (t_e-t_{23})}}{b_2} \frac{\partial t_{23}}{\partial x_2(I)} \left[ \begin{array}{c} T \\ x(I)=0 \end{array} \right] + c^T_2 A_2^{-1} (e^{A_2 t_e} - e^{A_2 t_{12}}) \]

Care must be taken in evaluating the above relationships to insure that \( (t_{1j} - t_{2k}) > 0 \), otherwise the expression is zero.
Now all the time derivatives with respect to $X_2(I)$ are known. Next consider the derivatives with respect to $\psi_2(I)$.

From (5-62):

$$1 + c_1^T x_1(t_{21}) \frac{\partial t_{21}}{\partial \psi_2(I)} \bigg|_{X(I)=0} = 0$$

$$c_1^T x_1(t_{22}) \frac{\partial t_{22}}{\partial \psi_2(I)} \bigg|_{X(I)=0} - c_1^T x_1(t_{21}) \frac{\partial t_{21}}{\partial \psi_2(I)} \bigg|_{X(I)=0} = 0$$

$$0 = c_1^T x_1(t_{23}) \frac{\partial t_{23}}{\partial \psi_2(I)} \bigg|_{X(I)=0} - c_1^T x_1(t_{22}) \frac{\partial t_{22}}{\partial \psi_2(I)} \bigg|_{X(I)=0}$$

$$- M_1 \sum_{k=1}^{2} b_{1k} \left( e_{A_1(t_{23}-t_{1k})} - e_{A_1(t_{22}-t_{1k})} \right) b_{1} \frac{\partial t_{1k}}{\partial \psi_2(I)} \bigg|_{X(I)=0}$$

From (5-61),

$$0 = (R - c_2^T x_2(t_{11})) \frac{\partial t_{11}}{\partial \psi_2(I)} \bigg|_{X(I)=0}$$

$$+ M_2 \sum_{k=1}^{2} b_{2k} e_{A_2(t_{11}-t_{2k})} b_{2} \frac{\partial t_{2k}}{\partial \psi_2(I)} \bigg|_{X(I)=0}$$

$$0 = (R - c_2^T x_2(t_{12})) \frac{\partial t_{12}}{\partial \psi_2(I)} \bigg|_{X(I)=0} - (R - c_2^T x_2(t_{11})) \frac{\partial t_{11}}{\partial \psi_2(I)} \bigg|_{X(I)=0}$$

$$+ M_2 \sum_{k=1}^{2} b_{2k} \left( e_{A_2(t_{12}-t_{2k})} - e_{A_2(t_{11}-t_{2k})} \right) b_{2} \frac{\partial t_{2k}}{\partial \psi_2(I)} \bigg|_{X(I)=0}$$
0 = (R - \frac{c_T X_{20}}{2}) \frac{\partial T}{\partial \psi_2(I)} \bigg|_{X(I)=0} - (R - \frac{c_T X_2(t_{12})}{2}) \frac{\partial t_{12}}{\partial \psi_2(I)} \bigg|_{X(I)=0}

+ M_2 c_T \sum_{k=1}^{3} b_{2k} \left( A_2 (T e^{-t_{2k}}) - e^{-A_2 (t_{12} - t_{2k})} \right) b_2 \frac{\partial t_{2k}}{\partial \psi_2(I)} \bigg|_{X(I)=0}

With the above expressions as motivation, the derivatives are written in their generalized form on the following pages. Note the negative sign in the \( x_1(t) \) and \( x_2(t) \) expressions. This follows from the derivation and is added in case the state experiences a discontinuity (impulse input) at the time in question. All the time derivatives can now be evaluated by sequentially evaluating in the order of the pulses; ie from the example the order would be \( t_{21}, t_{22}, t_{11}, t_{12}, t_{23}, T \). These relationships are easily programmed on the digital computer with only the system matrices, modulator characteristics and pulse times as input data.

The stability of the equilibrium for the case in which \( g_1(s) \) has a simple pole at the origin is found to be unstable. Since stability of motion is assumed not to occur if any condition can be found which causes the system never again to return to the original motion, the following arguments will show that the equilibrium is unstable.
\[
\begin{align*}
\frac{\partial t_{1k}}{\partial x_1(I)} & \bigg|_{x(I)=0} = (R - c_{2x_2}(t_{1(k-1)}^-)) \frac{\partial t_{1(k-1)}}{\partial x_1(I)} + \frac{\partial t_{1(k-1)}}{\partial x_1(I)} - c_{2M_2} \sum_{j=1}^{p_2} b_{2j} \left( e^{A_2(t_{1k-t_2j})} - e^{A_2(t_{1(k-1)-t_2j})} \right) b_2 \frac{\partial t_{2j}}{\partial x_1(I)} \\
\frac{\partial t_{1k}}{\partial x_2(I)} & \bigg|_{x(I)=0} = (R - c_{2x_2}(t_{1k}^-)) \frac{\partial t_{1(k-1)}}{\partial x_2(I)} + \frac{\partial t_{1(k-1)}}{\partial x_2(I)} - c_{2A_2} \left( e^{A_2(t_{1k-t_2j})} - e^{A_2(t_{1(k-1)-t_2j})} \right) b_2 \frac{\partial t_{2j}}{\partial x_2(I)} \\
\frac{\partial t_{1k}}{\partial \psi_2(I)} & \bigg|_{x(I)=0} = (R - c_{2x_2}(t_{1(k-1)}^-)) \frac{\partial t_{1(k-1)}}{\partial \psi_2(I)} - M_2 c_{x_2} \sum_{j=1}^{p_2} b_{2j} \left( e^{A_2(t_{1k-t_2j})} - e^{A_2(t_{1(k-1)-t_2j})} \right) b_2 \frac{\partial t_{2j}}{\partial \psi_2(I)} \\
\end{align*}
\]
\[ \frac{\partial \tau_{2k}}{\partial X_1(I)} \bigg|_{X(I)=0}^{T} = -c_1 A_1 \left( e^{A_1 \tau_{2k}} - e^{A_1 \tau_{2(k-1)}} \right) + c_1 x_1(t_{2(k-1)}) \frac{\partial t_{2(k-1)}}{\partial X_1(I)} \\
+ M_1 c_1 \sum_{j=1}^{P_1} b_{1j} \left( e^{A_1(t_{2k} - t_{1j})} - e^{A_1(t_{2(k-1)} - t_{1j})} \right) b_{1} \frac{\partial t_{1j}}{\partial X_1(I)} \bigg|_{X(I)=0}^{T} \\
(5-69) \]

\[ \frac{\partial \tau_{2k}}{\partial X_2(I)} \bigg|_{X(I)=0}^{T} = c_1 x_1(t_{2(k-1)}) \frac{\partial t_{2(k-1)}}{\partial X_2(I)} + M_1 c_1 \sum_{j=1}^{P_1} b_{1j} \left( e^{A_1(t_{2k} - t_{1j})} - e^{A_1(t_{2(k-1)} - t_{1j})} \right) b_{1} \frac{\partial t_{1j}}{\partial X_2(I)} \bigg|_{X(I)=0}^{T} \\
(5-70) \]

\[ \frac{\partial \tau_{2k}}{\partial \Psi_2(I)} \bigg|_{X(I)=0}^{T} = c_1 x_1(t_{2(k-1)}) \frac{\partial t_{2(k-1)}}{\partial \Psi_2(I)} + M_1 c_1 \sum_{j=1}^{P_1} b_{1j} \left( e^{A_1(t_{2k} - t_{1j})} - e^{A_1(t_{2(k-1)} - t_{1j})} \right) b_{1} \frac{\partial t_{1j}}{\partial \Psi_2(I)} \bigg|_{X(I)=0}^{T} \\
k=1, 2, \ldots, n \]
With $x_{11}(t)$ taken as the state which remains constant between impulses, the output of $G_1(s)$ can be written in the form:

