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Dynamic analysis of the generalized slider crank

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ABSTRACT

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by

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A numerical technique is used to analyze the kinematics of the generalized slider crank mechanism and an analytical technique to derive dynamic force equations for that mechanism has been formulated. The numerical technique used for displacement analysis is based on a combination of Newton-Raphson and Davidon-Fletcher-Powell optimization algorithm using dual-number coordinate-transformation matrices. Velocity analysis is performed by using a dual number method. Finally, dynamic force analysis is accomplished on the basis of the dual-Euler equation and D'Alembert's principle. The approach is developed in such a manner that a digital computer can detect when a solution is possible and then solve the whole problem.

In addition, kinematic displacements of slider and dynamic forces and torques at each of the joints have been graphed against input crank angles for different offsets. In all the graphs, possible cases have been compared with the ideal case, when the mechanism has zero offsets.

DYNAMIC ANALYSIS OF THE GENERALIZED SLIDER CRANK

by

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INTRODUCTION

1.1 Background and Motivation

In recent years mechanisms with multi-degree-of-freedom systems have commanded a great deal of research activity. The rapid advancements of the robot manipulator for industry is fueling the interest of many researchers. In 1964 Uicker, Denavit and Hartenberg developed an iterative technique for displacement analysis of spatial mechanisms using 4×4 transformation matrices (1). Later in 1967 Uicker did the dynamic force analysis of spatial linkages (2). Dual-number element coordinate transformation matrices were introduced by researchers such as Yang to study spatial mechanisms. In 1964 Yang and Freudenstein applied dual-number quaternion algebra to the analysis of spatial mechanisms (3). Based on dual vectors and screw calculus, Yang formulated inertia force equations for RCCC mechanisms (R stands for revolute joint and C stands for cylindrical joint) in 1971 (4). One year later Bagci did dynamic force and torque analysis of the planar 4R mechanism and spatial RCCC mechanism using dual vectors and 3×3 screw matrix (5). Pennock and Yang in 1983 developed the technique for dynamic analysis of a multi-rigid-body open chain system based on Newtonian mechanics with a composite 3×3 dual number, 6×1 Plucker coordinate method (6). In 1984 Fischer and Freudenstein did the complete derivation of internal force and torque components of a statically loaded cardan joint with manufacturing tolerances (7). As an extension of the earlier work, Chen and Freudenstein in 1986 developed a dynamic analysis of a universal joint with manufacturing tolerances (8). Fischer and Paul successfully extended the concept of Uicker-Denavit-Hartenberg in 1989 for kinematic displacement analysis of a double cardon joint driveline using 3×3 dual number transformation matrices (9). In 1990 Fischer investigated displacement errors in Cardan joints caused by coupler-link joint-axes offset (10). To model robot manipulators mathematically, Lagrange-Euler and Newton-Euler formulations using 4×4 transformation matrices have been used widely. The method described here serves a meaningful alternative to the existing procedures.

Sandor and Erdman did the displacement, velocity and acceleration analysis of a similar type of mechanism using 4×4 transformation matrix (11). The solution is based on Newton-Raphson iterative procedure commonly used for the solution of nonlinear equations. The present work owes much to pioneering work of Uicker, Denavit, Hartenberg and Yang on spatial mechanisms.

1.2 Outline of Thesis

Chapter 2 explains how displacement analysis is done numerically and the method of deriving expressions for joint velocities, accelerations and dynamic forces acting at each of the joints. Chapter 3 illustrates a physical example, discusses about the results and draws conclusions.

ANALYSIS AND FORMULATION OF THE METHOD

2.1 Basic Definitions:

Joint axis: A joint axis is established at the connection of two links. It is defined as the axis about which one link can either rotate or translate or both relative to the other.

Link axis and link length: Link length is defined as the shortest distance measured along the common normal between two joint axes of the link and the common normal is the link axis.

Link twist angle: Link twist angle is defined as the angle between two joint axis of the link.

Generalized slider crank: The generalized slider crank is actually a spatial mechanism with multi-degree-of-freedom joints, whereas the conventional slider crank universally used in reciprocating engins and compressors is a planar mechanism with single-degree-of-freedom joints. In a planar slider crank, the joint between frame and

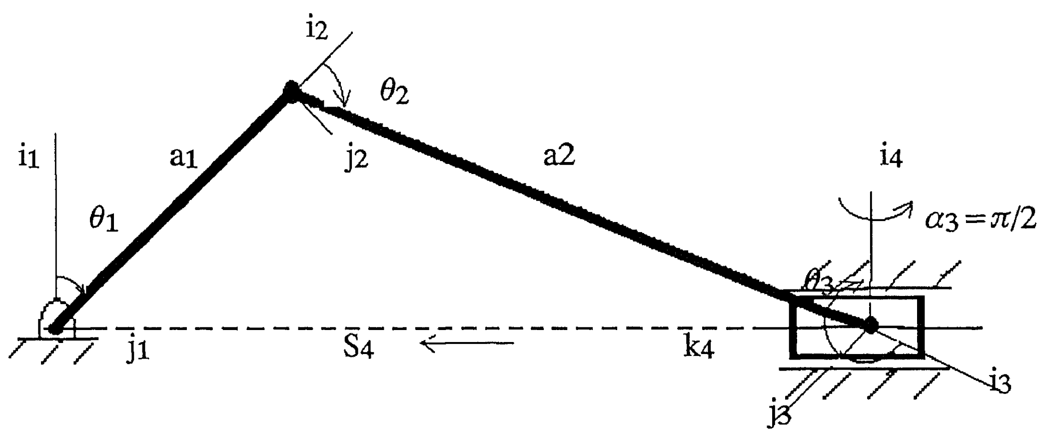

Figure 1 Planar(ideal) slider crank with no offset

crank is revolute and has one degree of freedom; i.e. rotation only. In generalized slider crank it is a cylindrical joint which has two degrees of freedom, one of translation and

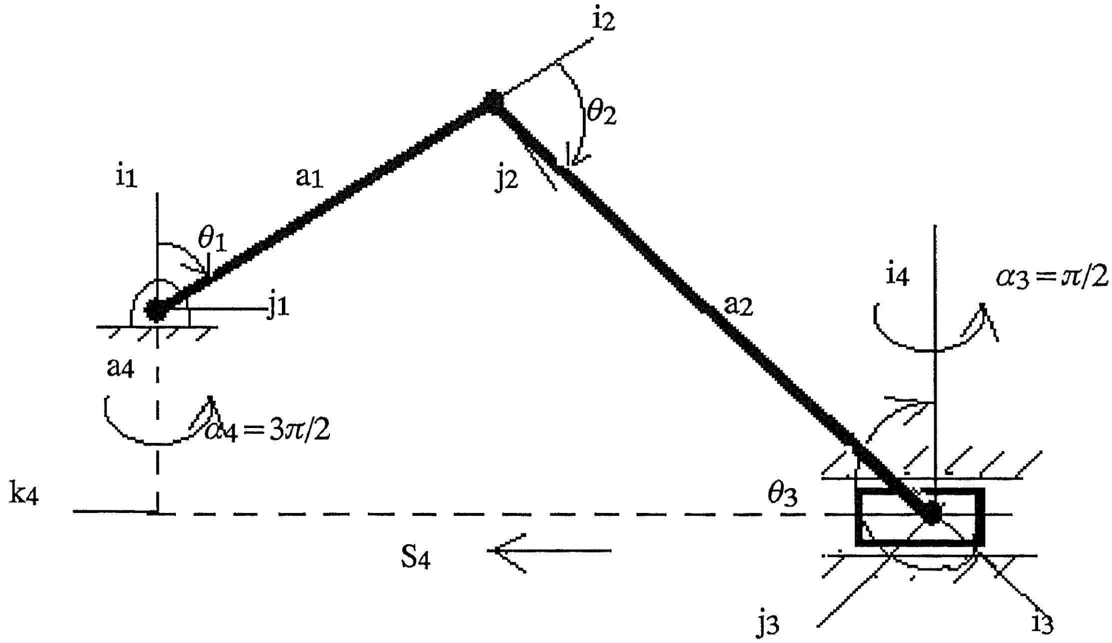

Figure 2 Planar slider crank with offset a4 only

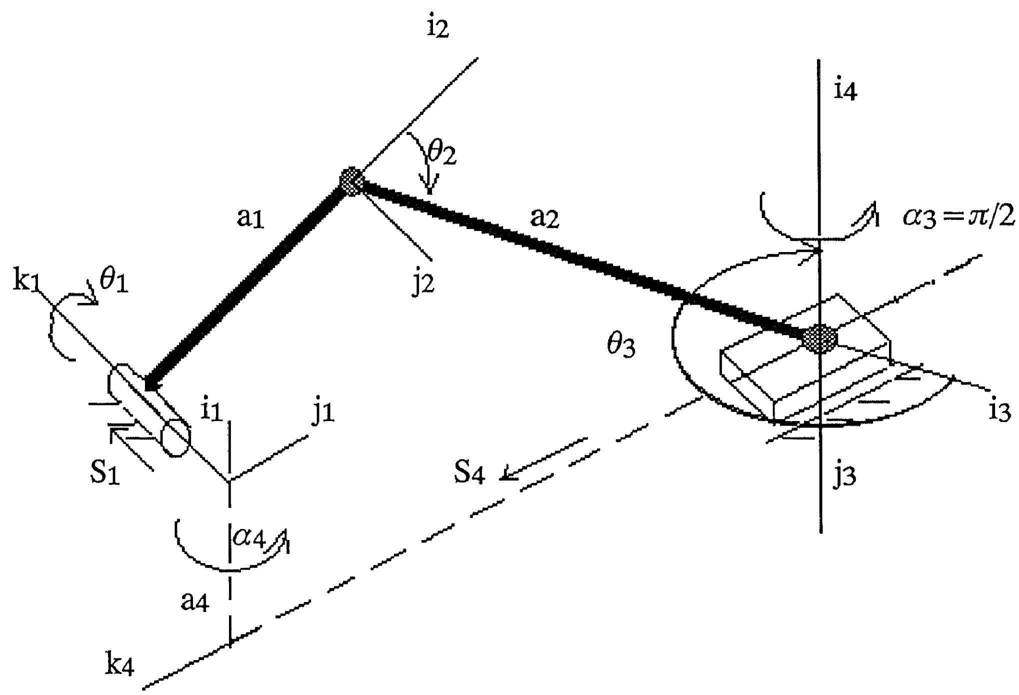

Figure 3 Generalized slider crank with all offsets

the other of rotation. In a planar slider crank, the joints between the crank and the connecting rod and between connecting rod and slider are both revolute joints. In generalized case both are ball (spherical) joints which have three rotational degrees of freedom about each one of three mutually perpendicular axes. For both the cases, the joint between slider and frame is prismatic i.e. joint having only one translational degree of freedom.

Rotation matrix: A rotation matrix can be defined as a transformation matrix which operates on a position vector in a three-dimensional euclidean space and maps its coordinates expressed in a rotated coordinate system (body-attached frame) to a reference coordinate system.

2.2 Notations:

Joint axis, link axis and the axis perpendicular to both of them in a right-handed coordinate system are respectively denoted by k, i and j. Rotations about these axes are respectively denoted by θ, α and η.

Basic 3×3 rotation matrices (13) are denoted by $[X(\alpha)]$, $[Y(\eta)]$ and $[Z(\theta)]$. They respectively represent rotations about i, j and k axes and can be written as:

$$[X(\alpha)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{bmatrix}$$

$$[Y(\eta)] = \begin{bmatrix} c\eta & 0 & s\eta \\ 0 & 1 & 0 \\ -s\eta & 0 & c\eta \end{bmatrix}$$

$$[Z(\theta)] = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

when dual-number operations are done α, η and θ are respectively replaced by $\hat{\alpha}$, $\hat{\eta}$ and $\hat{\theta}$.

2.3 Creation of link-joint parameter table:

Fig. 1 shows the planar slider crank mechanism without any offset; this is the ideal case without any tolerance error. Here, four coordinate frames are fixed on the distal end (fore-end) of each of the links at the joints. Each coordinate frame consists of three mutually perpendicular axes i, j and k. Joint *1* is between frame (fixed link) and crank, joint *2* is between crank and connecting rod, joint *3* is between connecting rod and slider and joint *4* is between slider and frame. For the planar case, joints *1*, *2* and *3* are revolute joints, each of which has one rotational degree of freedom, and joint *4* is prismatic with one sliding degree of freedom. Two axes of each frame are shown, from which one can see the position of the third axis, since each frame comprises a right-handed coordinate system. Fig. 2 shows the planar slider crank with offset a_4 only; there is no other difference between fig. 1 and fig. 2. Fig. 3 is the generalized slider crank in which joint *1* is cylindrical with one translational and one rotational degrees of freedom, joints *2* and *3* are ball joints, each of which has three rotational degrees of freedom, and joint *4* is prismatic, exactly the same as in the planar case.

The link parameters are:

a_1 = crank length

a_2 = connecting rod length

α_3 = twist angle between two joint axes - the axis of the joint between connecting rod and slider i.e. axis k_3 (not shown) and the axis of the joint between slider and frame

Table 1 Link-joint parameters

i	θ_i	Si	η_i	α_i	ai
1	θ_1	S_1	0	0	a_1
2	θ_2	0	η_2	0	a_2
3	θ_3	0	η_3	$\pi/2$	0
4	0	S_4	0	α_4	a_4

i.e. axis k_4; the angle is measured counterclockwise according to right-hand rule and is equal to 90 degrees.

a_4 = offset distance between two joint axes, one is the axis of the joint between frame and crank i.e. axis k_1 and the another is the axis of the joint between the slider and the frame i.e. axis k_4. Axes k_1 and k_4 are mutually perpendicular axes and the distance a_4 is measured from k_4 to k_1.

α_4 = twist angle between two joint axes k_1 and k_4; the angle is measured counterclockwise according to right-hand rule. For the ideal(planar) slider crank this angle is 270 degrees.

The joint variables are:

θ_1, θ_2 and θ_3 = angular displacements about joint axes at joints *1*, *2* and *3* respectively.

η_2 = angular displacement about the axis j_2, perpendicular to the joint axis as well as the link axis at joint *2* .

η_3 = angular displacement about the axis j_3, perpendicular to the joint axis as well as the link axis at joint *3* i.e. axis j_3 .

S_1 = displacement of the crank along the axis of joint *1* i.e. axis k_1.

S_4 = slider displacement which is measured about the axis k_4.

The link-joint parameter table is shown in table 1. This table describes the complete geometry of the mechanism.

2.4 Transformation Matrices:

Once the coordinate systems are rigidly fixed to each link of the mechanism and the link-joint parameter table is formed, coordinate-transformation matrices are specified. Coordinate-transformation matrices contain information about the links and the displacements (both sliding and rotation) between coordinate frames in the form of dual angles. The derivations of dual-number element coordinate transformation matrices, describing the links connecting joints *1* and *2*, *2* and *3*, *3* and *4* and *4* and *1*, are as follows.

The cosine and sine will be abbreviated by c and s respectively in the remainder of this work. The dual-number operator (3, 4), also called Clifford's screw operator, is denoted by ε ($\varepsilon^2 = 0$, $\varepsilon \neq 0$). The dual-number operator is used to express rotation and displacement about the same axis in combined way. It has the following properties:

$\hat{\theta} = \theta + \varepsilon S$ [θ and S are respectively rotation and displacement of any link about the same axis], then trigonometric functions of dual-angle can be obtained by using Taylor expansion. So, $s\hat{\theta} = s\theta + \varepsilon S \dfrac{d}{d\theta}(s\theta)$, $c\hat{\theta} = c\theta + \varepsilon S \dfrac{d}{d\theta}(c\theta)$, etc. All identities for ordinary trigonometry hold true for dual angle.