$$y_1(t) = c_{11}x_{11}(t) + \hat{c}_{11}^T\hat{x}_1(t)$$

where $c_{11}$ is the first element of the $c_1$ vector and $\hat{c}_1$ and $\hat{x}_1$ are the vectors which remain after removing $c_{11}$ and $x_{11}(t)$ from $c_1$ and $x_1(t)$ respectively. Since the motion is periodic the following relationship must hold:

$$A_2\gamma_2 = \int_0^T c_{11}x_{11}(t)\,dt + \int_0^T \hat{c}_{11}^T \hat{A}_1 \hat{x}_1(t) + \int_0^T \hat{b}_{11}(t-t_{1k})\hat{b}_{11}\,dt$$

$$= c_{11}x_{11}(0) + \hat{b}_{11}(t-t_{1k})\hat{b}_{11}\,dt$$  \hspace{1cm} (5-72)

where $\hat{A}_1$ is a nonsingular matrix obtained by eliminating the 1st row and 1st column from $A_1$, $\hat{b}_{11}$ is the first element in $b_1$ and $\hat{b}_1$ is the $b_1$ vector remaining after $\hat{b}_{11}$ has been removed. The above manipulations simply isolate the $x_{11}(t)$ state from the rest of the system. Continuing the development by noting the periodicity requirements and that from Theorem 5.4, $\gamma_1=0$ will allow equation (5-72) to be written,

$$A_2\gamma_2 = c_{11}x_{11}(0)T + M_1 \sum_{k=1}^m b_{1k}^T T_{b_{11}} - M_1 \sum_{k=1}^m b_{1k} t_{1k} \hat{b}_{11}$$

Thus,

$$A_2\gamma_2 = c_{11}(x_{11}(0)T - M_1 \sum_{k=1}^m b_{1k} t_{1k} \hat{b}_{11})$$  \hspace{1cm} (5-73)
Now assume that \( x_{11}(0) \) is perturbed to some new value \( x'_{11}(0) \). In order to satisfy the relationship (5-73), which is certainly a fundamental necessity for even a chance at stability, it will be necessary for the \( t_{1k}'s \) to make small adjustments, if they can, to balance the change caused by \( x'_{11}(0) \). However the \( t_{1k}'s \) represent pulse positions from the modulator IPFM 1. Any permanent change in these would certainly produce a new equilibrium motion and thus make the original motion unstable. This argument gives the following theorem:

**Theorem 5.8** For the double IPFM system with linear element \( G_1(s) \) containing a simple pole at the origin and all other poles in the left half of the s plane, and \( G_2(s) \) having only negative real part poles, the periodic motion described by Theorem 5.4 is unstable.

A similar argument is possible for the situation with \( G_2(s) \) containing the simple pole at the origin. The resulting key equation for the oscillatory case is:

\[
RT = c_{21}x_{21}(0)T - M_2 \sum_{k=1}^{n} b_{2k}t_{2k}^2k^2
\]

However in this case two equilibrium conditions must be considered; so instead of the same argument, a different one will be given to illustrate an alternative and a more intuitive approach. Consider again the state \( x_{21}(t) \) to be
that state which changes only when an impulse arrives at
the input of $G_2(s)$. Allow $x_{21}(t)$ to be perturbed by a
small amount $x_2$, $0 < x_2 < \frac{M_2}{b_{21}}$, at $t_1$. Since $x_{21}(t)$ can only
change by multiples of $\frac{M_2}{b_{21}}$, the impulsive change in
$x_{21}$, it will be impossible for $x_{21}(t)$ to ever again be
equal to $x_{21}(t_1)$. Thus the equilibrium condition will be
unstable for both cases considered in Theorem 5.6. The
following theorem may then be stated:

**Theorem 5.9** For the double IPFM system with all poles of
the linear elements $G_1(s)$ and $G_2(s)$ in the left half of
the $s$ plane except for a simple pole at the origin of $G_2(s)$,
the equilibrium states described in Theorem 5.6 are un-
stable.

5.3 Conclusions

Periodic forced oscillation is practical only for the
case when both linear elements have all their poles in the
left half of the complex $s$ plane. Equilibria for systems
with poles at the origin have been found to be unstable.

For the practical oscillatory case the stability of
the fundamental Dll motion has been investigated and found
to be dependent upon the relative firing times. This is
not surprising since Dll motion, when linearized about its
equilibrium motion (43), is not unlike an asynchronous
sampled data system. Such systems exhibit stability de­
pendence upon the relative intervals between sampling
instants (63).

Stability investigation for the general Dmn case are
quite difficult and since the exact periodic motion must
be known a priori, it is probably more practical to deal
with these general modes by direct simulation.
CHAPTER 6

CONCLUSIONS

The study of feedback control systems with multiple nonlinearities is a difficult structure to theoretically analyze. This is especially true if the nonlinearities contain hysteresis. The system analyzed in this dissertation contained two nonlinearities, both with memory, separated by arbitrary linear elements. The objective of the study was to identify those oscillatory states which are common to the class of nonlinearities considered; i.e., Integral Pulse Frequency Modulators.

Certain stability boundaries can be found, within which oscillation is likely to occur. One side of the boundary gives sufficient conditions for stability. This condition actually guarantees that there will be a finite number of impulses from the modulators in the time period $t=0$ to $t=\infty$. The number of impulses may in fact be quite large and occur over a considerable time period. The actual pattern depends upon the initial conditions. The other side of the boundary gives necessary conditions for
instability. Again the actual trajectory may even be stable for certain initial conditions even if this necessary condition is exceeded.

For free motion within the above boundaries it is convenient to consider three possibilities; a stable trajectory, an oscillatory trajectory and a periodic oscillatory motion. Each subsequent motion for a given system is dependent upon the initial conditions. These initial conditions may be grouped to form zones throughout the compound state space of the two linear elements. The zones are difficult to identify exactly because of their complex shape. Numerical methods are possible to obtain these initial condition regions; however only those corresponding to small numbers of pulses are practically calculated.

To identify those possible oscillatory modes, without regard to the initial conditions, the approximate describing function approach is possible. Because of the nonlinearities being separated by linear elements, it is necessary to satisfy two conditions to establish oscillation. Both of these conditions have been graphically identified and appear in Chapter 3. The resultant compound describing function is very complex, but the curves, once plotted, are valid for all systems with the structure studied; ie, only the linear frequency plot need be used for all such
systems. It must be remembered that the describing function analysis is an approximate technique which will give possible oscillatory modes. The actual motion may be stable; the initial conditions will establish the resultant type of trajectory.

Finally this same system when forced by a constant input signal may oscillate periodically. For this to occur for a certain class of linear elements, a condition on the first linear element has been established. Also the times of the possible periodic modes has been found in relation to the number of pulses and system parameters. Finally the stability of the resulting periodic motion is checked for the fundamental D11 mode. Higher Dmn modes are possible, but their stability involves the evaluation of a system of matrices.

The trajectories of this class of systems is very interesting, but difficult to predict because of the lack of a priori knowledge of the impulse instants. The solution of the equations describing the motion will involve the solution of systems of transcendental equations. Techniques for such solutions usually resort to numerical methods.
CHAPTER 7

SUGGESTIONS FOR FUTURE RESEARCH

The study of the IPFM system was originally motivated by its neural analogy. The IPFM is however only a first order approximation to the impulse generation in the neuron. A much closer approximation would be Neural Pulse Frequency Modulation which emits pulses of a single polarity. Using this modulator and identical linear elements, closer correlation with actual neural circuits could be achieved. Also threshold variation with time could be introduced as actually occurs within the physiological system. Finally the study can take on a more biological approach and search for an elementary neural network, probably in a simple insect, which would lend itself to theoretical analysis.