One can trace the path from joint *1* to joint *2* by taking the rotation through angle θ_1 and sliding through distance S_1 about and along the k_1 axis, followed by a link twist angle rotation through zero degrees with a translation through the link length a_1 about and along the i_2 axis. Therefore, the transformation matrix, specifying the location of coordinate frame {2} with respect to frame {1}, can be written as

$$ {}^{1}_{2}\hat{M} = [Z(\hat{\theta}_1)][X(\hat{\alpha}_1)] = \begin{bmatrix} c\hat{\theta}_1 & -s\hat{\theta}_1 & 0 \\ s\hat{\theta}_1 & c\hat{\theta}_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\hat{\alpha}_1 & -s\hat{\alpha}_1 \\ 0 & s\hat{\alpha}_1 & c\hat{\alpha}_1 \end{bmatrix} $$

$$ = \begin{bmatrix} c\theta_1 - \varepsilon S_1 s\theta_1 & -s\theta_1 - \varepsilon S_1 c\theta_1 & 0 \\ s\theta_1 + \varepsilon S_1 c\theta_1 & c\theta_1 - \varepsilon S_1 s\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_1 - \varepsilon a_1 s\alpha_1 & -s\alpha_1 - \varepsilon a_1 c\alpha_1 \\ 0 & s\alpha_1 + \varepsilon a_1 c\alpha_1 & c\alpha_1 - \varepsilon a_1 s\alpha_1 \end{bmatrix} $$

Since $\hat{\theta}_1 = \theta_1 + \varepsilon S_1$ and $\hat{\alpha}_1 = 0 + \varepsilon a_1$ as there is no link twist ($\alpha_1 = 0$), the above expression can be simplified as [In short ${}^{1}_{2}\hat{M}$ is replaced by $\hat{M}_1$],

$$ \hat{M}_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \varepsilon \begin{bmatrix} -S_1 s\theta_1 & -S_1 c\theta_1 & a_1 s\theta_1 \\ S_1 c\theta_1 & -S_1 s\theta_1 & -a_1 c\theta_1 \\ 0 & a_1 & 0 \end{bmatrix} \tag{1} $$

The path from joint *2* to joint *3* can be traced by taking a rotation through angle θ_2 and a translation through zero distance about and along the k_2 axis (not shown), followed by a rotation through angle η_2 with no translation about the j_2 axis, followed by no rotation and a translation through link length a_2 distance along the i_3 axis. Therefore, the derivation of the transformation matrix, specifying the location of coordinate frame $\{3\}$ with respect to frame $\{2\}$, is as follows:

$$ {}^{2}_{3}\hat{L} = [\,Z(\hat{\theta}_2)\,]\,[\,Y(\hat{\eta}_2)]\,[\,X(\hat{\alpha}_2)] $$

$$ = \begin{bmatrix} c\hat{\theta}_2 & -s\hat{\theta}_2 & 0 \\ s\hat{\theta}_2 & c\hat{\theta}_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\hat{\eta}_2 & 0 & s\hat{\eta}_2 \\ 0 & 1 & 0 \\ -s\hat{\eta}_2 & 0 & c\hat{\eta}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\hat{\alpha}_2 & -s\hat{\alpha}_2 \\ 0 & s\hat{\alpha}_2 & c\hat{\alpha}_2 \end{bmatrix} $$

$$ = \begin{bmatrix} c\theta_2 - \varepsilon S_2 s\theta_2 & -s\theta_2 - \varepsilon S_2 c\theta_2 & 0 \\ s\theta_2 + \varepsilon S_2 c\theta_2 & c\theta_2 - \varepsilon S_2 s\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\eta_2 - \varepsilon E_2 s\eta_2 & 0 & s\eta_2 + \varepsilon E_2 c\eta_2 \\ 0 & 1 & 0 \\ -s\eta_2 - \varepsilon E_2 c\eta_2 & 0 & c\eta_2 - \varepsilon E_2 s\eta_2 \end{bmatrix} $$

$$ \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_2 - \varepsilon a_2 s\alpha_2 & -s\alpha_2 - \varepsilon a_2 c\alpha_2 \\ 0 & s\alpha_2 + \varepsilon a_2 c\alpha_2 & c\alpha_2 - \varepsilon a_2 s\alpha_2 \end{bmatrix} $$

Since $S_2 = 0, E_2 = 0, \alpha_2 = 0$, and so $\hat{\theta}_2 = \theta_2 + \varepsilon S_2 = \theta_2, \hat{\eta}_2 = \eta_2 + \varepsilon E_2 = \eta_2$ and $\hat{\alpha}_2 = \alpha_2 + \varepsilon a_2 = \varepsilon a_2$. The above expression can be compressed as [In short, ${}^{2}_{3}\hat{L}$ is replaced by $\hat{L}_2$],

$$ \hat{L}_2 = \begin{bmatrix} c\theta_2 c\eta_2 & -s\theta_2 & c\theta_2 s\eta_2 \\ s\theta_2 c\eta_2 & c\theta_2 & s\theta_2 s\eta_2 \\ -s\eta_2 & 0 & c\eta_2 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & a_2 c\theta_2 s\eta_2 & a_2 s\theta_2 \\ 0 & a_2 s\theta_2 s\eta_2 & -a_2 c\theta_2 \\ 0 & a_2 c\eta_2 & 0 \end{bmatrix} \qquad (2) $$

The path from joint *3* to joint *4* can be traced by taking a rotation through angle θ_3 and a translation through zero distance about and along the k_3 axis (not shown), followed by a rotation through angle η_3 with no translation about the j_3 axis, followed by a 90-degree rotation with no translation about the i_4 axis. Therefore, the derivation of the transformation matrix, specifying the location of coordinate frame $\{4\}$ with respect to frame $\{3\}$, is as follows:

$$ {}^{3}_{4}\hat{L} = [\,Z(\hat{\theta}_3)\,]\,[\,Y(\hat{\eta}_3)]\,[\,X(\hat{\alpha}_3)] $$

$$ = \begin{bmatrix} c\hat{\theta}_3 & -s\hat{\theta}_3 & 0 \\ s\hat{\theta}_3 & c\hat{\theta}_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\hat{\eta}_3 & 0 & s\hat{\eta}_3 \\ 0 & 1 & 0 \\ -s\hat{\eta}_3 & 0 & c\hat{\eta}_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\hat{\alpha}_3 & -s\hat{\alpha}_3 \\ 0 & s\hat{\alpha}_3 & c\hat{\alpha}_3 \end{bmatrix} $$

$$ = \begin{bmatrix} c\theta_3 - \varepsilon S_3 s\theta_3 & -s\theta_3 - \varepsilon S_3 c\theta_3 & 0 \\ s\theta_3 + \varepsilon S_3 c\theta_3 & c\theta_3 - \varepsilon S_3 s\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\eta_3 - \varepsilon E_3 s\eta_3 & 0 & s\eta_3 + \varepsilon E_3 c\eta_3 \\ 0 & 1 & 0 \\ -s\eta_3 - \varepsilon E_3 c\eta_3 & 0 & c\eta_3 - \varepsilon E_3 s\eta_3 \end{bmatrix} $$

$$ \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_3 - \varepsilon a_3 s\alpha_3 & -s\alpha_3 - \varepsilon a_3 c\alpha_3 \\ 0 & s\alpha_3 + \varepsilon a_3 c\alpha_3 & c\alpha_3 - \varepsilon a_3 s\alpha_3 \end{bmatrix} $$

Since $S_3 = 0, E_3 = 0, a_3 = 0$ and so $\hat{\theta}_3 = \theta_3 + \varepsilon S_3 = \theta_3, \hat{\eta}_3 = \eta_3 + \varepsilon E_3 = \eta_3$ and $\hat{\alpha}_3 = \alpha_3 + \varepsilon a_3 = \frac{\pi}{2}$ (as the link-twist angle = 90 degrees), the above expression can be compressed as [in short, ${}^{3}_{4}\hat{L}$ is replaced by $\hat{L}_3$],

$$ \hat{L}_3 = \begin{bmatrix} c\theta_3 c\eta_3 & c\theta_3 s\eta_3 & s\theta 3 \\ s\theta_3 c\eta_3 & s\theta_3 s\eta_3 & -c\theta_3 \\ -s\eta_3 & c\eta_3 & 0 \end{bmatrix} \tag{3} $$

The path from joint *4* to joint *1* can be traced by sliding through distance S_4 with no rotation along the k_4 axis, followed by a rotation through angle α_4 and a translation through distance a_4 about and along the i_1 axis. Therefore, the transformation matrix, specifying the location of coordinate frame $\{1\}$ with respect to frame $\{4\}$, is written as follows:

$$ {}^{4}_{1}\hat{M} = [\,Z(\hat{\theta}_4)\,]\,[\,X(\hat{\alpha}_4)\,] = \begin{bmatrix} c\hat{\theta}_4 & -s\hat{\theta}_4 & 0 \\ s\hat{\theta}_4 & c\hat{\theta}_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\hat{\alpha}_4 & -s\hat{\alpha}_4 \\ 0 & s\hat{\alpha}_4 & c\hat{\alpha}_4 \end{bmatrix} $$

$$ = \begin{bmatrix} c\theta_4 - \varepsilon S_4 s\theta_4 & -s\theta_4 - \varepsilon S_4 c\theta_4 & 0 \\ s\theta_4 + \varepsilon S_4 c\theta_4 & c\theta_4 - \varepsilon S_4 s\theta_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_4 - \varepsilon a_4 s\alpha_4 & -s\alpha_4 - \varepsilon a_4 c\alpha_4 \\ 0 & s\alpha_4 + \varepsilon a_4 c\alpha_4 & c\alpha_4 - \varepsilon a_4 s\alpha_4 \end{bmatrix} $$

Since $\hat{\theta}_4 = 0 + \varepsilon S_4$, as there is no rotation ($\theta_4 = 0$), and $\hat{\alpha}_4 = \alpha_4 + \varepsilon a_4$, the above expression can be compressed as [In short, ${}^4_1\hat{M}$ is replaced by $\hat{M}_4$],

$$\hat{M}_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_4 & -s\alpha_4 \\ 0 & s\alpha_4 & c\alpha_4 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & -S_4 c\alpha_4 & S_4 s\alpha_4 \\ S_4 & -a_4 s\alpha_4 & -a_4 c\alpha_4 \\ 0 & a_4 c\alpha_4 & -a_4 s\alpha_4 \end{bmatrix} \quad (4)$$

2.5 Partial Derivatives of the Transformation Matrices:

Noting that this problem is to be adapted to computer operation, linear operator matrices will be introduced to perform differentiations of the transformation matrices. Taking the partial derivatives of the transformation matrices with respect to the variable quantity contained in the matrix is accomplished by premultiplying by an operator matrix. The derivation of partial differential operator of each of the four transformation matrices follows.

The first transformation matrix contains only input parameters (input crank angle and axial displacement) and crank length, which are all known quantities. Therefore this matrix need not be differentiated.

The second transformation matrix contains two unknown variables θ_2 and η_2. Upon taking the partial derivative of this matrix (equation 2) with respect to variable θ_2, the result is,

$$\frac{\delta \hat{L}_2}{\delta \theta_2} = \begin{bmatrix} -s\theta_2 c\eta_2 & -c\theta_2 & -s\theta_2 s\eta_2 \\ c\theta_2 c\eta_2 & -s\theta_2 & c\theta_2 s\eta_2 \\ 0 & 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & -a_2 s\theta_2 s\eta_2 & a_2 c\theta_2 \\ 0 & a_2 c\theta_2 s\eta_2 & a_2 s\theta_2 \\ 0 & 0 & 0 \end{bmatrix} = \hat{Q}_{L\theta_2} \hat{L}_2$$

where partial differential operator $Q_{L\theta_2}$ can be obtained in a number of ways; the easiest is perhaps by inspection. Hence operator $Q_{L\theta_2}$ can be written as follows:

$$\hat{Q}_{L\theta_2} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2a)$$

The partial derivative of matrix equation (2) with respect to variable η_2 is,

$$\frac{\delta\hat{L}_2}{\delta\eta_2} = \begin{bmatrix} -c\theta_2 s\eta_2 & 0 & c\theta_2 c\eta_2 \\ -s\theta_2 s\eta_2 & 0 & s\theta_2 c\eta_2 \\ -c\eta_2 & 0 & -s\eta_2 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & a_2 c\theta_2 c\eta_2 & 0 \\ 0 & a_2 s\theta_2 c\eta_2 & 0 \\ 0 & -a_2 s\eta_2 & 0 \end{bmatrix} = \hat{Q}_{L\eta_2}\hat{L}_2$$

hence partial differential operator $Q_{L\eta 2}$ can be written as,

$$\hat{Q}_{L\eta_2} = \begin{bmatrix} 0 & 0 & c\theta_2 \\ 0 & 0 & s\theta_2 \\ -c\theta_2 & -s\theta_2 & 0 \end{bmatrix} \tag{2b}$$

The partial derivative of matrix equation (3) with respect to variable θ_3 is,

$$\frac{\delta\hat{L}_3}{\delta\theta_3} = \begin{bmatrix} -s\theta_3 c\eta_3 & -s\theta_3 s\eta_3 & c\theta_3 \\ c\theta_3 c\eta_3 & c\theta_3 s\eta_3 & s\theta_3 \\ 0 & 0 & 0 \end{bmatrix} = \hat{Q}_{L\theta_3}\hat{L}_3$$

hence the partial differential operator $Q_{L\theta 3}$ can be written as,

$$\hat{Q}_{L\theta_3} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3a}$$

The partial derivative of matrix equation (3) with respect to variable η_3 is,

$$\frac{\delta\hat{L}_3}{\delta\eta_3} = \begin{bmatrix} -c\theta_3 s\eta_3 & c\theta_3 c\eta_3 & 0 \\ -s\theta_3 s\eta_3 & s\theta_3 c\eta_3 & 0 \\ -c\eta_3 & -s\eta_3 & 0 \end{bmatrix} = \hat{Q}_{L\eta_3}\hat{L}_3$$

hence the partial differential operator $Q_{L\eta 3}$ can be written as,

$$\hat{Q}_{L\eta_3} = \begin{bmatrix} 0 & 0 & c\theta_3 \\ 0 & 0 & s\theta_3 \\ -c\theta_3 & -s\theta_3 & 0 \end{bmatrix} \tag{3b}$$

The partial derivative of matrix equation (4) with respect to variable S_4 is,

$$\frac{\delta\hat{M}_4}{\delta S_4} = \varepsilon \begin{bmatrix} 0 & -c\alpha_4 & s\alpha_4 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \hat{Q}_{MS_4}\hat{M}_4$$

hence the partial differential operator Q_{MS4} can be written as,

$$\hat{Q}_{MS_4} = \begin{bmatrix} 0 & -\varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4a}$$

2.6 Displacement Analysis:

The relationship between transformation matrices which describes the four links of the mechanism and the joints connecting them is given by the Condition of Loop Closure. Because the links are connected end to end to form a closed loop, transforming from the frame to crank to the connecting rod to the slider brings one back to the coordinate system fixed on the frame as if no transformation at all has occured. Hence this can be written mathematically as:

$$ {}_{2}^{1}\hat{M}\,{}_{3}^{2}\hat{L}\,{}_{4}^{3}\hat{L}\,{}_{1}^{4}\hat{M} = \hat{I} \tag{5}$$

which is a nonlinear matrix equation, where $\hat{I}$ is dual identity matrix

$$\hat{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

2.6.1 Dual-number formulation of Numerical Method:

The numerical method being used to determine the displacements is based on the algorithm (12) introduced by Hall, Root & Sandgren (1977). This algorithm is the combination of Davidon-Fletcher-Powell optimization routine with the Newton-Raphson method as applied to kinematics by Uicker, Denavit and Hartenberg (1964). This procedure was developed for the use with 4×4 homogeneous transformation matrices, and was adopted to 3×3 dual-number matrices by Fischer and Paul(1988). Details of the Newton-Raphson method are as follows.