Engineering applications are scarce at present mainly because of a lack of comparison between PFM and other discrete methods such as sampled data systems or adaptive sampling systems. For a given system such comparisons considered could be noise immunity, hardware complexity and cost, power consumption etc. PFM when considered as a communication system is not unlike delta modulation; comparison of
such systems would be useful.

Limited work has been done on the optimal control of PFM systems. This particular area of theoretical research is essentially untapped.

The corruption of the signal in IPFM closed loop systems needs extensive study since noise immunity is one of the advantages of the system.

Adding a dead zone in the IPFM modulator, linear plants with transfer functions other than ratio of polynomials are also possibilities for further research.
APPENDIX A

DERIVATION OF THE STATE EQUATIONS

The following results to be derived will be done for one physical plant; in the system actually considered these results can easily be extended by adding the appropriate subscripts.

The linear differential equation governing the dynamics of the linear element is:

\[ y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0y = b_m u^{(m)} + \ldots + b_1 u^{(1)} + b_0 u \]  \hspace{1cm} (A-1)

where \( y^{(j)} \) and \( u^{(k)} \) are the jth derivative of the output and the kth derivative of the input respectively. The \( a_k \)'s and \( b_k \)'s are constants. This equation can be written with augmented dummy terms on the right hand side to make the results more general and easier to follow. Add to (A-1) derivatives of \( u \) with their coefficients equal to zero for the case under consideration; ie,

\[ y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0y = b_n u^{(n)} + \ldots + b_m u^{(m)} + \ldots + b_0 u \]  \hspace{1cm} (A-2)

where \( b_k = 0 \) for \( m < k < n \).

Since the normal form for representing the state of a system is the most popular (25), the derivation continues
with this matrix form as its objective. Also some of the
discussion in the body of this dissertation uses this form
for derivations. Assign the following state variables \( x \) to the system:

\[
y(t) = x_1 + K_0 u \\
x_1(1) = x_2 + K_1 u \\
x_2(1) = x_3 + K_2 u \\
\vdots \\
x_n(1) = x_{n+1} + K_n u
\]

Therefore \( x_1 \) may be written as,

\[
x_1 = y - K_0 u
\]

Differentiating both sides with respect to time gives:

\[
x_1(1) = y(1) - K_0 u(1)
\]

Then \( x_2 \) may be written as:

\[
x_2 = x_1(1) - K_1 u \\
= y(1) - K_0 u(1) - K_1 u
\]

Continuing in this manner the form of the \((i+1)\)st state
variable may be written as:

\[
x_{i+1} = x_i(1) - K_i u \\
= y(1) - \sum_{j=0}^{i-1} K_j u(i-j) \quad i \leq n
\]

or for \( x_i \),
From equation (A-8) \( x_n \) may also be written:

\[
x_n = y(n) - \sum_{j=0}^{n-1} K_j u(n-j) \tag{A-9}
\]

Differentiating the above equation with respect to time gives,

\[
x_n^{(1)} = y(n) - \sum_{j=0}^{n-1} K_j u(n-j) \tag{A-10}
\]

Equate equations (A-9) and (A-10):

\[
-a_0 x_1 - a_1 x_2 - \ldots - a_{n-1} x_n + K_n u = y(n) - K_0 u(n) - \ldots - K_{n-1} u^{(1)} \tag{A-10}
\]

Collecting terms on the left hand side under a summation yields:

\[
- \sum_{i=1}^{n} a_{i-1} x_i + K_n u = y(n) - \sum_{j=0}^{n-1} K_j u(n-j) \tag{A-11}
\]

For the general \( x_i \), equation (A-8) may be substituted into equation (A-11) yielding:

\[
- \sum_{i=1}^{n} a_{i-1} y^{(i-1)} - \sum_{j=0}^{i-1} K_j u(i-j-1) + K_n u = y(n) - \sum_{j=0}^{n-1} K_j u(n-j) \tag{A-12}
\]

Expanding by multiplying the first bracketed term in (A-12) gives:
Collecting the derivatives of the output, $y(t)$, on the right hand side of the equation and all other terms on the left hand side will give equation (A-13) in a form similar to the original differential equation:

\[ - \sum_{i=1}^{n} a_{i-1} y^{(i-1)} + \sum_{i=1}^{n} \sum_{j=0}^{i-1} k_{i} u^{(i-j-1)} + k_{n} u = y^{(n)} - \sum_{j=0}^{n-1} k_{j} u^{(n-j)} \]  

(A-13)

If the left hand side of equation (A-14) is expanded and equated to the terms in derivatives of $u$, the input, of the original equation; then the values of the $k$'s may be determined.

\[ a_{0}(K_{0}u) + a_{1}(K_{1}u + K_{0}u^{(1)}) + \ldots + a_{n-1}(K_{n-1}u + K_{n-2}u^{(1)} + \ldots + K_{0}u^{(n-1)}) + (K_{n}u + K_{n-1}u^{(1)} + K_{n-2}u^{(2)} + \ldots + K_{0}u^{(n)}) = b_{n}u^{(n)} + b_{n-1}u^{(n-1)} + \ldots + b_{1}u^{(1)} + b_{0}u \]

Now equate like coefficients of like derivatives in $u$; giving for the $n$th derivative,

\[ b_{n} = K_{0} \]
and for the other derivatives,
\[ b_{n-1} = K_1 + a_{n-1}K_0 \]
\[ b_{n-2} = K_2 + a_{n-1}K_1 + a_{n-2}K_0 \]
\[ \vdots \]
\[ b_0 = K_n + a_{n-1}K_{n-1} + \ldots + a_0K_0 \]

Now a recurrence relationship may be found to determine the \( K \)'s.

\[ K_0 = b_n \]
\[ K_1 = b_{n-1} - a_{n-1}K_0 \]
\[ K_2 = b_{n-2} - a_{n-1}K_1 - a_{n-2}K_0 \]
\[ \vdots \]
\[ K_n = b_0 - a_{n-1}K_{n-1} - a_{n-2}K_{n-2} - \ldots - a_0K_0 \]

Or in general terms for the \( i \)th \( K \) value:
\[ K_i = b_{n-i} - \sum_{m=0}^{i-1} a_{n-m-i}K_m \]

If the above values of \( K_i \) are used in the state variable normal form, the matrix equations may be written:

\[ A = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-a_0\alpha_1\alpha_2\alpha_3 & -a_n & -a_{n-1} & \cdots & \cdots & 0
\end{bmatrix} \]
Figure A.1 Flow chart for the normal representation of the state equations.
\[ b = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad d = K_0 \]  

(A-19)

With the resulting element operation then being described by the matrix equations:

\[ \dot{x} = A x + b u \\
    y = c^T x + d u \]  

(A-20)

A flow chart for this representation is given in figure A1. This chart is useful for the analog computer circuit.

Since in this case \( m < n \); i.e., the highest order of the derivative of \( y \) is greater than that of \( u \), the state equations will have the form:

\[ \dot{x} = A x + b u \\
    y = c^T x \]  

(A-21)

This is so because \( K_0 = b_n = 0 \). The general solution of the matrix differential equation (A-21) is then treated in a manner similar to that used in a normal ordinary first order differential equation.

First the homogeneous solution is obtained; i.e.,

\[ \dot{x}(t) = A x(t) \\
    u(t) = 0 \]  

(A-22)
In this case the $A$ matrix is constant; the solution is then most easily done with the Laplace Transform. Thus with $x_0 = x(t_0)$, the initial condition, the solution will be:

$$s \, X(s) - x_0 = A \, X(s)$$

$$(s \, I - A) \, X(s) = x_0$$

Therefore $X(s) = (s \, I - A)^{-1} \, x_0$

where $X(s)$ is the Laplace Transform of $x(t)$ and $I$ is the identity matrix. Then taking the inverse transform gives:

$$x(t) = L^{-1}((sI - A)^{-1}) \, x_0$$

$$= e^{A(t-t_0)} \, x_0 \quad t > t_0 \quad (A-23)$$

Take $t_0 = 0$ without loss of generality. Then the derivation continues by first noting a property of the transition matrix, $e^{At}$, that is:

$$\frac{d}{dt} e^{At} = A \, e^{At} \quad (A-24)$$

For the total solution of the state equation choose $(e^{At})^{-1}$ as an integrating factor. Then equation (A-21) may be written:

$$(e^{At})^{-1} \dot{x}(t) = (e^{At})^{-1} A x(t) + (e^{At})^{-1} b u(t) \quad (A-25)$$

If an exact differential can be found involving $x(t)$, the solution will be almost complete. Rearranging equation (A-25) gives:

* Note that $e^{At}$, the transition matrix, is sometimes written $\phi(t)$. 
Examining the left hand side of equation (A-26) will show an exact differential. This can be seen by considering the following steps:

\[
(e^{At})^{-1} x(t) - (e^{At})^{-1} A x(t) = (e^{At})^{-1} \frac{d}{dt} (e^{At})^{-1} x(t) + \frac{d}{dt} (e^{At})^{-1} x(t) \tag{A-27}
\]

The last term is true since:

\[
\frac{d}{dt} (e^{At})^{-1} = -(e^{At})^{-1} A (e^{At}) (e^{At})^{-1} = -(e^{At})^{-1} A
\]

The right hand side of equation (A-27) is an exact differential and thus may be written:

\[
(e^{At})^{-1} x(t) - (e^{At})^{-1} A x(t) = \frac{d}{dt} \{ (e^{At})^{-1} x(t) \} \tag{A-28}
\]

Substituting this result in the state equation (A-26) gives:

\[
\frac{d}{dt} \{ (e^{At})^{-1} x(t) \} = (e^{At})^{-1} b \ u(t)
\]

Integrating both sides from \( t=0 \) to \( t \) then gives the solution:

\[
(e^{At})^{-1} x(t) \bigg|_{t=0}^{t} = \int_{0}^{t} (e^{At})^{-1} b \ u(\tau) \ d\tau \tag{A-29}
\]

Expanding the left hand side,

\[
(e^{At})^{-1} x(t) - (e^{x0})^{-1} x_0 = (e^{At})^{-1} x(t) - x_0
\]

since \( e^0 = 1 \).
Premultiplying both sides of the equation by $e^{At}$ gives:

$$x(t) = e^{At} x_0 + e^{At} \int_0^t (e^{A\tau})^{-1} b u(\tau) \, d\tau \quad (A-30)$$

Since $e^{At}$ is not a function of $\tau$ it may be brought inside the integral; then using the property of the transition matrix:

$$(e^{At})^{-1} = (e^{-At})$$

will give (A-30) in the form:

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} b u(\tau) \, d\tau \quad (A-31)$$

For the particular case at hand $u(t)$ will be a series of impulses of strength $M$, polarity $b_k$ occurring at times $t_k$ as determined by the IPFM modulator. Thus the solution of the state equations may be written as:

$$x(t) = e^{At} x_0 + \sum_{k=1}^{K} M b_k e^{A(t-t_k)} b$$

where $K$ is defined in the relationship:

$$t_K = \sup_k (t_k : t > t_k)$$

Immediately after the first impulse firing,

$$x(t_1^+) = e^{At_1} x_0 + M b_1 b \quad (A-32)$$

If this value of $x(t)$ is now used for the initial condition of the next time interval, the problem can be solved as though $t_1$ where the starting time and not the time after the first impulse. This idea is extended in section 2.3.2.
APPENDIX B

DIGITAL COMPUTER FLOW CHART
FOR THE TOTAL RESPONSE

The input data to the computer is the transition, b and c matrices and the initial condition vectors \( x_0 \) for the linear plants \( G_1 \) and \( G_2 \). Also required are the modulator parameters \( A_1, A_2, M_1, M_2 \), the input signal and the control on the total running time and the printing interval. The program has been written for up to two 5th order linear elements (5x5 transition matrix).

The output will be a listing of the linear element output values at the time interval specified and at the instant prior to the impulse emission. The latter points are preceded by an asterisk for easy identification. The flow chart of the digital computer program begins on the following page.
READ: $N_1$, $N_2 = \text{Order of } G_1 \& G_2$

$A_1$, $A_2$, $M_1$, $M_2$.

TOT = Total time

DT = Print interval

WRITE: $A_1$, $A_2$, $M_1$, $M_2$

Initialize modulator integral

$V_{INT1} = \int e(t) \, dt = 0$

$V_{INT2} = \int y_1 \, dt = 0$

$T = \text{time} = 0$

DO 78

$I = 1, N_1$

$y_1 = c_1(I)x_{10}(I) + y_1$

Output $y_1$ and $y_2$

at time $t=0$

DO 79

$I = 1, N_2$

$y_2 = c_2(I)x_{20}(I) + y_2$

WRITE: $T$, $y_1$, $y_2$

$y_1 = 0$

$y_2 = 0$

$T_1 = 0 = \text{Time between IPFM1 pulse}$

$T_2 = 0 = \text{Time between IPFM2 pulse}$

$\text{INTIN} = DT/100 = \text{Integration int.}$

$T = T + \text{INTIN}$
System input

\[ R = \text{System input} \]
\[ T_1 = T_1 + \text{INTIN} \]
\[ IP = 0 \]

1. \[ e^{-A(T_1)} = \text{TRAN1}(T_1) \]

   DO 8
   \[ I = 1, \text{N1} \]
   \[ x_1(I) = 0 \]

   DO 3
   \[ I = 1, \text{N1} \]

   Calculate state \( x_1 \)

   DO 3
   \[ J = 1, \text{N1} \]

   \[ x_1(I) = x_1(I) + \text{TRAN1}(I,J)x_{10}(J) \]
   \[ y_1 = 0 \]

   DO 4
   \[ I = 1, \text{N1} \]

   Output \( y_1 \)

   \[ y_1 = y_1 + x_1(I)c_1(I) \]

   \[ \text{VINT2} = \text{VINT2} + y_1\text{INTIN} \]

   Modulator IPFM2

   Integral

   \[ \text{T2} = \text{T2} + \text{INTIN} \]

   \[ e^{-A(T2)} = \text{TRAN2}(T2) \]
DO 28  
I=1, N2  

x2(I)=0  

DO 23  
I=1, N2  

DO 23  
J=1, N2

Calculate state x2

\nx2(I)=x2(I)+TRAN2(I,J)x20(J)

y2 = 0

DO 24  
I=1, N2

Calculate output y2

\ny2 = y2+x2(I)c2(I)

VINT1 = VINT1+(R-y2)INTIN  
T1P=T1  
T2P=T2

Modulator IPFM1 Integral

IF VINT2-A2

Check IPFM2 for threshold

IF VINT2+A2

Polarity of impulse

WRITE:*  
SGN2 = -1

WRITE:*  
SGN2 = +1
Calculate new initial condition for $x_2$

$$x_{20}(I) = x_2(I) + \text{sgn}2(M_2)b_2(I)$$

$T_2 = 0$

$VINT2 = 0$

$IP = 1$

**Check IPPM1 for threshold**

**Polarity of impulse**

**WRITE:**

- IF $VINT1 < A_1$
  - NO = TOT/DT
  - DO 71
    - I = 1, NO
  - IF $T - I(DT) - \text{INTIN}$
    - CONTINUE
  - IF $T - (I)DT + \text{INTIN}$
    - WRITE: $T, Y_1, Y_2$
  - IF $T - TOT$
    - GO TO 73

- IF $VINT1 > A_1$
  - WRITE:* (SGN1 = -1)
  - DO 214
    - I = 1, N_1
  - IF $T_1 = 0$
    - WRITE:* (SGN1 = +1)
  - WRITE:T, Y_1, Y_2

Write Output
WRITE: "The time has reached TOT"

STOP

WRITE: $T, y_1, y_2, T_1P, T_2P$

GO TO 73

Write outputs
APPENDIX C

ANALOG COMPUTER CIRCUIT

FOR THE DOUBLE IPFM SYSTEM

Figure C1 shows the analog circuit used for the study of the double Integral Pulse Frequency Modulated feedback control system. The flow chart in figure A1, Appendix A, may be used for the linear element circuit.