A first order expansion of the loop closure equation is

$$[\hat{M}_1]\,[\hat{L}_2 + \hat{Q}_{L\theta}\hat{L}_2\,\delta\theta_2 + \hat{Q}_{L\eta}\hat{L}_2\,\delta\eta_2]\,[\hat{L}_3 + \hat{Q}_{L\theta}\hat{L}_3\,\delta\theta_3 + \hat{Q}_{L\eta}\hat{L}_3\,\delta\eta_3]$$
$$[\hat{M}_4 + \hat{Q}_{MS}\hat{M}_4\,\delta S_4] = \hat{I} \tag{7}$$

where θ_2, η_2, θ_3, η_3 and S_4 are guesses of displacement variables; $\delta\theta_2$, $\delta\eta_2$, $\delta\theta_3$, $\delta\eta_3$ and δS_4 are errors between guesses and exact values of the displacement variables; $Q_{L\theta}$, $Q_{L\eta}$ and Q_{MS} are partial differential operators such that (as shown before)

$$Q_{L\theta} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

$$Q_{L\eta} = \begin{bmatrix} 0 & 0 & c\theta \\ 0 & 0 & s\theta \\ -c\theta & -s\theta & 0 \end{bmatrix} \tag{9}$$

$$Q_{MS} = \begin{bmatrix} 0 & -\varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{10}$$

Expanding equation (7), and putting it into a form which is computationally advantageous, the following equation is obtained. Performance of the multiplication yields a very lengthy equation. However, keeping with the idea of the iteration process, all higher order terms of the form ($\delta\theta_2\delta\theta_3$, etc) are neglected.

$$\widehat{H}_2\delta\theta_2+\widehat{H'}_2\delta\eta_2+\widehat{H}_3\delta\theta_3+\widehat{H'}_3\delta\eta_3+\widehat{H''}_4\delta S_4 = \widehat{B}^T-\widehat{I} \tag{11}$$

where

$$\widehat{H}_2 = \widehat{M}_1\widehat{Q}_{L\theta}\widehat{M}_1^T \tag{12}$$

$$\widehat{H'}_2 = \widehat{M}_1\widehat{Q}_{L\eta}\widehat{M}_1^T \tag{13}$$

$$\widehat{H}_3 = (\widehat{M}_1\widehat{L}_2)\widehat{Q}_{L\theta}(\widehat{M}_1\widehat{L}_2)^T \tag{14}$$

$$\widehat{H'}_3 = (\widehat{M}_1\widehat{L}_2)\widehat{Q}_{L\eta}(\widehat{M}_1\widehat{L}_2)^T \tag{15}$$

$$\widehat{H''}_4 = (\widehat{M}_1\widehat{L}_2\widehat{L}_3)\widehat{Q}_{MS}(\widehat{M}_1\widehat{L}_2\widehat{L}_3)^T \tag{16}$$

In general,

$$\widehat{H}_i = (\widehat{T}_1\widehat{T}_2------\widehat{T}_{i-1})\widehat{Q}_X(\widehat{T}_1\widehat{T}_2------\widehat{T}_{i-1})^T \tag{17}$$

and

$$\widehat{B} = \widehat{M}_1\widehat{L}_2\widehat{L}_3\widehat{M}_4 \tag{18}$$

Equation (11) is condensed to

$$\widehat{F} = \widehat{B}^T-\widehat{I} \tag{19}$$

Equations (11) through (19) are all 3×3 dual-number matrix equations. So this equation can be written in a matrix form as follows:

$$\begin{bmatrix} \hat{F}_{11} & \hat{F}_{12} & \hat{F}_{13} \\ \hat{F}_{21} & \hat{F}_{22} & \hat{F}_{23} \\ \hat{F}_{31} & \hat{F}_{32} & \hat{F}_{33} \end{bmatrix} = \begin{bmatrix} \hat{B}_{11}-1 & \hat{B}_{21} & \hat{B}_{31} \\ \hat{B}_{12} & \hat{B}_{22}-1 & \hat{B}_{32} \\ \hat{B}_{13} & \hat{B}_{23} & \hat{B}_{33}-1 \end{bmatrix} \tag{19a}$$

where

$$\begin{aligned}
\hat{F}_{11} &= \hat{H}_{2,11}\,\delta\theta_2 + \hat{H}'_{2,11}\,\delta\eta_2 + \hat{H}_{3,11}\delta\theta_3 + \hat{H}'_{3,11}\,\delta\eta_3 + \hat{H}''_{4,11}\,\delta S_4 \\
\hat{F}_{12} &= \hat{H}_{2,12}\,\delta\theta_2 + \hat{H}'_{2,12}\,\delta\eta_2 + \hat{H}_{3,12}\delta\theta_3 + \hat{H}'_{3,12}\,\delta\eta_3 + \hat{H}''_{4,12}\,\delta S_4 \\
\hat{F}_{13} &= \hat{H}_{2,13}\,\delta\theta_2 + \hat{H}'_{2,13}\,\delta\eta_2 + \hat{H}_{3,13}\delta\theta_3 + \hat{H}'_{3,13}\,\delta\eta_3 + \hat{H}''_{4,13}\,\delta S_4 \\
\hat{F}_{21} &= \hat{H}_{2,21}\,\delta\theta_2 + \hat{H}'_{2,21}\,\delta\eta_2 + \hat{H}_{3,21}\delta\theta_3 + \hat{H}'_{3,21}\,\delta\eta_3 + \hat{H}''_{4,21}\,\delta S_4 \\
\hat{F}_{22} &= \hat{H}_{2,22}\,\delta\theta_2 + \hat{H}'_{2,22}\,\delta\eta_2 + \hat{H}_{3,22}\delta\theta_3 + \hat{H}'_{3,22}\,\delta\eta_3 + \hat{H}''_{4,22}\,\delta S_4 \\
\hat{F}_{23} &= \hat{H}_{2,23}\,\delta\theta_2 + \hat{H}'_{2,23}\,\delta\eta_2 + \hat{H}_{3,23}\delta\theta_3 + \hat{H}'_{3,23}\,\delta\eta_3 + \hat{H}''_{4,23}\,\delta S_4 \\
\hat{F}_{31} &= \hat{H}_{2,31}\,\delta\theta_2 + \hat{H}'_{2,31}\,\delta\eta_2 + \hat{H}_{3,31}\delta\theta_3 + \hat{H}'_{3,31}\,\delta\eta_3 + \hat{H}''_{4,31}\,\delta S_4 \\
\hat{F}_{32} &= \hat{H}_{2,32}\,\delta\theta_2 + \hat{H}'_{2,32}\,\delta\eta_2 + \hat{H}_{3,32}\delta\theta_3 + \hat{H}'_{3,32}\,\delta\eta_3 + \hat{H}''_{4,32}\,\delta S_4 \\
\hat{F}_{33} &= \hat{H}_{2,33}\,\delta\theta_2 + \hat{H}'_{2,33}\,\delta\eta_2 + \hat{H}_{3,33}\delta\theta_3 + \hat{H}'_{3,33}\,\delta\eta_3 + \hat{H}''_{4,33}\,\delta S_4
\end{aligned} \tag{19b}$$

So from equation (19a) nine dual equations can be written from this matrix equation, out of which only three are independent. Using the three terms in the upper triangle of each matrix and separating them into a set of real equations and a set of dual equations:

$$\begin{aligned} F_{p12} &= B_{p21} \\ F_{p13} &= B_{p31} \\ F_{p23} &= B_{p32} \end{aligned} \tag{20}$$

and

$$
\begin{aligned}
F_{d12}&=B_{d21}\\
F_{d13}&=B_{d31}\\
F_{d23}&=B_{d32}
\end{aligned}
\tag{21}
$$

where the symbols p and d represent primary and dual components respectively. Equation set (20) does not include the constraint that the each diagonal element must be equal to unity at closure. This constraint is included through modification of the current set of equations as discussed in Sandor and Erdman (1984). So, this set can be modified as,

$$
\begin{aligned}
F_{p12}&=B_{p21}+B_{p11}+B_{p22}-2\\
F_{p13}&=B_{p31}+B_{p11}+B_{p33}-2\\
F_{p23}&=B_{p32}+B_{p22}+B_{p33}-2
\end{aligned}
\tag{22}
$$

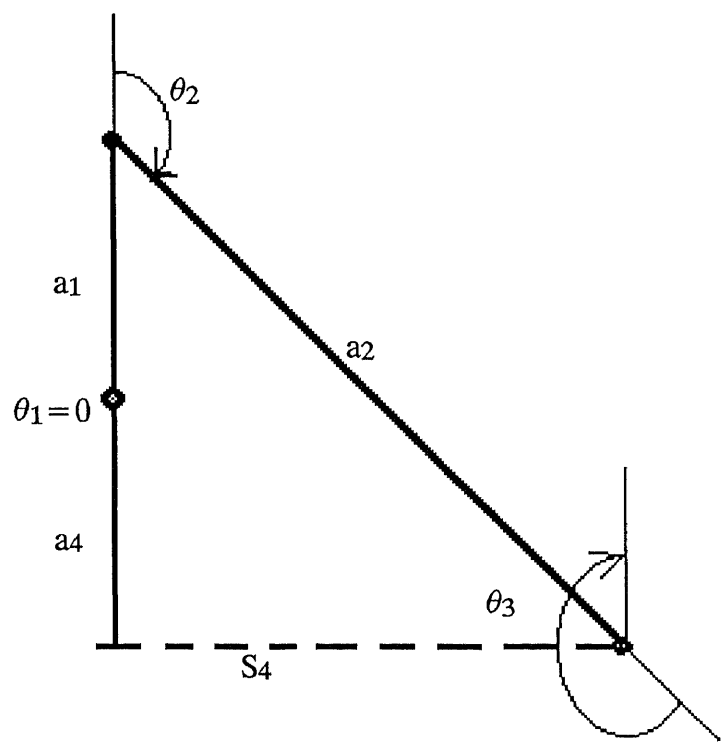

Figure 4 Planar slider crank when crank angle is zero

The left-hand sides of equations (21) & (22) contain the linear combination of respective elements of the H matrices and the error terms are contained in the left hand side of equation (11). Hence, combining equations (21) & (22) as partitions of a single matrix of real-number equations, the resulting equation is

$$\begin{bmatrix} A_p \\ --- \\ A_d \end{bmatrix} [\,\Delta] = \begin{bmatrix} V_p \\ --- \\ V_d \end{bmatrix} \tag{23}$$

where

$$A = \begin{bmatrix} H_{2p12} & H'_{2p12} & H_{3p12} & H'_{3p12} & H''_{4p12} \\ H_{2p13} & H'_{2p13} & H_{3p13} & H'_{3p13} & H''_{4p13} \\ H_{2p23} & H'_{2p23} & H_{3p23} & H'_{3p23} & H''_{4p23} \\ \hline H_{2d12} & H'_{2d12} & H_{3d12} & H'_{3d12} & H''_{4d12} \\ H_{2d13} & H'_{2d13} & H_{3d13} & H'_{3d13} & H''_{4d13} \\ H_{2d23} & H'_{2d23} & H_{3d23} & H'_{3d23} & H''_{4d23} \end{bmatrix} \tag{24}$$

$$V = \begin{bmatrix} B_{p21}+B_{p11}+B_{p22}-2 \\ B_{p31}+B_{p11}+B_{p33}-2 \\ B_{p32}+B_{p22}+B_{p33}-2 \\ B_{d21} \\ B_{d31} \\ B_{d32} \end{bmatrix} \tag{25}$$

and

$$\Delta = \begin{bmatrix} \delta\theta_2 \\ \delta\eta_2 \\ \delta\theta_3 \\ \delta\eta_3 \\ \delta S_4 \end{bmatrix} \tag{26}$$

Rearranging equation (23) to solve for the error terms, the Newton-Raphson iteration technique can be achieved. But the matrix A, which is 6×5, cannot be inverted in usual method in order to do so, unless manipulations are performed as follows.

$$A^{-1} = (A^T A)^{-1} A^T$$

Hence the modified form of equation (23) is as follows:

$$\Delta = (A^T A)^{-1} A^T V \tag{23a}$$

which is exactly same as $\Delta = A^{-1}V$.

The guesses of displacement variables are revised by adding the error terms, Δ, to the guesses and the new set of joint variables is used in the next iteration to compute a new set of error terms. The iterative process is continued until the absolute values of all error terms are less than or equal to the desired accuracy limits. At this point the displacements are determined only for the first position of the input link. To solve for the displacements for a new position of the input link displacement, angle θ_1 is increased by a small amount and the previously calculated displacements are used as initial estimates for the next position. This process is continued for a complete rotation of the input link, the crank.

Another input variable, i.e. length S_1, the axial displacement of the crank, can be introduced in two following ways:

(1) it is constant throughout the complete rotation of the crank, i.e. the crank is axially displaced to a certain distance and is then rotated.

(2) the crank is constantly moving to and fro in simple harmonic motion along its own joint axis (the joint connecting the frame and the crank, i.e. joint *1*). A new parameter α is introduced which can mathematically be defined as follows:

$$\alpha = \frac{\textit{number of cycles the crank slides through along its own joint axis}}{\textit{number of complete rotations of the crank}} \tag{27}$$

Hence relating the axial displacement of the crank to the rotation, translation S_1 can be written as

$$S_1 = S_{S1}\sin(\alpha\theta_1) \tag{28}$$

where symbol S_{S1} is the amplitude of the sliding cyclic motion of the crank.

2.6.2 Plane Geometry Method for Initial Estimations of the Joint Variables:

Initial estimations of the joint variables θ_2, η_2, θ_3, η_3, S_4 have been done by following simple plane geometry method. Fig. 4 shows planar slider crank with offset a_4 only when input crank angle is zero. From geometry the following formulas for the joint variables can be written.

$$S_4 = [\, a_2^2 - (\, a_1 + a_4\,)^2 \,]^{\frac{1}{2}}$$

$$\theta_2 = \pi - \tan^{-1}\left[\frac{S_4}{a_1+a_4}\right]$$

$$\theta_3 = 2\pi - \theta_2$$

$$\eta_2 = 0 \quad \text{and} \quad \eta_3 = 0$$

2.7 Velocity Analysis:

The dual velocity of link *1* (i.e., the crank) relative to link *4* (i.e., the frame) with respect to point *1* in terms of the unit vectors of system $\{1\}$ is represented by the vector

$$^{1}\hat{V}^{1}_{14} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \varepsilon \dot{S}_1 \end{Bmatrix} \tag{29}$$

where symbols $\dot{\theta}_1$ and $\dot{S}_1$ denote the rotational and translational velocities about k_1 axis at joint *1* respectively. There is no other velocity about the other two axes at this joint.

Similarly the vector equations for other links can be written as

$$^{2}\hat{V}^{2}_{21} = \begin{Bmatrix} 0 \\ \dot{\eta}_2 \\ \dot{\theta}_2 \end{Bmatrix} \tag{30}$$

where symbols $\dot{\eta}_2$ and $\dot{\theta}_2$ denote rotational velocities about the j_2 and k_2 axes at joint *2* respectively.

$$^{3}\hat{V}^{3}_{32} = \begin{Bmatrix} 0 \\ \dot{\eta}_3 \\ \dot{\theta}_3 \end{Bmatrix} \tag{31}$$

where symbols $\dot{\eta}_3$ and $\dot{\theta}_3$ denote rotational velocities about the j_3 and k_3 axes at joint 3 respectively.

$$
{}^4\hat{V}^4_{43} = \begin{Bmatrix} 0 \\ 0 \\ \varepsilon \dot{S}_4 \end{Bmatrix} \tag{32}
$$

where symbol $\dot{S}_4$ denotes actually the slider velocity about the k_4 axis.