Scaling must be based on the available impulse strength. Values of 0.06 volt-seconds were obtained on the Electronic Associates computer TR-48. The impulse was then multiplied by a factor of ten to give M=0.6. The approximate width of the pulse was 6 msec. Compared to the time constants of the examples (about 1 second) this was a satisfactory approximation to an impulse. Relay bounce was noted in the circuit, however it did not effect the results to any noticeable degree.
Polarities have been added at the appropriate places to insure correct operation. Terminals 1-2 and 3-4 are set across integrators 1 and 2 reset.
The following is the derivation of the stability theorem (Theorem 2.2) given in Chapter 2 without proof. The IPFM system is redrawn in figure D.1 and the theorem restated below for convenience. This theorem is an extension of an unpublished result by Meyer (43) for the single modulator system.

Figure D.1 Double IPFM feedback control system.

**Theorem 2.2** If the absolute values of the initial condition and impulse response of both linear elements have finite integrals for all time; ie,

\[ y_{10}(t), y_{20}(t), g_1(t), g_2(t) \in L_1(0, \infty) \]  

(D-1)

where \( y_{i0}(t) \) is the initial condition response of the \( i \)th linear element; then for arbitrary initial conditions, if
where \( L_G_1 = \int_{0}^{\infty} |g_1(t)| \, dt \) and \( L_G_2 = \int_{0}^{\infty} |g_2(t)| \, dt \)

the unforced system will be stable in the sense of the definition given in section 2.4.

**Proof:** From the system configuration, see figure D.1, the error and output signals may be written:

\[
e(t) = -y_{20}(t) - M_2 \sum_{j=1}^{\infty} b_{2j} g_2(t-t_{2j}) \quad (D-3)
\]

\[
y_{11}(t) = y_{10}(t) + M_1 \sum_{k=1}^{\infty} b_{1k} g_1(t-t_{1k}) \quad (D-4)
\]

If the modulators are assumed to fire an impulse at some arbitrary time, \( t_{1n} \) for modulator IPFM 1 and \( t_{2m} \) for modulator IPFM 2, then the integral values of the modulators at the firing times must be:

\[
A_1 b_{1n} = \int_{t_{1(n-1)}}^{t_{1n}} -y_{20}(t) \, dt + \int_{t_{1(n-1)}}^{t_{1n}} -M_2 \sum_{j=1}^{J} b_{2j} g_2(t-t_{2j}) \, dt \quad (D-5)
\]

\[
A_2 b_{2m} = \int_{t_{2(m-1)}}^{t_{2m}} y_{10}(t) \, dt + \int_{t_{2(m-1)}}^{t_{2m}} M_1 \sum_{k=1}^{K} b_{1k} g_1(t-t_{1k}) \, dt \quad (D-6)
\]

where \( J \) and \( K \) are given by

\[
t_{2j} = \max_{j} (t_{2j} : t_{2j} < t_{1n}) \quad (D-7)
\]

\[
t_{1k} = \max_{k} (t_{1k} : t_{1k} < t_{2m})
\]
Continuing the manipulation by removing the summations from inside the integrals of equations (D-5) and (D-6) will give:

\[ A_1b_{ln} = \int_{t_1(n-1)} t_{1n} y_{20}(t)dt - M_2 \sum_{j=1}^{J} b_{2j} \int_{t_1(n-1)} t_{1n} g_2(t-t_{2j})dt \quad (D-8) \]

\[ A_2b_{2m} = \int_{t_2(m-1)} t_{2m} y_{10}(t)dt + M_1 \sum_{k=1}^{K} b_{1k} \int_{t_2(m-1)} t_{2m} g_1(t-t_{1k})dt \quad (D-9) \]

The derivation will be continued for equation (D-8) only. Equation (D-9) has exactly a similar development which will merely be stated at the appropriate time. In equation (D-8) to change all left hand side terms to +A_l, multiply by b_{ln}; this is true since b_{ln}b_{ln} = +1 always. This gives:

\[ A_1 = -b_{ln}\int_{t_1(n-1)} t_{1n} y_{20}(t)dt - M_2b_{ln} \sum_{j=1}^{J} b_{2j} \int_{t_1(n-1)} t_{1n} g_2(t-t_{2j})dt \quad (D-10) \]

Sum up to the Kth pulse,

\[ KA_1 = -\sum_{k=1}^{K} b_{1k} \int_{t_1(k-1)} t_{1k} y_{20}(t)dt - M_2\sum_{k=1}^{K} b_{1k} \sum_{j=1}^{J} b_{2j} \int_{t_1(k-1)} t_{1k} g_2(t-t_{2j})dt \quad (D-11) \]

Then the development continues as follows:

\[ KA_1 = -\sum_{k=1}^{K} b_{1k} \int_{t_1(k-1)} t_{1k} y_{20}(t)dt + M_2\sum_{k=1}^{K} b_{1k} \sum_{j=1}^{J} b_{2j} \int_{t_1(k-1)} t_{1k} g_2(t-t_{2j})dt \quad | \quad (D-12) \]

\[ \leq -\sum_{k=1}^{K} b_{1k} \int_{t_1(k-1)} t_{1k} y_{20}(t)dt + M_2|b_{1k}| \sum_{j=1}^{J} b_{2j} \int_{t_1(k-1)} t_{1k} g_2(t-t_{2j})dt | \quad (D-13) \]
\[ \left\lfloor \sum_{j=1}^{t_{lk}} b_{2j} \sum_{k=1}^{t_{lk}} b_{1k} \int_{t_{l(k-1)} + t_{lk}}^{t_{lk}} g_2(t) \, dt \right\rfloor \leq \int_{0}^{t_{lk}} |y_{20}(t)| \, dt + M_2 \int_{0}^{t_{lk}} |g_2(t)| \, dt \]  
\hfill (D-13)

Now let time approach infinity and because of the assumptions of the theorem,

\[ KA_1 \leq \int_{0}^{\infty} |y_{20}(t)| \, dt + M_2 J L_{G_2} \]

Also for the other modulator IPFM 2,

\[ JA_2 \leq \int_{0}^{\infty} |y_{10}(t)| \, dt + M_1 K L_{G_1} \]

Solving for either \( K \) or \( J \) will give similar results as shown below. Only \( K \) will be used.

\[ K \leq \frac{\int_{0}^{\infty} |y_{20}(t)| \, dt + M_2 L_{G_2} \int_{0}^{\infty} |y_{10}(t)| \, dt}{A_2} \]  
\hfill (D-14)