The dual velocity of point 2 on the crank relative to the frame in terms of the unit vectors of system $\{2\}$, i.e. ${}^2\hat{V}^2_{14}$, can be derived by premultiplying equation (29) by the transpose of equation (1):

$$
{}^2\hat{V}^2_{14} = {}^2_1\hat{M}\,{}^1\hat{V}^1_{14} \qquad (\; {}^2_1\hat{M} = ({}^1_2\hat{M})^T \;)
$$

$$
= \begin{bmatrix} c\theta_1 - \varepsilon S_1 s\theta_1 & s\theta_1 + \varepsilon S_1 c\theta_1 & 0 \\ -s\theta_1 - \varepsilon S_1 c\theta_1 & c\theta_1 - \varepsilon S_1 s\theta_1 & \varepsilon a_1 \\ \varepsilon a_1 s\theta_1 & -\varepsilon a_1 c\theta_1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \varepsilon \dot{S}_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \varepsilon a_1 \dot{\theta}_1 \\ \dot{\theta}_1 + \varepsilon \dot{S}_1 \end{Bmatrix} \tag{33}
$$

The dual velocity of point 2 on the connecting rod relative to the frame in terms of the unit vectors of the system $\{2\}$ is determined by adding the equations (33) and (30):

$$
{}^2\hat{V}^2_{24} = {}^2\hat{V}^2_{14} + {}^2\hat{V}^2_{21} = \begin{Bmatrix} 0 \\ \varepsilon a_1 \dot{\theta}_1 \\ \dot{\theta}_1 + \varepsilon \dot{S}_1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \dot{\eta}_2 \\ \dot{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dot{\eta}_2 + \varepsilon \dot{\theta}_1 a_1 \\ \dot{\theta}_1 + \dot{\theta}_2 + \varepsilon \dot{S}_1 \end{Bmatrix} \tag{34}
$$

The dual velocity of point 3 on the connecting rod relative to the frame in terms of the unit vectors of the system $\{3\}$ is determined by using this equation and the transpose of equation (2):

$$
{}^3\hat{V}^3_{24} = {}^3_2\hat{L}\,{}^2\hat{V}^2_{24} \qquad (\; {}^3_2\hat{L} = ({}^2_3\hat{L})^T \;)
$$

$$
= \begin{bmatrix} c\theta_2 c\eta_2 & s\theta_2 c\eta_2 & -s\eta_2 \\ -s\theta_2 + \varepsilon a_2 c\theta_2 s\eta_2 & c\theta_2 + \varepsilon a_2 s\theta_2 s\eta_2 & \varepsilon a_2 c\eta_2 \\ c\theta_2 s\eta_2 + \varepsilon a_2 s\theta_2 & s\theta_2 s\eta_2 - \varepsilon a_2 c\theta_2 & c\eta_2 \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{\eta}_2 + \varepsilon \dot{\theta}_1 a_1 \\ \dot{\theta}_1 + \dot{\theta}_2 + \varepsilon \dot{S}_1 \end{Bmatrix}
$$

$$= \begin{Bmatrix} -(\dot{\theta}_1+\dot{\theta}_2)s\eta_2+\dot{\eta}_2 s\theta_2 c\eta_2+\varepsilon(\dot{\theta}_1 a_1 s\theta_2 c\eta_2-\dot{S}_1 s\eta_2) \\ \dot{\eta}_2 c\theta_2+\varepsilon[\dot{\theta}_1(a_1 c\theta_2+a_2 c\eta_2)+\dot{\theta}_2 a_2 c\eta_2+\dot{\eta}_2 a_2 s\theta_2 s\eta_2] \\ (\dot{\theta}_1+\dot{\theta}_2)c\eta_2+\dot{\eta}_2 s\theta_2 s\eta_2+\varepsilon(\dot{\theta}_1 a_1 s\theta_2 s\eta_2-\dot{\eta}_2 a_2 c\theta_2+\dot{S}_1 c\eta_2) \end{Bmatrix} \tag{35}$$

The matrix equation for dual velocity of point *3* on the connecting rod relative to the frame in terms of unit vector of system {*3*} can also be formulated by using the backward path of the loop. The procedure is as follows:

From equations (31) and (32) respectively,

$$^3\hat{V}^3_{23} = -^3\hat{V}^3_{32} = \begin{Bmatrix} 0 \\ -\dot{\eta}_3 \\ -\dot{\theta}_3 \end{Bmatrix} \tag{31a}$$

$$^4\hat{V}^4_{34} = -^4\hat{V}^4_{43} = \begin{Bmatrix} 0 \\ 0 \\ -\varepsilon\dot{S}_4 \end{Bmatrix} \tag{32a}$$

$$^3\hat{V}^3_{24} = {}^3_4\hat{L}\,{}^4\hat{V}^4_{34} + {}^3\hat{V}^3_{23}$$

$$= \begin{bmatrix} c\theta_3 c\eta_3 & c\theta_3 s\eta_3 & s\theta_3 \\ s\theta_3 c\eta_3 & s\theta_3 s\eta_3 & -c\theta_3 \\ -s\eta_3 & c\eta_3 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -\varepsilon\dot{S}_4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\dot{\eta}_3 \\ -\dot{\theta}_3 \end{Bmatrix} = \begin{Bmatrix} -\varepsilon\dot{S}_4 s\theta_3 \\ -\dot{\eta}_3+\varepsilon\dot{S}_4 c\theta_3 \\ -\dot{\theta}_3 \end{Bmatrix} \tag{36}$$

[This has been obtained using equations (3), (31a) and (32a)]

Equating the two expressions (35) & (36) yields

$$\begin{Bmatrix} -(\dot{\theta}_1+\dot{\theta}_2)s\eta_2+\dot{\eta}_2 s\theta_2 c\eta_2+\varepsilon(\dot{\theta}_1 a_1 s\theta_2 c\eta_2-\dot{S}_1 s\eta_2) \\ \dot{\eta}_2 c\theta_2+\varepsilon[\dot{\theta}_1(a_1 c\theta_2+a_2 c\eta_2)+\dot{\theta}_2 a_2 c\eta_2+\dot{\eta}_2 a_2 s\theta_2 s\eta_2] \\ (\dot{\theta}_1+\dot{\theta}_2)c\eta_2+\dot{\eta}_2 s\theta_2 s\eta_2+\varepsilon(\dot{\theta}_1 a_1 s\theta_2 s\eta_2-\dot{\eta}_2 a_2 c\theta_2+\dot{S}_1 c\eta_2) \end{Bmatrix} = \begin{Bmatrix} -\varepsilon\dot{S}_4 s\theta_3 \\ -\dot{\eta}_3+\varepsilon\dot{S}_4 c\theta_3 \\ -\dot{\theta}_3 \end{Bmatrix} \tag{37}$$

This matrix equation actually contains six linear equations (3 primary and 3 dual) with five unknowns. The next step is to rewrite equation (37) as a system of linear equations as follows:

$$-(\dot{\theta}_1+\dot{\theta}_2)s\eta_2+\dot{\eta}_2 s\theta_2 c\eta_2 = 0 \tag{37a}$$

$$\dot{\theta}_1 a_1 s\theta_2 c\eta_2 - \dot{S}_1 s\eta_2 = -\dot{S}_4 s\theta_3 \tag{37b}$$

$$\dot{\eta}_2 c\theta_2 = -\dot{\eta}_3 \tag{37c}$$

$$\dot{\theta}_1(a_1 c\theta_2 + a_2 c\eta_2) + \dot{\theta}_2 a_2 c\eta_2 + \dot{\eta}_2 a_2 s\theta_2 s\eta_2 = \dot{S}_4 c\theta_3 \tag{37d}$$

$$(\dot{\theta}_1 + \dot{\theta}_2) c\eta_2 + \dot{\eta}_2 s\theta_2 s\eta_2 = -\dot{\theta}_3 \tag{37e}$$

$$\dot{\theta}_1 a_1 s\theta_2 s\eta_2 - \dot{\eta}_2 a_2 c\theta_2 + \dot{S}_1 c\eta_2 = 0 \tag{37f}$$

Now separating the unknowns to form a column matrix and a coefficient matrix, the product of these two matrices must be equal to some column matrix as shown below:

$$\begin{bmatrix} -s\eta_2 & 0 & s\theta_2 c\eta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & s\theta_3 \\ 0 & 0 & c\theta_2 & 1 & 0 \\ a_2 c\eta_2 & 0 & a_2 s\theta_2 s\eta_2 & 0 & -c\theta_3 \\ c\eta_2 & 1 & s\theta_2 s\eta_2 & 0 & 0 \\ 0 & 0 & -a_2 c\theta_2 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \\ \dot{S}_4 \end{Bmatrix} = \begin{Bmatrix} \dot{\theta}_1 s\eta_2 \\ -\dot{\theta}_1 a_1 s\theta_2 c\eta_2 + \dot{S}_1 s\eta_2 \\ 0 \\ -\dot{\theta}_1(a_1 c\theta_2 + a_2 c\eta_2) \\ -\dot{\theta}_1 c\eta_2 \\ -\dot{\theta}_1 a_1 s\theta_2 - \dot{S}_1 c\eta_2 \end{Bmatrix} \tag{38}$$

The coefficient matrix is 6×5 and the column matrix of the unknowns is 5×1, therefore it is a system of six equations with five unknowns. Since rank of the coefficient matrix equals 5, the number of columns, a unique solution can be obtained. The final solutions for all the unknowns are as follows:

$$\dot{\theta}_2 = \frac{\dot{S}_1 c\theta_3 s\eta_2 c\eta_2 - a_1(\dot{\theta}_1 s\theta_2 c\theta_3 + \dot{\theta}_1 c\theta_2 s\theta_3 c\eta_2) - a_2 \dot{\theta}_1 s\theta_3}{a_2 s\theta_3} \tag{39}$$

$$\dot{\theta}_3 = \frac{a_1(\dot{\theta}_1 c\theta_2 s\theta_3 + \dot{\theta}_1 s\theta_2 c\theta_3 c\eta_2) - \dot{S}_1 c\theta_3 s\eta_2}{a_2 s\theta_3} \tag{40}$$

$$\dot{\eta}_2 = \frac{-s\eta_2[a_1(\dot{\theta}_1 c\theta_2 s\theta_3 + \dot{\theta}_1 s\theta_2 c\theta_3 c\eta_2) - \dot{S}_1 s\eta_2 c\theta_3]}{a_2 s\theta_2 s\theta_3} \tag{41}$$

$$\dot{\eta}_3 = \frac{c\theta_2 s\eta_2[\dot{\theta}_1 a_1(s\theta_2 c\theta_3 c\eta_2 + c\theta_2 s\theta_3) - \dot{S}_1 s\eta_2 c\theta_3]}{a_2 s\theta_2 s\theta_3} \tag{42}$$

$$\dot{S}_4 = \frac{\dot{S}_1 s\eta_2 - \dot{\theta}_1 a_1 s\theta_2 c\eta_2}{s\theta_3} \tag{43}$$

All these equations involve the sliding velocity of the crank along its own joint axis (the joint connecting frame and crank, i.e. joint *1*), which is given by either

$$\dot{S}_1 = 0 \qquad [\, when\ S_1 = constant \,]$$

or

$$\dot{S}_1 = \alpha S_{S1} \cos(\alpha\theta_1)\dot{\theta}_1 \qquad [\, when\ S_1\ is\ sinusoidal \,] \tag{28a}$$

[equation (28a) is obtained by differentiating equation (28) with respect to time when S_1 is variable]

For dynamic analysis, equations (33) and (35) which respectively formulate the dual velocities of point *2* on the crank relative to the stationary frame in terms of the unit vectors of the system {*2*} and of point *3* on the connecting rod relative to the stationary frame in terms of the unit vectors of the system {*3*}, will be used. In addition to that, the dual velocity of point *4* on the slider relative frame in terms of the unit vectors of the system {*4*}, i.e. ${}^4\hat{V}^4_{34}$,is also needed. The derivation of that velocity vector is obtained by adding equations (31) & (35)

$${}^3\hat{V}^3_{34} = {}^3\hat{V}^3_{24} + {}^3\hat{V}^3_{32}$$

$$= \begin{Bmatrix} \dot{\eta}_2 s\theta_2 c\eta_2 - (\dot{\theta}_1 + \dot{\theta}_2)s\eta_2 + \varepsilon(\dot{\theta}_1 a_1 s\theta_2 c\eta_2 - \dot{S}_1 s\eta_2) \\ \dot{\eta}_2 c\theta_2 + \dot{\eta}_3 + \varepsilon[\dot{\theta}_1 a_1 c\theta_2 + \dot{\eta}_2 a_2 s\theta_2 s\eta_2 + (\dot{\theta}_1 + \dot{\theta}_2)a_2 c\eta_2] \\ (\dot{\theta}_1 + \dot{\theta}_2)c\eta_2 + \dot{\eta}_2 s\theta_2 s\eta_2 + \dot{\theta}_3 + \varepsilon(\dot{S}_1 c\eta_2 + a_1\dot{\theta}_1 s\theta_2 s\eta_2 - \dot{\eta}_2 a_2 c\theta_2) \end{Bmatrix} \tag{44}$$

using equations (3) & (44) ${}^4\hat{V}^4_{34} = {}^4_3\hat{L}\,{}^3\hat{V}^3_{34}$

$$= \begin{Bmatrix} c\theta_3 c\eta_3(\dot{\eta}_2 s\theta_2 c\eta_2 - (\dot{\theta}_1+\dot{\theta}_2)s\eta_2) + s\theta_3 c\eta_3(\dot{\eta}_2 c\theta_2 + \dot{\eta}_3) - s\eta_3((\dot{\theta}_1+\dot{\theta}_2)c\eta_2 + \\ \dot{\eta}_2 s\theta_2 s\eta_2 + \dot{\theta}_3) + \varepsilon[c\theta_3 c\eta_3(\dot{\theta}_1 a_1 s\theta_2 c\eta_2 - \dot{S}_1 s\eta_2) + s\theta_3 c\eta_3(\dot{\theta}_1 a_1 c\theta_2 + \dot{\eta}_2 a_2 s\theta_2 s\eta_2 + \\ (\dot{\theta}_1+\dot{\theta}_2)a_2 c\eta_2) - s\eta_3(\dot{S}_1 c\eta_2 + a_1\dot{\theta}_1 s\theta_2 s\eta_2 - \dot{\eta}_2 a_2 c\theta_2)] \\ \text{-----------------------} \\ c\theta_3 s\eta_3(\dot{\eta}_2 s\theta_2 c\eta_2 - (\dot{\theta}_1+\dot{\theta}_2)s\eta_2) + s\theta_3 s\eta_3(\dot{\eta}_2 c\theta_2 + \dot{\eta}_3) + c\eta_3((\dot{\theta}_1+\dot{\theta}_2)c\eta_2 + \\ \dot{\eta}_2 s\theta_2 s\eta_2 + \dot{\theta}_3) + \varepsilon[c\theta_3 s\eta_3(\dot{\theta}_1 a_1 s\theta_2 c\eta_2 - \dot{S}_1 s\eta_2) + s\theta_3 s\eta_3(\dot{\theta}_1 a_1 c\theta_2 + \dot{\eta}_2 a_2 s\theta_2 s\eta_2 + \\ (\dot{\theta}_1+\dot{\theta}_2)a_2 c\eta_2) + c\eta_3(\dot{S}_1 c\eta_2 + a_1\dot{\theta}_1 s\eta_2 s\theta_2 - \dot{\eta}_2 a_2 c\theta_2)] \\ \text{-----------------------} \\ s\theta_3(\dot{\eta}_2 s\theta_2 c\eta_2 - (\dot{\theta}_1+\dot{\theta}_2)s\eta_2) - c\theta_3(\dot{\eta}_2 c\theta_2 + \dot{\eta}_3) + \varepsilon[s\theta_3(\dot{\theta}_1 a_1 s\theta_2 c\eta_2 - \dot{S}_1 s\eta_2) \\ - c\theta_3(\dot{\theta}_1 a_1 c\theta_2 + \dot{\eta}_2 a_2 s\theta_2 s\eta_2 + (\dot{\theta}_1+\dot{\theta}_2)a_2 c\eta_2)] \end{Bmatrix} \tag{45}$$

2.8 Dynamic Force and Torque Analysis:

STEP (1) Formulation of inertia binors:

Assuming that the links are symmetrical about their respective lengths, the location of centers of mass of link *1* in terms of frame {*2*}, of link *2* in terms of frame {*3*} and of link *3* in terms of {*4*} can be represented by the vector forms as follows:

$$^2\vec{G}_1 = \begin{Bmatrix} g_1 \\ 0 \\ 0 \end{Bmatrix} \qquad ^3\vec{G}_2 = \begin{Bmatrix} g_2 \\ 0 \\ 0 \end{Bmatrix} \qquad ^4\vec{G}_3 = \begin{Bmatrix} g_3 \\ 0 \\ 0 \end{Bmatrix}$$

where symbols g_1, g_2 and g_3 are respectively the distances along the lengths of links *1*, *2* and *3* from their distal joints to their centers of mass. If symbols m_1, m_2 and m_3 represent their respective masses, then the inertia matrices of link *1* in terms of frame {*2*}, of link *2* in terms of frame {*3*}, of link *3* in terms of frame {*4*} can be written as

$$\left[{}^2S_1\right] = m_1\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -g_1 \\ 0 & g_1 & 0 \end{bmatrix} \quad \left[{}^3S_2\right] = m_2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -g_2 \\ 0 & g_2 & 0 \end{bmatrix} \quad \left[{}^4S_3\right] = m_3\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -g_3 \\ 0 & g_3 & 0 \end{bmatrix}$$

Assuming that the principal axes of each link are oriented the same as the frame fixed upon it, the mass moments of inertia for each link can be written as

$$[{}^{2}J_{1}] = m_1\begin{bmatrix} K_{1x}^2 & 0 & 0 \\ 0 & K_{1y}^2 & 0 \\ 0 & 0 & K_{1z}^2 \end{bmatrix} \quad [{}^{3}J_{2}] = m_2\begin{bmatrix} K_{2x}^2 & 0 & 0 \\ 0 & K_{2y}^2 & 0 \\ 0 & 0 & K_{2z}^2 \end{bmatrix} \quad [{}^{4}J_{3}] = m_3\begin{bmatrix} K_{3x}^2 & 0 & 0 \\ 0 & K_{3y}^2 & 0 \\ 0 & 0 & K_{3z}^2 \end{bmatrix}$$

where symbols K_{ax}, K_{ay} and K_{az} represent the radii of gyration of link a in terms of frame $\{a+1\}$ fixed on that link (a = 1, 2 and 3).

The general form of the inertia binor of link a in terms of frame {B} is written as

$$[{}^{B}\varphi_{a}] = \left[\begin{array}{c|c} [{}^{B}S_{a}]^{T} & m_{a}[I] \\ \hline [{}^{B}J_{a}]^{T} & [{}^{B}S_{a}] \end{array}\right]$$

This is a 6×6 matrix where symbol m_a is the mass of link a, $[{}^{B}J_a]$ is the mass moment of inertia as described above and $[I]$ is identity matrix,

$$[I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and matrix $[{}^{B}S_a]$ is the first moment of mass matrix of link 'a' in terms of frame {B}, which can be generalized as

$$[{}^{B}S_{a}] = m_{a}\begin{bmatrix} 0 & -g_z & g_y \\ g_z & 0 & -g_x \\ -g_y & g_x & 0 \end{bmatrix}$$

Therefore the inertia binors of link *1* in terms of frame {*2*}, of link *2* in terms of frame {*3*}, of link *3* in terms of frame {*4*} can be formulated as follows:

$$[{}^{2}\varphi_{1}] = m_1 \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & g_1 & 0 & 1 & 0 \\ 0 & -g_1 & 0 & 0 & 0 & 1 \\ \hline K_{1x}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{1y}^2 & 0 & 0 & 0 & -g_1 \\ 0 & 0 & K_{1z}^2 & 0 & g_1 & 0 \end{array}\right] \tag{46}$$

$$[{}^{3}\varphi_{2}] = m_2 \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & g_2 & 0 & 1 & 0 \\ 0 & -g_2 & 0 & 0 & 0 & 1 \\ \hline K_{2x}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{2y}^2 & 0 & 0 & 0 & -g_2 \\ 0 & 0 & K_{2z}^2 & 0 & g_2 & 0 \end{array}\right] \tag{47}$$

$$[{}^{4}\varphi_{3}] = m_3 \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & g_3 & 0 & 1 & 0 \\ 0 & -g_3 & 0 & 0 & 0 & 1 \\ \hline K_{3x}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{3y}^2 & 0 & 0 & 0 & -g_3 \\ 0 & 0 & K_{3z}^2 & 0 & g_3 & 0 \end{array}\right] \tag{48}$$

STEP (2) Formulation of dual momentum vectors:

In this step, some of the velocity vector equations are used to formulate the momenta of the moving links. In order to do that we can either use the velocity equations formulated through the forward path only or some velocity equation obtained through the backward path and some through the forward path. In either case, the same numerical results are obtained, but mixed path equations set is much simpler than only the forward path equations set.

First the forward path equations are used to formulate the momenta. Equations (33), (35) and (45) can respectively be written in the form of 6×1 column matrices by separating primary and dual components as shown below (equations (33), (35) and (45) are obtained from the forward path only):

$$ {}^{2}\tilde{V}^{2}_{14} = \begin{Bmatrix} V_{1xp} \\ V_{1yp} \\ V_{1zp} \\ V_{1xd} \\ V_{1yd} \\ V_{1zd} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \\ 0 \\ a_1\dot{\theta}_1 \\ \dot{S}_1 \end{Bmatrix} \tag{33a} $$

$$ {}^{3}\tilde{V}^{3}_{24} = \begin{Bmatrix} V_{2xp} \\ V_{2yp} \\ V_{2zp} \\ V_{2xd} \\ V_{2yd} \\ V_{2zd} \end{Bmatrix} = \begin{Bmatrix} \dot{\eta}_2 s\theta_2 c\eta_2 - (\dot{\theta}_1+\dot{\theta}_2)s\eta_2 \\ \dot{\eta}_2 c\theta_2 \\ (\dot{\theta}_1+\dot{\theta}_2)c\eta_2 + \dot{\eta}_2 s\theta_2 s\eta_2 \\ \dot{\theta}_1 a_1 s\theta_2 c\eta_2 - \dot{S}_1 s\eta_2 \\ \dot{\theta}_1 a_1 c\theta_2 + \dot{\eta}_2 a_2 s\theta_2 s\eta_2 + (\dot{\theta}_1+\dot{\theta}_2)a_2 c\eta_2 \\ \dot{S}_1 c\eta_2 + a_1\dot{\theta}_1 s\theta_2 s\eta_2 - \dot{\eta}_2 a_2 c\theta_2 \end{Bmatrix} \tag{35a} $$

$$
{}^{4}\widetilde{V}_{34}^{4} = \begin{Bmatrix} V_{3xp} \\ V_{3yp} \\ V_{3zp} \\ V_{3xd} \\ V_{3yd} \\ V_{3zd} \end{Bmatrix} = \begin{Bmatrix} c\theta_3 c\eta_3[\dot{\eta}_2 s\theta_2 c\eta_2 - (\dot{\theta}_1+\dot{\theta}_2)s\eta_2] + s\theta_3 c\eta_3(\dot{\eta}_2 c\theta_2 + \dot{\eta}_3) \\ -s\eta_3[(\dot{\theta}_1+\dot{\theta}_2)c\eta_2 + \dot{\eta}_2 s\theta_2 s\eta_2 + \dot{\theta}_3] \\ ----------- \\ c\theta_3 s\eta_3[\dot{\eta}_2 s\theta_2 c\eta_2 - (\dot{\theta}_1+\dot{\theta}_2)s\eta_2] + s\theta_3 s\eta_3(\dot{\eta}_2 c\theta_2 + \dot{\eta}_3) \\ +c\eta_3[(\dot{\theta}_1+\dot{\theta}_2)c\eta_2 + \dot{\eta}_2 s\theta_2 s\eta_2 + \dot{\theta}_3] \\ ----------- \\ s\theta_3[\dot{\eta}_2 s\theta_2 c\eta_2 - (\dot{\theta}_1+\dot{\theta}_2)s\eta_2] - c\theta_3(\dot{\eta}_2 c\theta_2 + \dot{\eta}_3) \\ ----------- \\ c\theta_3 c\eta_3(\dot{\theta}_1 a_1 s\theta_2 c\eta_2 - \dot{S}_1 s\eta_2) + s\theta_3 c\eta_3[a_1\dot{\theta}_1 c\theta_2 + a_2\dot{\eta}_2 s\theta_2 s\eta_2 \\ +(\dot{\theta}_1+\dot{\theta}_2)a_2 c\eta_2] - s\eta_3(\dot{S}_1 c\eta_2 + a_1\dot{\theta}_1 s\theta_2 s\eta_2 - a_2\dot{\eta}_2 c\theta_2) \\ ----------- \\ c\theta_3 s\eta_3(a_1\dot{\theta}_1 s\theta_2 c\eta_2 - \dot{S}_1 s\eta_2) + s\theta_3 s\eta_3[a_1\dot{\theta}_1 c\theta_2 + a_2\dot{\eta}_2 s\theta_2 s\eta_2 \\ +(\dot{\theta}_1+\dot{\theta}_2)a_2 c\eta_2] + c\eta_3(\dot{S}_1 c\eta_2 + a_1\dot{\theta}_1 s\theta_2 s\eta_2 - a_2\dot{\eta}_2 c\theta_2) \\ ----------- \\ s\theta_3(a_1\dot{\theta}_1 s\theta_2 c\eta_2 - \dot{S}_1 s\eta_2) - c\theta_3[a_1\dot{\theta}_1 c\theta_2 + a_2\dot{\eta}_2 s\theta_2 s\eta_2 \\ +(\dot{\theta}_1+\dot{\theta}_2)a_2 c\eta_2] \end{Bmatrix} \tag{45a}
$$

Premultiplying equation (33a) by equations (46), (35a) by (47) and (45a) by (48) the following column matrices are obtained

$$
{}^{2}\widetilde{H}_1 = \left[{}^{2}\varphi_1\right] {}^{2}\widetilde{V}_{14}^{2} = m_1 \begin{Bmatrix} 0 \\ g_1\dot{\theta}_1 + a_1\dot{\theta}_1 \\ \dot{S}_1 \\ 0 \\ -g_1\dot{S}_1 \\ K_{1z}^2\dot{\theta}_1 + g_1 a_1\dot{\theta}_1 \end{Bmatrix} \tag{49}
$$

$$
{}^{3}\widetilde{H}_2 = \left[{}^{3}\varphi_2\right] {}^{3}\widetilde{V}_{24}^{3}
$$

$$
= m_2\left\{\begin{array}{c}
\dot\theta_1 a_1 s\theta_2 c\eta_2 - \dot S_1 s\eta_2 \\
\hdashline
g_2(\dot\theta_1+\dot\theta_2)c\eta_2 + g_2\dot\eta_2 s\theta_2 s\eta_2 + \dot\theta_1 a_1 c\theta_2 + \dot\eta_2 a_2 s\theta_2 s\eta_2 + (\dot\theta_1+\dot\theta_2)a_2 c\eta_2 \\
\hdashline
-g_2\dot\eta_2 c\theta_2 + \dot S_1 c\eta_2 + a_1\dot\theta_1 s\theta_2 s\eta_2 - \dot\eta_2 a_2 c\theta_2 \\
\hdashline
K_{2x}^2\dot\eta_2 s\theta_2 c\eta_2 - K_{2x}^2(\dot\theta_1+\dot\theta_2)s\eta_2 \\
\hdashline
K_{2y}^2\dot\eta_2 c\theta_2 - g_2\dot S_1 c\eta_2 - g_2 a_1\dot\theta_1 s\theta_2 s\eta_2 + g_2\dot\eta_2 a_2 c\theta_2 \\
\hdashline
K_{2z}^2(\dot\theta_1+\dot\theta_2)c\eta_2 + K_{2z}^2\dot\eta_2 s\theta_2 s\eta_2 + g_2\dot\theta_1 a_1 c\theta_2 + g_2\dot\eta_2 a_2 s\theta_2 s\eta_2 + g_2(\dot\theta_1+\dot\theta_2)a_2 c\eta_2
\end{array}\right\} \qquad (50)
$$

$$
{}^4\tilde H_3 = \left[{}^4\varphi_3\right] {}^4\tilde V_{34}^4
$$

$$
= m_3\left\{\begin{array}{c}
c\theta_3 c\eta_3(a_1\dot\theta_1 s\theta_2 c\eta_2 - \dot S_1 s\eta_2) + s\theta_3 c\eta_3[a_1\dot\theta_1 c\theta_2 + a_2\dot\eta_2 s\theta_2 s\eta_2 \\
+(\dot\theta_1+\dot\theta_2)a_2 c\eta_2] - s\eta_3(\dot S_1 c\eta_2 + a_1\dot\theta_1 s\theta_2 s\eta_2 - a_2\dot\eta_2 c\theta_2) \\
\hdashline
g_3[s\theta_3[\dot\eta_2 s\theta_2 c\eta_2 - (\dot\theta_1+\dot\theta_2)s\eta_2] - c\theta_3(\dot\eta_2 c\theta_2 + \dot\eta_3)] + c\theta_3 s\eta_3 \\
(a_1\dot\theta_1 s\theta_2 c\eta_2 - \dot S_1 s\eta_2) + s\theta_3 s\eta_3[a_1\dot\theta_1 c\theta_2 + a_2\dot\eta_2 s\theta_2 s\eta_2 + \\
(\dot\theta_1+\dot\theta_2)a_2 c\eta_2] + c\eta_3(\dot S_1 c\eta_2 + a_1\dot\theta_1 s\theta_2 s\eta_2 - a_2\dot\eta_2 c\theta_2) \\
\hdashline
-g_3[c\theta_3 s\eta_3[\dot\eta_2 s\theta_2 c\eta_2 - (\dot\theta_1+\dot\theta_2)s\eta_2] + s\theta_3 s\eta_3(\dot\eta_2 c\theta_2 + \dot\eta_3) + \\
c\eta_3[(\dot\theta_1+\dot\theta_2)c\eta_2 + \dot\eta_2 s\theta_2 s\eta_2 + \dot\theta_3]] + s\theta_3(a_1\dot\theta_1 s\theta_2 c\eta_2 - \dot S_1 s\eta_2) \\
-c\theta_3[a_1\dot\theta_1 c\theta_2 + a_2\dot\eta_2 s\theta_2 s\eta_2 + (\dot\theta_1+\dot\theta_2)a_2 c\eta_2] \\
\hdashline
K_{3x}^2[c\theta_3 c\eta_3[\dot\eta_2 s\theta_2 c\eta_2 - (\dot\theta_1+\dot\theta_2)s\eta_2] + s\theta_3 c\eta_3(\dot\eta_2 c\theta_2 + \dot\eta_3) \\
-s\eta_3[(\dot\theta_1+\dot\theta_2)c\eta_2 + \dot\eta_2 s\theta_2 s\eta_2 + \dot\theta_3]] \\
\hdashline
K_{3y}^2[c\theta_3 s\eta_3[\dot\eta_2 s\theta_2 c\eta_2 - (\dot\theta_1+\dot\theta_2)s\eta_2] + s\theta_3 s\eta_3(\dot\eta_2 c\theta_2 + \dot\eta_3) + \\
c\eta_3[(\dot\theta_1+\dot\theta_2)c\eta_2 + \dot\eta_2 s\theta_2 s\eta_2 + \dot\theta_3]] - g_3[s\theta_3(a_1\dot\theta_1 s\theta_2 c\eta_2 - \dot S_1 s\eta_2) \\
-c\theta_3[a_1\dot\theta_1 c\theta_2 + a_2\dot\eta_2 s\theta_2 s\eta_2 + (\dot\theta_1+\dot\theta_2)a_2 c\eta_2]] \\
\hdashline
K_{3z}^2[s\theta_3[\dot\eta_2 s\theta_2 c\eta_2 - (\dot\theta_1+\dot\theta_2)s\eta_2] - c\theta_3(\dot\eta_2 c\theta_2 + \dot\eta_3)] + \\
g_3[c\theta_3 s\eta_3(a_1\dot\theta_1 s\theta_2 c\eta_2 - \dot S_1 s\eta_2) + s\theta_3 s\eta_3[a_1\dot\theta_1 c\theta_2 + a_2\dot\eta_2 s\theta_2 s\eta_2 + \\
(\dot\theta_1+\dot\theta_2)a_2 c\eta_2] + c\eta_3(\dot S_1 c\eta_2 + a_1\dot\theta_1 s\theta_2 s\eta_2 - a_2\dot\eta_2 c\theta_2)]
\end{array}\right\} \qquad (51)
$$