For \( K \) to be finite the denominator of the above fraction must be greater than zero, or,

\[ L_{G_1} L_{G_2} < A_1 A_2 / M_1 M_2 \]
Corollary 2.2.2 If $g_1(t)$ and $g_2(t)$ are single signed, then for stability it is sufficient that:

$$\int_0^\infty g_1(t) dt < A_2/M_1$$

and

$$\int_0^\infty g_2(t) dt < A_1/M_2$$

Proof: This is merely a decomposition of the theorem, but there is a physical interpretation that can be used here. This corollary says that if the integral of the impulse responses of the linear elements are less than the threshold values of the modulators into which they feed, the system will be stable. Thus each impulse response will not be able to produce, by itself, an impulse from the modulator. The accumulative effect of this will eventually mean that all impulses will cease.
APPENDIX E

COMPUTER PROGRAM FOR THE DESCRIBING
FUNCTION OF THE IPFM MODULATOR

No data input is needed; the program is self generating. The output will be a listing for incremented values of V and β of the magnitude and phase of the describing function, the magnitude and phase, both numerical and in db, for -1/N_D for the Nyquist and Nichols chart plotting. The flow chart is given below.

```
DO 1 I=5,50
V=A_1=1.0(I)

IF
A_1-1

For N=2 only

β_c
Critical Angle

IF
A_1-1.01

β_c=π/2

β_c
Critical Angle

DO 2 K=1,J

β_c
Critical Angle

β_c
Index for Phase

J

Index for phase

L

L

Index for Phase
```
The derivation of the harmonic content of the describing function for a single IPFM modulator is important if the describing function is to be used intelligently. The steps below pick up at equations (3.16) and (3.17) and develop the end results given in Chapter 3 in the form of equations (3.22) and (3.26). The final form of the relationships contain only $\sin(\beta)$ and $\cos(\beta)$ terms.

For convenience equations (3.16) and (3.17) are repeated for completeness:

\[
\begin{align*}
    a_n &= \frac{2\pi}{\omega} \int_0^\frac{N}{2} M\left\{ \sum_{k=1}^{N} \delta(t-t_k) - \sum_{k=N+1}^{2N} \delta(t-t_k) \right\} \cos(n\omega t) \, dt \\
    &= \frac{\omega}{\pi} M\left\{ \sum_{k=1}^{N} \cos(n\omega t_k) - \sum_{k=N+1}^{2N} \cos(n\omega t_k) \right\}
\end{align*}
\]
\[ \cos(n\omega_k) = \sum_{j=0}^{J} (-1)^j \binom{n}{2j} \cos^{n-2j}(\omega_k) \sin^{2j}(\omega_k) \] (F-1)

where the upper limit on the summation is \( n/2 \) if \( n \) is even and \( (n-1)/2 \) if \( n \) is odd, and the notation \( \binom{n}{2j} \) is the standard for the binomial coefficients. This relationship can then be substituted into equation (3.16) and the result will be a function of the fundamental sine and cosine terms only. Proceeding in this manner the derivation of the harmonic content follows:

\[
\begin{align*}
\quad a_n &= \frac{\omega M}{\pi} \sum_{k=1}^{N} \sum_{j=0}^{J} (-1)^j \binom{n}{2j} (\cos^{n-2j}(\omega_k) \sin^{2j}(\omega_k)) \\
&\quad - \sum_{k=N+1}^{2N} \sum_{j=0}^{J} (-1)^j \binom{n}{2j} (\cos^{n-2j}(\omega_k) \sin^{2j}(\omega_k)) \\
&= \quad (F-2)
\end{align*}
\]

Since,
\[ \cos(\omega_k) = \cos(\omega_{N-k}) \quad k=1,2,..(N-1) \]
\[ \cos(\omega_N) = \cos(\beta) \]
and
\[ \sin(\omega_k) = -\sin(\omega_{N-k}) \]
\[ \sin(\omega_N) = -\sin(\beta) \] (F-3)

because of the pulse pattern symmetry, the above substitutions can be made into equation (F-2). Note that the sine term will always be raised to an even power and thus will always be positive. Continuing the derivation:
\[ a_n = \frac{\omega M}{\pi} \left\{ \sum_{k=1}^{N-1} \sum_{j=0}^{J} (-1)^j \binom{n}{2j} \cos^{n-2j}(\omega t_k) \sin^{2j}(\omega t_k) \right\} \]

Because of the cancellation of the like terms, equation (F-4) becomes:

\[ a_n = \frac{\omega M}{\pi} \left\{ \sum_{j=0}^{J} (-1)^j \binom{n}{2j} \cos^{n-2j}(\omega t_N) \sin^{2j}(\omega t_N) \right\} \]

Simplifying:

\[ a_n = \frac{\omega M}{\pi} \left\{ \sum_{j=0}^{J} (-1)^j \binom{n}{2j} \cos^{n-2j}(\omega t_N) \sin^{2j}(\omega t_N) \right\} - \cos^{n-2j}(\omega t_N) \sin^{2j}(\omega t_N) \]

Using equations (3.14) and (F-3) then gives:

\[ \cos(\omega t_N) = \cos(\beta) - \frac{N/2}{\omega} \]

\[ \sin(\omega t_N) = (1 - \cos^2(\omega t_N))^{1/2} \]
thus,

\[ a_n = \frac{\omega M}{\pi} \sum_{j=0}^{J} (-1)^j \binom{n}{2j} \left( \frac{\cos(\beta) - N/2}{V} \right)^{n-2j} \left( 1 - \frac{\cos(\beta) - N/2}{V} \right)^j \]

\[ -\cos^{n-2j}(\beta) \sin^{2j}(\beta) \]

(F-5)

In particular for \( a_2 \),

\[ a_2 = \frac{\omega M}{\pi} \sum_{j=0}^{J} (-1)^j \binom{2}{2j} \left( \frac{\cos(\beta) - N/2}{V} \right)^{2-2j} \left( 1 - \frac{\cos(\beta) - N/2}{V} \right)^j \]

\[ -\cos^{2-2j}(\beta) \sin^{2j}(\beta) \]

Expanding the summation:

\[ a_2 = \frac{\omega M}{\pi} \left( \cos^2(\beta) - N \cos(\beta) + \left[ \frac{N}{V} \right]^2 - \cos^2(\beta) + \cos^2(\beta) - 1 \right) \]

\[ -N \cos(\beta) + \left[ \frac{N}{V} \right]^2 + 1 - \cos^2(\beta) \]

Expanding,

\[ a_2 = \frac{\omega M}{\pi} \left[ \cos^2(\beta) - N \cos(\beta) + \left[ \frac{N}{V} \right]^2 - \cos^2(\beta) + \cos^2(\beta) - 1 \right] \]

Collecting terms:

\[ a_2 = \frac{\omega M}{\pi} \left[ 2 \left[ \frac{N/2}{V} \right]^2 - \frac{4 N/2}{V} \cos(\beta) \right] \]

(F-6)

For the third harmonic term \( a_3 \), the computation is as follows:

\[ a_3 = \frac{\omega M}{\pi} \sum_{j=0}^{1} (-1)^j \binom{3}{2j} \left( \frac{\cos(\beta) - N/2}{V} \right)^{3-2j} \left( 1 - \frac{\cos(\beta) - N/2}{V} \right)^j \]

\[ -\cos^{3-2j}(\beta) \sin^{2j}(\beta) \]
Expanding the summation:

\[ a_3 = \frac{\omega M}{\pi} \left[ \frac{(\cos\beta-N/2)^3 - \cos^3(\beta) - 3 \left( (\cos\beta-N/2) (1-(\cos\beta-N/2)^2) \right)}{V} \right] \]

Now expand the cubic term:

\[ a_3 = \frac{\omega M}{\pi} \left[ \frac{\cos^3(\beta) - 3N/2 \cos^2(\beta) + 3 N/2 \cos(\beta) - \cos^3(\beta)}{V} \right] \]

Continuing the expansion:

\[ a_3 = \frac{\omega M}{\pi} \left( \frac{3 \cos^3(\beta) - 12N/2 \cos^2(\beta) + 12 \left[ \frac{N/2}{V} \right]^2 \cos(\beta) - 4 \left[ \frac{N/2}{V} \right]^3}{V} \right) \]