Formulations for the dual momentum of link 1, i.e. the crank in terms of frame $\{2\}$, of link 2, i.e. the connecting rod, in terms frame $\{3\}$ and of link 3, i.e. the slider, in terms of frame $\{4\}$, can respectively be obtained by modifying equations (49), (50) and (51) to 3×1 dual-number matrix equations as shown below

$$
{}^{2}\hat{H}_1 = \begin{Bmatrix} \hat{H}_{1x} \\ \hat{H}_{1y} \\ \hat{H}_{1z} \end{Bmatrix} = \begin{Bmatrix} H_{1xp} + \varepsilon H_{1xd} \\ H_{1yp} + \varepsilon H_{1yd} \\ H_{1zp} + \varepsilon H_{1zd} \end{Bmatrix} = m_1 \begin{Bmatrix} 0 \\ \dot{\theta}_1(a_1+g_1) - \varepsilon g_1 \dot{S}_1 \\ \dot{S}_1 + \varepsilon(K_{1z}^2 \dot{\theta}_1 + g_1 a_1 \dot{\theta}_1) \end{Bmatrix} \tag{49a}
$$

$$
{}^{3}\hat{H}_2 = \begin{Bmatrix} \hat{H}_{2x} \\ \hat{H}_{2y} \\ \hat{H}_{2z} \end{Bmatrix} = \begin{Bmatrix} H_{2xp} + \varepsilon H_{2xd} \\ H_{2yp} + \varepsilon H_{2yd} \\ H_{2zp} + \varepsilon H_{2zd} \end{Bmatrix}
$$

$$
= m_2 \begin{Bmatrix} a_1\dot{\theta}_1 s\theta_2 c\eta_2 - \dot{S}_1 s\eta_2 + \varepsilon[K_{2x}^2 \dot{\eta}_2 s\theta_2 c\eta_2 - K_{2x}^2(\dot{\theta}_1+\dot{\theta}_2) s\eta_2] \\ ------------ \\ [(g_2 c\eta_2 + a_2 c\eta_2)(\dot{\theta}_1+\dot{\theta}_2) + (g_2+a_2)(\dot{\eta}_2 s\theta_2 s\eta_2) + \dot{\theta}_1 a_1 c\theta_2] \\ +\varepsilon[(K_{2y}^2 + g_2 a_2)\dot{\eta}_2 c\theta_2 - g_2 \dot{S}_1 c\eta_2 - g_2 a_1 \dot{\theta}_1 s\theta_2 s\eta_2] \\ ------------ \\ [\dot{S}_1 c\eta_2 + a_1\dot{\theta}_1 s\theta_2 s\eta_2 - (a_2+g_2)\dot{\eta}_2 c\theta_2] + \varepsilon[(K_{2z}^2 c\eta_2 + g_2 a_2 c\eta_2) \\ (\dot{\theta}_1+\dot{\theta}_2) + (K_{2z}^2 + g_2 a_2)\dot{\eta}_2 s\theta_2 s\eta_2 + a_1 g_2 \dot{\theta}_1 c\theta_2] \end{Bmatrix} \tag{50a}
$$

$$
{}^{4}\hat{H}_3 = \begin{Bmatrix} \hat{H}_{3x} \\ \hat{H}_{3y} \\ \hat{H}_{3z} \end{Bmatrix} = \begin{Bmatrix} H_{3xp} + \varepsilon H_{3xd} \\ H_{3yp} + \varepsilon H_{3yd} \\ H_{3zp} + \varepsilon H_{3zd} \end{Bmatrix}
$$

$$
= m_3 \left\{
\begin{array}{c}
c\theta_3 c\eta_3(a_1\dot{\theta}_1 s\theta_2 c\eta_2 - \dot{S}_1 s\eta_2) + s\theta_3 c\eta_3[a_1\dot{\theta}_1 c\theta_2 + a_2\dot{\eta}_2 s\theta_2 s\eta_2 + (\dot{\theta}_1+\dot{\theta}_2) \\
a_2 c\eta_2] - s\eta_3(\dot{S}_1 c\eta_2 + a_1\dot{\theta}_1 s\theta_2 s\eta_2 - a_2\dot{\eta}_2 c\theta_2) + \varepsilon[K_{3x}^2[c\theta_3 c\eta_3[\dot{\eta}_2 s\theta_2 c\eta_2 - \\
(\dot{\theta}_1+\dot{\theta}_2)s\eta_2] + s\theta_3 c\eta_3(\dot{\eta}_2 c\theta_2 + \dot{\eta}_3) - s\eta_3[(\dot{\theta}_1+\dot{\theta}_2)c\eta_2 + \dot{\eta}_2 s\theta_2 s\eta_2 + \dot{\theta}_3]]] \\
----------- \\
g_3[s\theta_3[\dot{\eta}_2 s\theta_2 c\eta_2 - (\dot{\theta}_1+\dot{\theta}_2)s\eta_2] - c\theta_3(\dot{\eta}_2 c\theta_2 + \dot{\eta}_3)] + [c\theta_3 s\eta_3(a_1\dot{\theta}_1 s\theta_2 c\eta_2 \\
- \dot{S}_1 s\eta_2) + s\theta_3 s\eta_3[a_1\dot{\theta}_1 c\theta_2 + a_2\dot{\eta}_2 s\theta_2 s\eta_2 + (\dot{\theta}_1+\dot{\theta}_2)a_2 c\eta_2] + c\theta_3(\dot{S}_1 c\eta_2 \\
+ a_1\dot{\theta}_1 s\theta_2 s\eta_2 - a_2\dot{\eta}_2 c\theta_2)] + \varepsilon[K_{3y}^2[c\theta_3 s\eta_3[\dot{\eta}_2 s\theta_2 c\eta_2 - (\dot{\theta}_1+\dot{\theta}_2)s\eta_2] \\
+ s\theta_3 s\eta_3(\dot{\eta}_2 c\theta_2 + \dot{\eta}_3) + c\eta_3[(\dot{\theta}_1+\dot{\theta}_2)c\eta_2 + \dot{\eta}_2 s\theta_2 s\eta_2 + \dot{\theta}_3]] - g_3[s\theta_3 \\
(a_1\dot{\theta}_1 s\theta_2 c\eta_2 - \dot{S}_1 s\eta_2) - c\theta_3[a_1\dot{\theta}_1 c\theta_2 + a_2\dot{\eta}_2 s\theta_2 s\eta_2 + (\dot{\theta}_1+\dot{\theta}_2)a_2 c\eta_2]]] \\
----------- \\
-g_3[c\theta_3 s\eta_3[\dot{\eta}_2 s\theta_2 c\eta_2 - (\dot{\theta}_1+\dot{\theta}_2)s\eta_2] + s\theta_3 s\eta_3(\dot{\eta}_2 c\theta_2 + \dot{\eta}_3) + c\eta_3 \\
[(\dot{\theta}_1+\dot{\theta}_2)c\eta_2 + \dot{\eta}_2 s\theta_2 s\eta_2 + \dot{\theta}_3]] + [s\theta_3(a_1\dot{\theta}_1 s\theta_2 c\eta_2 - \dot{S}_1 s\eta_2) - c\theta_3[a_1\dot{\theta}_1 c\theta_2 \\
+ a_2\dot{\eta}_2 s\theta_2 s\eta_2 + (\dot{\theta}_1+\dot{\theta}_2)a_2 c\eta_2]] + \varepsilon[K_{3z}^2[s\theta_3[\dot{\eta}_2 s\theta_2 c\eta_2 - (\dot{\theta}_1+\dot{\theta}_2)s\eta_2] \\
- c\theta_3(\dot{\eta}_2 c\theta_2 + \dot{\eta}_3)] + g_3[c\theta_3 s\eta_3(a_1\dot{\theta}_1 s\theta_2 c\eta_2 - \dot{S}_1 s\eta_2) + s\theta_3 s\eta_3[a_1\dot{\theta}_1 c\theta_2 + \\
a_2\dot{\eta}_2 s\theta_2 s\eta_2 + (\dot{\theta}_1+\dot{\theta}_2)a_2 c\eta_2] + c\eta_3(\dot{S}_1 c\eta_2 + a_1\dot{\theta}_1 s\theta_2 s\eta_2 - a_2\dot{\eta}_2 c\theta_2)]]
\end{array}
\right\} \qquad (51a)
$$

Dual momentum equations (49a), (50a) and (51a) are purely forward path equations, since all the velocity equations (33a, 35a and 45a) used to derive these equations are forward path velocity equations. Out of these three momentum equations, two equations (50a) and (51a) are very large consisting of many sine and cosine functions and their derivatives, which may cause inaccurate results. In order to have more concise equations and better accuracy, two backward path velocity equations are used to derive two momentum equations as the alternative to equations (50a) and (51a).

Now, the backward path velocity equations (36) and (32a) can be written in the form of 6×1 column matrices by the same method as before, and going through exactly the same procedure, simpler and concise equations are obtained which can be used as replacements for the above-mentioned large equations.

$$
{}^3\tilde{V}_{24}^3 = \begin{Bmatrix} V_{2xp} \\ V_{2yp} \\ V_{2zp} \\ V_{2xd} \\ V_{2yd} \\ V_{2zd} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\dot{\eta}_3 \\ -\dot{\theta}_3 \\ -\dot{S}_4 s\theta_3 \\ \dot{S}_4 c\theta_3 \\ 0 \end{Bmatrix} \tag{36a}
$$

$$
{}^4\tilde{V}_{34}^4 = \begin{Bmatrix} V_{3xp} \\ V_{3yp} \\ V_{3zp} \\ V_{3xd} \\ V_{3yd} \\ V_{3zd} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\dot{S}_4 \end{Bmatrix} \tag{32b}
$$

Premultiplying equation (36a) by equation (47) and (32b) by (48) the following column matrices are obtained.

$$
{}^3\tilde{H}_2 = \left[{}^3\varphi_2\right] {}^3\tilde{V}_{24}^3 = m_2 \begin{Bmatrix} -\dot{S}_4 s\theta_3 \\ -g_2\,\dot{\theta}_3 + \dot{S}_4 c\theta_3 \\ g_2\,\dot{\eta}_3 \\ 0 \\ -K_{2y}^2\,\dot{\eta}_3 \\ -K_{2z}^2\,\dot{\theta}_3 + g_2\dot{S}_4 c\theta_3 \end{Bmatrix} \tag{52}
$$

$$
{}^4\tilde{H}_3 = \left[{}^4\varphi_3\right] {}^4\tilde{V}_{34}^4 = m_3 \begin{Bmatrix} 0 \\ 0 \\ -\dot{S}_4 \\ 0 \\ g_3\dot{S}_4 \\ 0 \end{Bmatrix} \tag{53}
$$

Now, alternative expressions for (50a) and (51a) using the backward path are respectively

$$
{}^3\hat{H}_2 = \begin{Bmatrix} \hat{H}_{2x} \\ \hat{H}_{2y} \\ \hat{H}_{2z} \end{Bmatrix} = \begin{Bmatrix} H_{2xp} + \varepsilon H_{2xd} \\ H_{2yp} + \varepsilon H_{2yd} \\ H_{2zp} + \varepsilon H_{2zd} \end{Bmatrix} = m_2 \begin{Bmatrix} -\dot{S}_4 s\theta_3 \\ (-g_2\dot{\theta}_3 + \dot{S}_4 c\theta_3) - \varepsilon K_{2y}^2 \dot{\eta}_3 \\ g_3\dot{\eta}_3 + \varepsilon(g_2\dot{S}_4 c\theta_3 - K_{2z}^2\dot{\theta}_3) \end{Bmatrix} \tag{52a}
$$

$$
{}^4\hat{H}_3 = \begin{Bmatrix} \hat{H}_{3x} \\ \hat{H}_{3y} \\ \hat{H}_{3z} \end{Bmatrix} = \begin{Bmatrix} H_{3xp} + \varepsilon H_{3xd} \\ H_{3yp} + \varepsilon H_{3yd} \\ H_{3zp} + \varepsilon H_{3zd} \end{Bmatrix} = m_3 \begin{Bmatrix} 0 \\ \varepsilon g_3 \dot{S}_4 \\ -\dot{S}_4 \end{Bmatrix} \tag{53a}
$$

In the next steps and in the computer program either equations set { (33a), (35a), (45a), (49a), (50a) and (51a) } or set { (33a), (36a), (32b), (49a), (52a) and (53a) } can be used. The first set consists of purely forward path equations and the second set consists of mixed path equations; equations (33a) and (49a) are forward path equations and rest of the equations in the second set are backward path equations. Both the sets were tested by the program for a number of cases and resulted in same outcome. Finally the second set is used in the program to have better accuracy.

STEP (3) Time derivatives of dual momentum Equations:

Differentiating forward path equations (49a), (50a) and (51a) with respect to time the following equations are obtained respectively:

$$
{}^2\dot{\hat{H}}_1 = \begin{Bmatrix} \dot{\hat{H}}_{1x} \\ \dot{\hat{H}}_{1y} \\ \dot{\hat{H}}_{1z} \end{Bmatrix} = \begin{Bmatrix} \dot{H}_{1xp} + \varepsilon \dot{H}_{1xd} \\ \dot{H}_{1yp} + \varepsilon \dot{H}_{1yd} \\ \dot{H}_{1zp} + \varepsilon \dot{H}_{1zd} \end{Bmatrix} = m_1 \begin{Bmatrix} 0 \\ \ddot{\theta}_1(a_1+g_1) - \varepsilon g_1 \ddot{S}_1 \\ \ddot{S}_1 + \varepsilon(K_{1z}^2 \ddot{\theta}_1 + g_1 a_1 \ddot{\theta}_1) \end{Bmatrix} \tag{54}
$$

$$
{}^3\dot{\hat{H}}_2 = \begin{Bmatrix} \dot{\hat{H}}_{2x} \\ \dot{\hat{H}}_{2y} \\ \dot{\hat{H}}_{2z} \end{Bmatrix} = \begin{Bmatrix} \dot{H}_{2xp} + \varepsilon \dot{H}_{2xd} \\ \dot{H}_{2yp} + \varepsilon \dot{H}_{2yd} \\ \dot{H}_{2zp} + \varepsilon \dot{H}_{2zd} \end{Bmatrix}
$$

$$
= m_2 \left\{ \begin{array}{c}
a_1\ddot{\theta}_1 s\theta_2 c\eta_2 + a_1\dot{\theta}_1\dot{\theta}_2 c\theta_2 c\eta_2 - a_1\dot{\theta}_1\dot{\eta}_2 s\theta_2 s\eta_2 - \ddot{S}_1 s\eta_2 - \dot{S}_1\dot{\eta}_2 c\eta_2 \\
+\varepsilon[K_{2x}^2(\ddot{\eta}_2 s\theta_2 c\eta_2 + \dot{\eta}_2\dot{\theta}_2 c\theta_2 c\eta_2 - \dot{\eta}_2^2 s\theta_2 s\eta_2) - K_{2x}^2(\ddot{\theta}_1 + \ddot{\theta}_2) s\eta_2 \\
-K_{2x}^2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\eta}_2 c\eta_2] \\
------------ \\
[(-g_2\dot{\eta}_2 s\eta_2 - a_2\dot{\eta}_2 s\eta_2)(\dot{\theta}_1 + \dot{\theta}_2) + (g_2 c\eta_2 + a_2 c\eta_2)(\ddot{\theta}_1 + \ddot{\theta}_2) + (g_2 + a_2) \\
(\ddot{\eta}_2 s\theta_2 s\eta_2 + \dot{\eta}_2\dot{\theta}_2 c\theta_2 s\eta_2 + \dot{\eta}_2^2 s\theta_2 c\eta_2) + a_1\ddot{\theta}_1 c\theta_2 - a_1\dot{\theta}_1\dot{\theta}_2 s\theta_2] + \\
\varepsilon[(K_{2y}^2 + g_2 a_2)(\ddot{\eta}_2 c\theta_2 - \dot{\eta}_2\dot{\theta}_2 s\theta_2) - g_2(\ddot{S}_1 c\eta_2 - \dot{S}_1\dot{\eta}_2 s\eta_2) - \\
g_2 a_1(\ddot{\theta}_1 s\theta_2 s\eta_2 + \dot{\theta}_1\dot{\theta}_2 c\theta_2 s\eta_2 + \dot{\theta}_1\dot{\eta}_2 s\theta_2 c\eta_2)] \\
------------ \\
[\ddot{S}_1 c\eta_2 - \dot{S}_1\dot{\eta}_2 s\eta_2 + a_1(\ddot{\theta}_1 s\theta_2 s\eta_2 + \dot{\theta}_1\dot{\theta}_2 c\theta_2 s\eta_2 + \dot{\theta}_1\dot{\eta}_2 s\theta_2 c\eta_2) \\
-(a_2 + g_2)(\ddot{\eta}_2 c\theta_2 - \dot{\eta}_2\dot{\theta}_2 s\theta_2)] + \varepsilon[(-K_{2z}^2\dot{\eta}_2 s\eta_2 - g_2 a_2\dot{\eta}_2 s\eta_2)(\dot{\theta}_1 + \dot{\theta}_2) + \\
(K_{2z}^2 c\eta_2 + g_2 a_2 c\eta_2)(\ddot{\theta}_1 + \ddot{\theta}_2) + (K_{2z}^2 + g_2 a_2)(\ddot{\eta}_2 s\theta_2 s\eta_2 + \dot{\eta}_2\dot{\theta}_2 c\theta_2 s\eta_2 \\
+\dot{\eta}_2^2 s\theta_2 c\eta_2) + a_1 g_2(\ddot{\theta}_1 c\theta_2 - \dot{\theta}_1\dot{\theta}_2 s\theta_2)]
\end{array} \right\} \tag{55}
$$

$$
{}^4\dot{\hat{H}}_3 = \left\{ \begin{array}{c} \dot{\hat{H}}_{3x} \\ \dot{\hat{H}}_{3y} \\ \dot{\hat{H}}_{3z} \end{array} \right\} = \left\{ \begin{array}{c} \dot{H}_{3xp} + \varepsilon\dot{H}_{3xd} \\ \dot{H}_{3yp} + \varepsilon\dot{H}_{3yd} \\ \dot{H}_{3zp} + \varepsilon\dot{H}_{3zd} \end{array} \right\} \tag{56}
$$

where

$$
\begin{aligned}
\dot{H}_{3xp} = m_3[&(-\dot{\theta}_3 s\theta_3 c\eta_3 - \dot{\eta}_3 c\theta_3 s\eta_3)(a_1\dot{\theta}_1 s\theta_2 c\eta_2 - \dot{S}_1 s\eta_2) + c\theta_3 c\eta_3(a_1\ddot{\theta}_1 s\theta_2 c\eta_2 + \\
&a_1\dot{\theta}_1\dot{\theta}_2 c\theta_2 c\eta_2 - a_1\dot{\theta}_1\dot{\eta}_2 s\theta_2 s\eta_2 - \ddot{S}_1 s\eta_2 - \dot{S}_1\dot{\eta}_2 c\eta_2) + (\dot{\theta}_3 c\theta_3 c\eta_3 - \dot{\eta}_3 s\theta_3 s\eta_3) \\
&[a_1\dot{\theta}_1 c\theta_2 + a_2\dot{\eta}_2 s\theta_2 s\eta_2 + (\dot{\theta}_1 + \dot{\theta}_2) a_2 c\eta_2] + s\theta_3 c\eta_3[a_1\ddot{\theta}_1 c\theta_2 - a_1\dot{\theta}_1\dot{\theta}_2 s\theta_2 \\
&+ a_2\ddot{\eta}_2 s\theta_2 s\eta_2 + a_2\dot{\eta}_2\dot{\theta}_2 c\theta_2 s\eta_2 + a_2\dot{\eta}_2^2 s\theta_2 c\eta_2 - (\dot{\theta}_1 + \dot{\theta}_2) a_2\dot{\eta}_2 s\eta_2 + (\ddot{\theta}_1 + \ddot{\theta}_2) a_2 c\eta_2] \\
&-\dot{\eta}_3 c\eta_3(\dot{S}_1 c\eta_2 + a_1\dot{\theta}_1 s\theta_2 s\eta_2 - a_2\dot{\eta}_2 c\theta_2) - s\eta_3(\ddot{S}_1 c\eta_2 - \dot{S}_1\dot{\eta}_2 s\eta_2 \\
&+ a_1\ddot{\theta}_1 s\theta_2 s\eta_2 + a_1\dot{\theta}_1\dot{\theta}_2 c\theta_2 s\eta_2 + a_1\dot{\theta}_1\dot{\eta}_2 s\theta_2 c\eta_2 - a_2\ddot{\eta}_2 c\theta_2 + a_2\dot{\eta}_2\dot{\theta}_2 s\theta_2)]
\end{aligned} \tag{56a}
$$

$$
\begin{aligned}
\dot{H}_{3xd} = m_3[K_{3x}^2[&(-\dot{\theta}_3 s\theta_3 c\eta_3 - \dot{\eta}_3 c\theta_3 s\eta_3)[\dot{\eta}_2 s\theta_2 c\eta_2 - (\dot{\theta}_1 + \dot{\theta}_2) s\eta_2] + c\theta_3 c\eta_3[\ddot{\eta}_2 s\theta_2 c\eta_2 \\
&+ \dot{\eta}_2\dot{\theta}_2 c\theta_2 c\eta_2 - \dot{\eta}_2^2 s\theta_2 s\eta_2 - (\dot{\theta}_1 + \dot{\theta}_2)\dot{\eta}_2 c\eta_2 - (\ddot{\theta}_1 + \ddot{\theta}_2) s\eta_2] + (\dot{\theta}_3 c\theta_3 c\eta_3 - \\
&\dot{\eta}_3 s\theta_3 s\eta_3)(\dot{\eta}_2 c\theta_2 + \dot{\eta}_3) + s\theta_3 c\eta_3(\ddot{\eta}_2 c\theta_2 - \dot{\eta}_2\dot{\theta}_2 s\theta_2 + \ddot{\eta}_3) \\
&-\dot{\eta}_3 c\eta_3[(\dot{\theta}_1 + \dot{\theta}_2) c\eta_2 + \dot{\eta}_2 s\theta_2 s\eta_2 + \dot{\theta}_3] - s\eta_3[(\ddot{\theta}_1 + \ddot{\theta}_2) c\eta_2 - \\
&(\dot{\theta}_1 + \dot{\theta}_2)\dot{\eta}_2 s\eta_2 + \ddot{\eta}_2 s\theta_2 s\eta_2 + \dot{\eta}_2\dot{\theta}_2 c\theta_2 s\eta_2 + \dot{\eta}_2^2 s\theta_2 c\eta_2 + \ddot{\theta}_3]]]
\end{aligned} \tag{56b}
$$

$$\begin{aligned}
\dot{H}_{3yp} = m_3[&g_3[\dot\theta_3 c\theta_3[\dot\eta_2 s\theta_2 c\eta_2-(\dot\theta_1+\dot\theta_2)s\eta_2]+s\theta_3[\ddot\eta_2 s\theta_2 c\eta_2+\dot\eta_2\dot\theta_2 c\theta_2 c\eta_2 \\
&-\dot\eta_2^2 s\theta_2 s\eta_2-(\ddot\theta_1+\ddot\theta_2)s\eta_2-\dot\eta_2 c\eta_2(\dot\theta_1+\dot\theta_2)]+\dot\theta_3 s\theta_3(\dot\eta_2 c\theta_2+\dot\eta_3)-c\theta_3 \\
&(\ddot\eta_2 c\theta_2-\dot\eta_2\dot\theta_2 s\theta_2+\ddot\eta_3)]+[(\dot\eta_3 c\eta_3 c\theta_3-\dot\theta_3 s\theta_3 s\eta_3)(a_1\dot\theta_1 s\theta_2 c\eta_2-\dot S_1 s\eta_2) \\
&+c\theta_3 s\eta_3(a_1\ddot\theta_1 s\theta_2 c\eta_2+a_1\dot\theta_1\dot\theta_2 c\theta_2 c\eta_2-a_1\dot\theta_1\dot\eta_2 s\theta_2 s\eta_2-\ddot S_1 s\eta_2-\dot S_1\dot\eta_2 c\eta_2) \\
&+(\dot\theta_3 c\theta_3 s\eta_3+\dot\eta_3 s\theta_3 c\eta_3)[a_1\dot\theta_1 c\theta_2+a_2\dot\eta_2 s\theta_2 s\eta_2+(\dot\theta_1+\dot\theta_2)a_2 c\eta_2]+s\theta_3 s\eta_3 \\
&[a_1\ddot\theta_1 c\theta_2-a_1\dot\theta_1\dot\theta_2 s\theta_2+a_2\ddot\eta_2 s\theta_2 s\eta_2+a_2\dot\eta_2\dot\theta_2 c\theta_2 s\eta_2+a_2\dot\eta_2^2 s\theta_2 c\eta_2+ \\
&(\ddot\theta_1+\ddot\theta_2)a_2 c\eta_2-(\dot\theta_1+\dot\theta_2)a_2\dot\eta_2 s\eta_2]-\dot\eta_3 s\eta_3(\dot S_1 c\eta_2+a_1\dot\theta_1 s\theta_2 s\eta_2-a_2\dot\eta_2 c\theta_2)+ \\
&c\eta_3(\ddot S_1 c\eta_2-\dot S_1\dot\eta_2 s\eta_2+a_1\dot\theta_1\dot\theta_2 c\theta_2 s\eta_2+a_1\dot\theta_1\dot\eta_2 s\theta_2 c\eta_2+a_1\ddot\theta_1 s\theta_2 s\eta_2 \\
&-a_2\ddot\eta_2 c\theta_2+a_2\dot\theta_2\dot\eta_2 s\theta_2)]]
\end{aligned} \tag{56c}$$

$$\begin{aligned}
\dot{H}_{3yd} = m_3[&K_{3y}^2[(\dot\eta_3 c\theta_3 c\eta_3-\dot\theta_3 s\theta_3 s\eta_3) \\
&[\dot\eta_2 s\theta_2 c\eta_2-(\dot\theta_1+\dot\theta_2)s\eta_2]+c\theta_3 s\eta_3[\ddot\eta_2 s\theta_2 c\eta_2+\dot\eta_2\dot\theta_2 c\theta_2 c\eta_2-\dot\eta_2^2 s\theta_2 s\eta_2 \\
&-(\ddot\theta_1+\ddot\theta_2)s\eta_2-(\dot\theta_1+\dot\theta_2)\dot\eta_2 c\eta_2]+(\dot\theta_3 c\theta_3 s\eta_3+\dot\eta_3 s\theta_3 c\eta_3)(\dot\eta_2 c\theta_2+\dot\eta_3)+ \\
&s\theta_3 s\eta_3(\ddot\eta_3+\ddot\eta_2 c\theta_2-\dot\eta_2\dot\theta_2 s\theta_2)+c\eta_3[(\ddot\theta_1+\ddot\theta_2)c\eta_2-(\dot\theta_1+\dot\theta_2)\dot\eta_2 s\eta_2 \\
&+\ddot\eta_2 s\theta_2 s\eta_2+\dot\eta_2\dot\theta_2 c\theta_2 s\eta_2+\dot\eta_2^2 s\theta_2 c\eta_2+\ddot\theta_3]-\dot\eta_3 s\eta_3[(\dot\theta_1+\dot\theta_2)c\eta_2+\dot\eta_2 s\theta_2 s\eta_2+\dot\theta_3]] \\
&-g_3[s\theta_3[a_1\dot\theta_1\dot\theta_2 c\theta_2 c\eta_2+a_1\ddot\theta_1 s\theta_2 c\eta_2-a_1\dot\theta_1\dot\eta_2 s\theta_2 s\eta_2-\ddot S_1 s\eta_2-\dot S_1\dot\eta_2 c\eta_2] \\
&+\dot\theta_3 c\theta_3(a_1\dot\theta_1 s\theta_2 c\eta_2-\dot S_1 s\eta_2)+\dot\theta_3 s\theta_3[a_1\dot\theta_1 c\theta_2+a_2\dot\eta_2 s\theta_2 s\eta_2+(\dot\theta_1+\dot\theta_2)a_2 c\eta_2] \\
&-c\theta_3[a_1\ddot\theta_1 c\theta_2-a_1\dot\theta_1\dot\theta_2 s\theta_2+a_2\ddot\eta_2 s\theta_2 s\eta_2+a_2\dot\eta_2\dot\theta_2 c\theta_2 s\eta_2+ \\
&a_2\dot\eta_2^2 s\theta_2 c\eta_2+(\ddot\theta_1+\ddot\theta_2)a_2 c\eta_2-(\dot\theta_1+\dot\theta_2)a_2\dot\eta_2 s\eta_2]]]
\end{aligned} \tag{56d}$$

$$\begin{aligned}
\dot{H}_{3zp} = m_3[&-g_3[(\dot\eta_3 c\theta_3 c\eta_3-\dot\theta_3 s\theta_3 s\eta_3)[\dot\eta_2 s\theta_2 c\eta_2-(\dot\theta_1+\dot\theta_2)s\eta_2]+c\theta_3 s\eta_3[\ddot\eta_2 s\theta_2 c\eta_2+ \\
&\dot\eta_2\dot\theta_2 c\theta_2 c\eta_2-\dot\eta_2^2 s\theta_2 s\eta_2-(\dot\theta_1+\dot\theta_2)\dot\eta_2 c\eta_2-(\ddot\theta_1+\ddot\theta_2)s\eta_2]+(\dot\theta_3 c\theta_3 s\eta_3+\dot\eta_3 s\theta_3 c\eta_3) \\
&(\dot\eta_2 c\theta_2+\dot\eta_3)+s\theta_3 s\eta_3(\ddot\eta_2 c\theta_2-\dot\eta_2\dot\theta_2 s\theta_2+\ddot\eta_3)+c\eta_3[(\ddot\theta_1+\ddot\theta_2)c\eta_2-(\dot\theta_1+\dot\theta_2)\dot\eta_2 s\eta_2 \\
&+\ddot\eta_2 s\theta_2 s\eta_2+\dot\eta_2\dot\theta_2 c\theta_2 s\eta_2+\dot\eta_2^2 s\theta_2 c\eta_2+\ddot\theta_3]-\dot\eta_3 s\eta_3[(\dot\theta_1+\dot\theta_2)c\eta_2+\dot\eta_2 s\theta_2 s\eta_2+\dot\theta_3]] \\
&+[\dot\theta_3 c\theta_3(a_1\dot\theta_1 s\theta_2 c\eta_2-\dot S_1 s\eta_2)+s\theta_3(a_1\ddot\theta_1 s\theta_2 c\eta_2+a_1\dot\theta_1\dot\theta_2 c\theta_2 c\eta_2 \\
&-a_1\dot\theta_1\dot\eta_2 s\theta_2 s\eta_2-\ddot S_1 s\eta_2-\dot S_1\dot\eta_2 c\eta_2)+\dot\theta_3 s\theta_3[a_1\dot\theta_1 c\theta_2+a_2\dot\eta_2 s\theta_2 s\eta_2+ \\
&(\dot\theta_1+\dot\theta_2)a_2 c\eta_2]-c\theta_3[a_1\ddot\theta_1 c\theta_2-a_1\dot\theta_1\dot\theta_2 s\theta_2+a_2\ddot\eta_2 s\theta_2 s\eta_2+a_2\dot\eta_2\dot\theta_2 c\theta_2 s\eta_2+a_2\dot\eta_2^2 s\theta_2 c\eta_2+ \\
&(\ddot\theta_1+\ddot\theta_2)a_2 c\eta_2-(\dot\theta_1+\dot\theta_2)a_2\dot\eta_2 s\eta_2]]]
\end{aligned} \tag{56e}$$

$$\dot{H}_{3zd} = m_3[K_{3z}^2[\dot{\theta}_3 c\theta_3[\dot{\eta}_2 s\theta_2 c\eta_2 - (\dot{\theta}_1 + \dot{\theta}_2)s\eta_2] +$$
$$s\theta_3[\ddot{\eta}_2 s\theta_2 c\eta_2 + \dot{\eta}_2\dot{\theta}_2 c\theta_2 c\eta_2 - \dot{\eta}_2^2 s\theta_2 s\eta_2 - (\ddot{\theta}_1 + \ddot{\theta}_2)s\eta_2 -$$
$$(\dot{\theta}_1 + \dot{\theta}_2)\dot{\eta}_2 c\eta_2] + \dot{\theta}_3 s\theta_3(\dot{\eta}_2 c\theta_2 + \dot{\eta}_3) - c\theta_3(\ddot{\eta}_2 c\theta_2 - \dot{\eta}_2\dot{\theta}_2 s\theta_2 + \ddot{\eta}_3)] + g_3[(\dot{\eta}_3 c\theta_3 c\eta_3 -$$
$$\dot{\theta}_3 s\theta_3 s\eta_3)(a_1\dot{\theta}_1 s\theta_2 c\eta_2 - \dot{S}_1 s\eta_2) + c\theta_3 s\eta_3(a_1\ddot{\theta}_1 s\theta_2 c\eta_2 + a_1\dot{\theta}_1\dot{\theta}_2 c\theta_2 c\eta_2 -$$
$$a_1\dot{\theta}_1\dot{\eta}_2 s\theta_2 s\eta_2 - \ddot{S}_1 s\eta_2 - \dot{S}_1\dot{\eta}_2 c\eta_2) + (\dot{\theta}_3 c\theta_3 s\eta_3 + \dot{\eta}_3 s\theta_3 c\eta_3)[a_1\dot{\theta}_1 c\theta_2 + a_2\dot{\eta}_2 s\theta_2 s\eta_2 +$$
$$(\dot{\theta}_1 + \dot{\theta}_2)a_2 c\eta_2] + s\theta_3 s\eta_3[a_1\ddot{\theta}_1 c\theta_2 - a_1\dot{\theta}_1\dot{\theta}_2 s\theta_2 + a_2\ddot{\eta}_2 s\theta_2 s\eta_2 +$$
$$a_2\dot{\eta}_2^2 s\theta_2 c\eta_2 + a_2\dot{\eta}_2\dot{\theta}_2 c\theta_2 s\eta_2 + (\ddot{\theta}_1 + \ddot{\theta}_2)a_2 c\eta_2 - (\dot{\theta}_1 + \dot{\theta}_2)a_2\dot{\eta}_2 s\eta_2] -$$
$$\dot{\eta}_3 s\eta_3(\dot{S}_1 c\eta_2 + a_1\dot{\theta}_1 s\theta_2 s\eta_2 - a_2\dot{\eta}_2 c\theta_2) + c\eta_3(\ddot{S}_1 c\eta_2 - \dot{S}_1\dot{\eta}_2 s\eta_2 + a_1\ddot{\theta}_1 s\theta_2 s\eta_2$$
$$+ a_1\dot{\theta}_1\dot{\theta}_2 c\theta_2 s\eta_2 + a_1\dot{\theta}_1\dot{\eta}_2 s\theta_2 c\eta_2 - a_2\ddot{\eta}_2 c\theta_2 + a_2\dot{\eta}_2\dot{\theta}_2 s\theta_2)]] \tag{56f}$$