Collecting terms:

\[ a_3 = \frac{\omega M}{\pi} \left( 4 \left( \frac{\cos^3(\beta) - 3N/2 \cos^2(\beta) + 3 \left[ \frac{N/2}{V} \right]^2 \cos(\beta) - \left[ \frac{N/2}{V} \right]^3}{V} \right) \right) \]

Adding and subtracting \( \cos^3(\beta) \) allows for some simplification:

\[ a_3 = \frac{\omega M}{\pi} \left( 4 \left( \frac{\cos^3(\beta) - 3N/2 \cos^2(\beta) + 3 \left[ \frac{N/2}{V} \right]^2 \cos(\beta) - \left[ \frac{N/2}{V} \right]^3}{V} \right) \right) \]
In general the relationship between the \( a_n \) term and the \( a_1 \) term, due to the first harmonic can be written as:

\[
\frac{a_n}{a_1} = \sum_{j=0}^{n} \left( \frac{1}{2j} \right) (-1)^j \frac{(\cos \frac{\pi N}{2})^{n-2j} (1-(\cos \frac{\pi N}{2})^2)^j}{(\frac{\pi}{2})^j (\cos \frac{\pi N}{2}) \sin^2 j(\beta)}
\]

When the number of pulses per period is high, \( \beta \) approaches zero and \( \frac{N}{2} \) approaches 1.0; which means that the value of the terms in the brackets will be small. For large \( N \) the ratio of \( a_n/a_1 \) becomes small, thus only small \( N \) need be investigated for harmonic distortion.

Following the same development for the sinusoidal component will yield similar results if one starts with equation (3.17). The results are stated below.

\[
b_n = \frac{\omega M}{\pi} \left( \sum_{k=1}^{\frac{N-1}{2}} \sum_{j=0}^{J} (-1)^j \left( \frac{n}{2j+1} \right) (\cos k)^{n-2j-1} \left( 1-(\cos k)^2 \right)^{j+\frac{1}{2}} \right.
\]

\[
+ \sum_{j=0}^{J} (-1)^j \left( \frac{n}{2j+1} \right) (\cos \frac{\pi N}{2})^{n-2j-1} \left( 1-(\cos \frac{\pi N}{2})^2 \right)^{j+\frac{1}{2}}
\]

\[
+ \cos^{n-2j-1}(\beta) \sin^{2j+1}(\beta)
\]
In particular for the second and third harmonic terms:

\[ b_2 = \frac{\omega M}{\pi} \sum_{k=1}^{N-1} \left\{ 2 \left( \frac{\cos\beta-k}{V} \right)^2 \left( 1 - \left( \frac{\cos\beta-k}{V} \right)^2 \right)^{1/2} \right. \]

\[ + 2 \left( \frac{\cos\beta-N/2}{V} \right) \left( 1 - \left( \frac{\cos\beta-N/2}{V} \right)^2 \right)^{1/2} \]

\[ + 2 \sin(\beta) \cos(\beta) \} \]

\[ b_3 = \frac{\omega M}{\pi} \left[ \sum_{k=1}^{N-1} \left\{ 3 \left( \frac{\cos\beta-k}{V} \right)^2 \left( 1 - \left( \frac{\cos\beta-k}{V} \right)^2 \right)^{1/2} \left( 1 - \left( \frac{\cos\beta-k}{V} \right)^2 \right)^{3/2} \right\} \right. \]

\[ + 3 \left( \frac{\cos\beta-N/2}{V} \right)^2 \left( 1 - \left( \frac{\cos\beta-N/2}{V} \right)^2 \right)^{1/2} \]

\[ - \left( 1 - \left( \frac{\cos\beta-N/2}{V} \right)^2 \right)^{3/2} + 3 \cos^2(\beta) \sin(\beta) - \sin^3(\beta) \] \]

As in the case of \( a_n/a_1 \) the investigation of \( b_n/b_1 \) need only be made for small numbered pulse patterns.
APPENDIX G

FLOW CHART FOR THE COMPOUND IPFM
DESCRIBING FUNCTION

The input data is the Dmn pattern to be investigated. The output will be $-1/N_{1D}N_{2D}$ for the two IPFM describing function. The flow chart follows.

```
READ Dmn

Estimate range $|G_1|$  (3.31)

DO $|G_1|$ range

Range $V_1$  (3.30)

DO $V_1$ range

Calculate $N_{1D}$ (Appendix E) over $\beta_1$ range
```
\( V_2 = V_1 |G_1| |N_{1D}| \)

In proper range (3.30)

Calculate \( N_{2D} \)

(Appendix E)

Over \( \theta_2 \) range

\[-\frac{1}{N_{1D}N_{2D}}\]

WRITE

\[-\frac{1}{N_{1D}N_{2D}}\]

CONTINUE

STOP
Consider the system shown in figure H.1. Let \( G_1 \) be described by the linear time invariant state equations:

\[
\begin{align*}
\dot{x} &= Ax + bu \\
y &= c^T x
\end{align*}
\]  \hspace{1cm} (H-1)

where "T" indicates the transpose. Further assume that all poles of \( G_1(s) \) are in the left hand of the \( s \) plane, this guarantees that \( A^{-1} \) will exist. Let the initial condition solution of the above equations be:

\[
\begin{align*}
x &= e^{At} x_0 \\
y &= c^T x
\end{align*}
\]  \hspace{1cm} (H-2)

where \( x_0 \) is the initial condition state vector.

There will be a capture zone around the origin in which an initial condition will be unable to produce an impulse because the integral of the modulator will never reach the threshold firing value, \( A \). This zone is determined by all points satisfying the relationship:
Let all the initial conditions, $x_0$, which satisfy this relationship denote a region, $R_0$. Thus for an initial condition in this capture zone, there will be $N=0$ impulses emitted from the modulator, see figure H.2.

All trajectories which are to go to the origin; ie, be stable, must switch into this region, $R_0$. This means that $x(t)$ must move into $R_0$ at an impulse instant so that the modulator integrator has zero value just after the impulse and the system has a new initial condition within $R_0$. If the total number of impulses emitted to enter $R_0$ is $N=1$, then the state at time $t_1^-$ just prior to the impulse must be:

$$x(t_1^-) = x_0 - Mb_1b$$

$$x_0 \in R_0$$
Figure H.2 "R_n" regions defining initial conditions for which the resulting motion is stable.
Since the system is continuous between impulses it must be that

\[
\text{sgn } y(t_1^-) = \text{sgn } \int_{t_0}^{t_1} y(t) \, dt
\]

\[
= -b_1
\]  \hspace{1cm} (H-5)

This is because the modulator fires the first time the threshold value is reached. If cast into the standard normal form \(y=x_1\) equation (H-5) can be written as:

\[
\text{sgn } x_1(t_1^-) = -b_1
\]  \hspace{1cm} (H-6)

Let the region denoted by equation (H-4) be \(R_0^\prime\); and it is a simple linear translation of \(R_0\). See figure H.2, the dotted curve. Trajectories passing through \(R_0^\prime\) will determine the initial conditions \(x_{01}\) for which the modulator will emit \(N=1\) impulses before entering the capture zone, \(R_0\). Two conditions must be satisfied by \(x_{01}\): (1) it must be on the trajectory passing through \(x(t_1^-)\) and (2) the modulator integral must reach threshold, \(A\), between \(x_{01}\) and \(x(t_1^-)\). These conditions give the following relationships; for requirement (1):

\[
x(t_1^-) = e^{-A t_1} x_{01}
\]  \hspace{1cm} (H-7)

equation (H-7) may be written:

\[
x_{01} = e^{-A t_1} x(t_1^-)
\]  \hspace{1cm} (H-8)
which is the backward mapping along the trajectory passing through \( x(t^-_1) \). For requirement (2) the integral must satisfy the relationship:

\[
 b_1 A = \int_{t_0}^{t_1} c^T e^{At} x_{01} \, dt \\
= c^T A^{-1} (e^{A t_1} - I) x_{01} \\
= c^T A^{-1} (x(t^-_1) - x_{01}) \tag{H-9}
\]

\( x(t^-_1) \in R^1_0 \)

When equation (H-4) gives \( x(t^-_1) \), equation (H-9) represents a hyperplane (straight line in two dimensions). Thus the intersection of the trajectory of (H-8) with the hyperplane determines \( x_{01} \). If there are multiple intersections, that one corresponding to the minimum time, \( t^-_1 \), is chosen.

Let \( R_1 \) designate the region defined by all the \( x_{01} \), see figure H.2.

Note that in general there will be two \( R_1 \) regions corresponding to \( b_1 = +1 \) and \( b_1 = -1 \), and that only the boundary of \( R_0^1 \) need be considered in the backward mapping.

If the total number of pulses emitted before entering \( R_0 \) is \( N=2 \), the \( R_1 \) region defines the zone into which the trajectory must be switched if it is to enter \( R_0 \). Thus \( R_1 \)
is mapped backwards to obtain \( R_j \) as in (H-4), then (H-8) and (H-9) are used to determine \( x_{02} \). In general for the \( n \)th backward impulse zone:

\[
x_0(n-1) - Mb_n = x(t_n^+)
\]

\[
x_0(n-1) \quad R_{n-1}
\]

\[
\text{sgn } y(t_n^+) = -b_n
\]

also

\[
c^T A^{-1}(x(t_n^+) - x_{0n}) = b_n A^\prime
\]

\[
x(t_n^+) \quad R_{n-1}^\prime
\]

and finally

\[
x_{0n} = e^{-At_n} x(t_n^+)
\]

The above primed equations establish an algorithm which can be used to map the initial conditions in state space which guarantee stability, see figure H.2. From each region the backward mapping must consider both \( b_n = +1 \). This would indicate a possible \( 2^N \) regions. Since the system is open loop between impulses and linear, the trajectories will not intersect and \( 2^{N-1} \) regions need be considered since there will be symmetry through the origin.

**Example H.1** For dimensions greater than 2 a digital computer must be used, but for systems of order 1 or 2, graphical techniques have been developed. Consider the system
with the following state equations:

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y &= (1, 0)^T x \\
A &= 1 \text{ and } M = 7
\end{align*}
\]

The region $R_0$ can be determined algebraically and is bounded by the four curves:

\[
\begin{align*}
3x_{10} + x_{20} &= 2 \\
3x_{10} + x_{20} &= -2 \\
x_{10}^2 &= 2(x_{10} + x_{20}) \\
x_{10}^2 &= -2(x_{10} + x_{20})
\end{align*}
\]

This region defines $R_0$ and is shown in figure H.3.

Next the $N=1$ zone is found. First translate the $N=0$ region by $M_b^1b$. This is done for $b_1=-1$ in figure H.3 and is indicated in dashed lines. All trajectories passing through the translated zone are candidates; however note that for $b_1=-1$, $x_1(t_1^-)$ must be positive and thus that translated part with $x_1(t_1^-)<0$ can be eliminated from consideration.

Construct a few trajectories from the boundaries of the translated zone and note the $x(t_1^-)$ coordinate. The
Figure H.3 \( R_0 \) and \( R_1 \) region for example H.1.
integral condition (H-9) in this case gives:

\[-1 = (-3x_1(t_{1^-}) - x_2(t_{1^-})) - (-3x_{101} - x_{201})\]

where

\[\dot{x}(t_{1^-}) = \begin{bmatrix} x_1(t_{1^-}) \\ x_2(t_{1^-}) \end{bmatrix}\] and \[x_{01} = \begin{bmatrix} x_{101} \\ x_{201} \end{bmatrix}\]

This is the equation of a straight line. The intersection of this line with the trajectory emanating from \(\dot{x}(t_{1^-})\) will be the require \(\dot{x}_{01}\). This backward mapping is carried on until a new sector (N=1) is satisfactorily defined.
APPENDIX I

PHASE PLANE TRAJECTORY CONSTRUCTION

FOR SPECIAL SYSTEM

For the special system shown in figure 1.1 a graphical construction technique for the system trajectory is possible. Note that the linear element outputs will be given by:

\[ x_1(t) = e^{-at} x_{10} \quad (I-1) \]

\[ x_2(t) = e^{-bt} x_{20} \]

and an impulse will be fired every time a modulator reaches its threshold value. Thus:

\[ \int_{0}^{t_1} e^{-bt} x_{20} \, dt = b_1 A_1 \quad (I-2) \]

\[ \int_{0}^{t_2} e^{-at} x_{10} \, dt = b_2 A_2 \]

For modulator IPFM 1 for example (I-2) gives:

\[ (1 - e^{-bt_1}) x_{20} = b b_1 A_1 \]

but since \( e^{-bt_1} x_{20} = x_2(t_1) \) is the point at which firing takes place, the relationship may be written:
Figure I.1 Special system.

Figure I.2 Phase plane trajectory for system in figure I.1.
\[ x_{20} - x_2 = bb_1A_1 \]  \hspace{1cm} (I-3)

also for modulator IPFM 2,

\[ x_{10} - x_1 = ab_2A_2 \]  \hspace{1cm} (I-4)

where \( x_1 \) is the firing point on the trajectory.

Thus a linear difference between the states determines the firing points on the phase plane. These points can easily be determined by a scale or templet.

If for example \( x_{10} - x_1 = +A_2a \), a positive pulse will be fired from IPFM 2 and the subsequent motion will only effect \( x_2 \). The variable \( x_2 \) will move vertically (increasing) by an amount \( M_2 \), see figure I.2. At this new point modulator IPFM 2 must start the integration process over again. Modulator IPFM 1 has some residual value in its integrator which must be retained and used during the next time period. Since the integrator value is on a linear scale this can easily be done in the following manner:

Construct a linear grid as shown in the insert in figure I.2. Place \( x_{10}, x_{20} \) at the origin of the templet. If the trajectory crosses one of the \( A_1b \) or \( A_2a \) lines, that corresponding modulator has reached its threshold value. The accumulated value of the other modulator in-
integral is indicated on the other axis. At point "1", figure I.2, modulator IPFM 2 has reached its threshold value and will fire. The accumulated modulator IPFM 1 integral is indicated by the point "x". Point "x" is the origin for the next time interval and would be placed at point "2" on figure I.2 and the process repeated.

Since the plants are linear and the operation between the impulses is due to these elements only, the trajectories on the state plane need be constructed only once by some appropriate technique (isoclines for example). The resultant trajectory is then easily found for this special case.

Example I.1 Let $a=1$, $b=2$, $A_1=A_2=1.0$ and $M_1=1.0$ and $M_2=3$ with $x_{10}=7.5$ and $x_{20}=10.0$. Find the resultant trajectory. The result is shown in figure I.3 with the templet insert.
Figure I.3 Trajectory for example I.1.
REFERENCES


VITA

Warren J. Guy was born in on . He attended Drexel Institute of Technology, Phila., Pa., from September 1954 to June 1959 when he received a B.S. in Electrical Engineering. He received an M.A. (Physics) from Temple University, Phila., Pa., in February 1961. He has been employed by Philco Corp. and the Temple University Research Foundation. After five years in the U.S. Army Signal Corps he joined Lafayette College as a teacher in the Department of Electrical Engineering, where he is currently employed.

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