Now, backward path equations (52a) and (53a) are differentiated with respect to time. The following equations are replacements for equations (55) and (56) respectively.

$$^3\dot{\hat{H}}_2 = \begin{Bmatrix} \dot{\hat{H}}_{2x} \\ \dot{\hat{H}}_{2y} \\ \dot{\hat{H}}_{2z} \end{Bmatrix} = \begin{Bmatrix} \dot{H}_{2xp} + \varepsilon\dot{H}_{2xd} \\ \dot{H}_{2yp} + \varepsilon\dot{H}_{2yd} \\ \dot{H}_{2zp} + \varepsilon\dot{H}_{2zd} \end{Bmatrix}$$

$$= m_2 \begin{Bmatrix} -\ddot{S}_4 s\theta_3 - \dot{S}_4 c\theta_3\dot{\theta}_3 \\ (\ddot{S}_4 c\theta_3 - \dot{S}_4 s\theta_3\dot{\theta}_3 - g_2\ddot{\theta}_3) - \varepsilon K_{2y}^2\ddot{\eta}_3 \\ g_2\ddot{\eta}_3 + \varepsilon(g_2\ddot{S}_4 c\theta_3 - g_2\dot{S}_4 s\theta_3\dot{\theta}_3 - K_{2z}^2\ddot{\theta}_3) \end{Bmatrix} \tag{57}$$

$$^4\dot{\hat{H}}_3 = \begin{Bmatrix} \dot{\hat{H}}_{3x} \\ \dot{\hat{H}}_{3y} \\ \dot{\hat{H}}_{3z} \end{Bmatrix} = \begin{Bmatrix} \dot{H}_{3xp} + \varepsilon\dot{H}_{3xd} \\ \dot{H}_{3yp} + \varepsilon\dot{H}_{3yd} \\ \dot{H}_{3zp} + \varepsilon\dot{H}_{3zd} \end{Bmatrix} = m_3 \begin{Bmatrix} 0 \\ \varepsilon g_3\ddot{S}_4 \\ -\ddot{S}_4 \end{Bmatrix} \tag{58}$$

In the program either forward path equations set { (54), (55) and (56) } or mixed path equations set { (54), (57) and (58) } can be used. To have better results the second set is used.

STEP (4) Formulation of the Dual-Euler equations:

In this step, equations for the inertia forces of link 1 (i.e. the crank) relative to frame {2}, for link 2 (i.e. the connecting rod) relative to frame {3} and for link 3 (i.e. the slider) relative to frame {4} are derived. The derivations are shown as follows:

$$ {}^{2}\hat{f}_1 = \begin{Bmatrix} \dot{\hat{H}}_{1x} - \hat{V}_{1z}\hat{H}_{1y} + \hat{V}_{1y}\hat{H}_{1z} \\ \dot{\hat{H}}_{1y} - \hat{V}_{1x}\hat{H}_{1z} + \hat{V}_{1z}\hat{H}_{1x} \\ \dot{\hat{H}}_{1z} - \hat{V}_{1y}\hat{H}_{1x} + \hat{V}_{1x}\hat{H}_{1y} \end{Bmatrix} $$

Expanding both sides,

$$ \begin{Bmatrix} f_{1xp} + \varepsilon f_{1xd} \\ f_{1yp} + \varepsilon f_{1yd} \\ f_{1zp} + \varepsilon f_{1zd} \end{Bmatrix} = \begin{Bmatrix} (\dot{H}_{1xp} - V_{1zp}H_{1yp} + V_{1yp}H_{1zp}) + \varepsilon(\dot{H}_{1xd} - V_{1zp}H_{1yd} - V_{1zd}H_{1yp} + \\ V_{1yp}H_{1zd} + V_{1yd}H_{1zp}) \\ \\ (\dot{H}_{1yp} - V_{1xp}H_{1zp} + V_{1zp}H_{1xp}) + \varepsilon(\dot{H}_{1yd} - V_{1xp}H_{1zd} - V_{1xd}H_{1zp} + \\ V_{1zp}H_{1xd} + V_{1zd}H_{1xp}) \\ \\ (\dot{H}_{1zp} - V_{1yp}H_{1xp} + V_{1xp}H_{1yp}) + \varepsilon(\dot{H}_{1zd} - V_{1yp}H_{1xd} - V_{1yd}H_{1xp} + \\ V_{1xp}H_{1yd} + V_{1xd}H_{1yp}) \end{Bmatrix} \tag{59} $$

$$ {}^{3}\hat{f}_2 = \begin{Bmatrix} \dot{\hat{H}}_{2x} - \hat{V}_{2z}\hat{H}_{2y} + \hat{V}_{2y}\hat{H}_{2z} \\ \dot{\hat{H}}_{2y} - \hat{V}_{2x}\hat{H}_{2z} + \hat{V}_{2z}\hat{H}_{2x} \\ \dot{\hat{H}}_{2z} - \hat{V}_{2y}\hat{H}_{2x} + \hat{V}_{2x}\hat{H}_{2y} \end{Bmatrix} $$

Expanding both sides,

$$ \begin{Bmatrix} f_{2xp} + \varepsilon f_{2xd} \\ f_{2yp} + \varepsilon f_{2yd} \\ f_{2zp} + \varepsilon f_{2zd} \end{Bmatrix} = \begin{Bmatrix} (\dot{H}_{2xp} - V_{2zp}H_{2yp} + V_{2yp}H_{2zp}) + \varepsilon(\dot{H}_{2xd} - V_{2zp}H_{2yd} - V_{2zd}H_{2yp} + \\ V_{2yp}H_{2zd} + V_{2yd}H_{2zp}) \\ \\ (\dot{H}_{2yp} - V_{2xp}H_{2zp} + V_{2zp}H_{2xp}) + \varepsilon(\dot{H}_{2yd} - V_{2xp}H_{2zd} - V_{2xd}H_{2zp} + \\ V_{2zp}H_{2xd} + V_{2zd}H_{2xp}) \\ \\ (\dot{H}_{2zp} - V_{2yp}H_{2xp} + V_{2xp}H_{2yp}) + \varepsilon(\dot{H}_{2zd} - V_{2yp}H_{2xd} - V_{2yd}H_{2xp} + \\ V_{2xp}H_{2yd} + V_{2xd}H_{2yp}) \end{Bmatrix} \tag{60} $$

$$
{}^{4}\hat{f}_3 = \begin{Bmatrix} \dot{\hat{H}}_{3x} - \hat{V}_{3z}\hat{H}_{3y} + \hat{V}_{3y}\hat{H}_{3z} \\ \dot{\hat{H}}_{3y} - \hat{V}_{3x}\hat{H}_{3z} + \hat{V}_{3z}\hat{H}_{3x} \\ \dot{\hat{H}}_{3z} - \hat{V}_{3y}\hat{H}_{3x} + \hat{V}_{3x}\hat{H}_{3y} \end{Bmatrix}
$$

Expanding both sides,

$$
\begin{Bmatrix} f_{3xp} + \varepsilon f_{3xd} \\ f_{3yp} + \varepsilon f_{3yd} \\ f_{3zp} + \varepsilon f_{3zd} \end{Bmatrix} = \begin{Bmatrix} (\dot{H}_{3xp} - V_{3zp}H_{3yp} + V_{3yp}H_{3zp}) + \varepsilon(\dot{H}_{3xd} - V_{3zp}H_{3yd} - V_{3zd}H_{3yp} + \\ V_{3yp}H_{3zd} + V_{3yd}H_{3zp}) \\ \\ (\dot{H}_{3yp} - V_{3xp}H_{3zp} + V_{3zp}H_{3xp}) + \varepsilon(\dot{H}_{3yd} - V_{3xp}H_{3zd} - V_{3xd}H_{3zp} + \\ V_{3zp}H_{3xd} + V_{3zd}H_{3xp}) \\ \\ (\dot{H}_{3zp} - V_{3yp}H_{3xp} + V_{3xp}H_{3yp}) + \varepsilon(\dot{H}_{3zd} - V_{3yp}H_{3xd} - V_{3yd}H_{3xp} + \\ V_{3xp}H_{3yd} + V_{3xd}H_{3yp}) \end{Bmatrix} \quad (61)
$$

STEP (5) Application of D'Alembert's principles:

The force equations for links 1, 2 and 3 are respectively as follows:

$$
{}^{2}_{1}\hat{M}\,{}^{1}\hat{F}_1 + {}^{2}\hat{F}_2 + {}^{2}\hat{f}_1 = 0 \quad (62)
$$

$$
{}^{3}_{2}\hat{L}\,{}^{2}\hat{F}_2 + {}^{3}\hat{F}_3 + {}^{3}\hat{f}_2 = 0 \quad (63)
$$

$$
{}^{4}_{3}\hat{L}\,{}^{3}\hat{F}_3 + {}^{4}\hat{F}_4 + {}^{4}\hat{f}_3 = 0 \quad (64)
$$

Dual forces acting on each joint can be written in vector form as follows:

$$
{}^{1}\hat{F}_1 = \begin{Bmatrix} {}^{1}F_{1i} + \varepsilon\,{}^{1}T_{1i} \\ {}^{1}F_{1j} + \varepsilon\,{}^{1}T_{1j} \\ {}^{1}F_{1k} + \varepsilon\,{}^{1}T_{1k} \end{Bmatrix} \quad (65)
$$

$$
{}^{2}\hat{F}_2 = \begin{Bmatrix} {}^{2}F_{2i} + \varepsilon\,{}^{2}T_{2i} \\ {}^{2}F_{2j} + \varepsilon\,{}^{2}T_{2j} \\ {}^{2}F_{2k} + \varepsilon\,{}^{2}T_{2k} \end{Bmatrix} \quad (66)
$$

where torques ${}^{2}T_{2i} = {}^{2}T_{2j} = {}^{2}T_{2k} = 0$, since an intermediate ball joint cannot support torque.

$$
{}^{3}\hat{F}_{3}=\begin{Bmatrix} {}^{3}F_{3i}+\varepsilon\,{}^{3}T_{3i} \\ {}^{3}F_{3j}+\varepsilon\,{}^{3}T_{3j} \\ {}^{3}F_{3k}+\varepsilon\,{}^{3}T_{3k} \end{Bmatrix} \tag{67}
$$

where torques ${}^{3}T_{3i}={}^{3}T_{3j}={}^{3}T_{3k}=0$, since an intermediate ball joint cannot support torque.

$$
{}^{4}\hat{F}_{4}=\begin{Bmatrix} {}^{4}F_{4i}+\varepsilon\,{}^{4}T_{4i} \\ {}^{4}F_{4j}+\varepsilon\,{}^{4}T_{4j} \\ {}^{4}F_{4k}+\varepsilon\,{}^{4}T_{4k} \end{Bmatrix} \tag{68}
$$

Expanding equations (62), (63), (64) and separating primary and dual parts, the following equations are obtained:

$$
{}^{1}F_{1i}c\theta_1+{}^{1}F_{1j}s\theta_1+{}^{2}F_{2i}+f_{1xp}=0 \tag{62a}
$$

$$
({}^{1}T_{1i}c\theta_1-{}^{1}F_{1i}S_1s\theta_1+{}^{1}F_{1j}S_1c\theta_1+{}^{1}T_{1j}s\theta_1+f_{1xd})=0 \tag{62b}
$$

$$
-{}^{1}F_{1i}s\theta_1+{}^{1}F_{1j}c\theta_1+{}^{2}F_{2j}+f_{1yp}=0 \tag{62c}
$$

$$
(-{}^{1}T_{1i}s\theta_1-{}^{1}F_{1i}S_1c\theta_1+{}^{1}T_{1j}c\theta_1-{}^{1}F_{1j}S_1s\theta_1+a_1{}^{1}F_{1k}+f_{1yd})=0 \tag{62d}
$$

$$
{}^{1}F_{1k}+{}^{2}F_{2k}+f_{1zp}=0 \tag{62e}
$$

$$
({}^{1}F_{1i}a_1s\theta_1-{}^{1}F_{1j}a_1c\theta_1+{}^{1}T_{1k}+f_{1zd})=0 \tag{62f}
$$

$$
{}^{2}F_{2i}c\theta_2c\eta_2+{}^{2}F_{2j}s\theta_2c\eta_2-{}^{2}F_{2k}s\eta_2+{}^{3}F_{3i}+f_{2xp}=0 \tag{63a}
$$

$$
f_{2xd}=0 \tag{63b}
$$

$$
-{}^{2}F_{2i}s\theta_2+{}^{2}F_{2j}c\theta_2+{}^{3}F_{3j}+f_{2yp}=0 \tag{63c}
$$

$$
({}^{2}F_{2i}a_2c\theta_2s\eta_2+{}^{2}F_{2j}a_2s\theta_2s\eta_2+{}^{2}F_{2k}a_2c\eta_2+f_{2yd})=0 \tag{63d}
$$

$$
{}^{2}F_{2i}c\theta_2s\eta_2+{}^{2}F_{2j}s\theta_2s\eta_2+{}^{2}F_{2k}c\eta_2+{}^{3}F_{3k}+f_{2zp}=0 \tag{63e}
$$

$$
({}^{2}F_{2i}a_2s\theta_2-{}^{2}F_{2j}a_2c\theta_2+f_{2zd})=0 \tag{63f}
$$

$$
{}^{3}F_{3i}c\theta_3c\eta_3+{}^{3}F_{3j}s\theta_3c\eta_3-{}^{3}F_{3k}s\eta_3+{}^{4}F_{4i}+f_{3xp}=0 \tag{64a}
$$

$$({}^4T_{4i}+f_{3xd}) = 0 \tag{64b}$$

$$ {}^3F_{3i}c\theta_3s\eta_3+{}^3F_{3j}s\theta_3s\eta_3+{}^3F_{3k}c\eta_3+{}^4F_{4j}+f_{3yp} = 0 \tag{64c}$$

$$({}^4T_{4j}+f_{3yd}) = 0 \tag{64d}$$

$$ {}^3F_{3i}s\theta_3-{}^3F_{3j}c\theta_3+{}^4F_{4k}+f_{3zp} = 0 \tag{64e}$$

$$({}^4T_{4k}+f_{3zd}) = 0 \tag{64f}$$

There are eighteen equations out of which one does not give any solution (equation (63b)), since it is of the form $0=0$; confirmation of this was done by evaluating it in the computer program. The known force and torque are respectively ${}^1F_{1k}$ and ${}^1T_{1k}$ which are the crank inputs. Axial force ${}^1F_{1k}$ acts on the crank at joint 1 and equals mass of the crank times its axial acceleration, and torque ${}^1T_{1k}$ is applied constantly. Force ${}^1F_{1k}$ can be written mathematically as either

$$ {}^1F_{1k} = 0 \qquad [\,when\ S_1 = constant\,]$$

or

$$ {}^1F_{1k} = m_1\ddot{S}_1$$

where symbol m_1 = mass of the crank and

$$\ddot{S}_1 = \alpha S_{S1}\,[\cos(\alpha\theta_1)\ddot{\theta}_1-\alpha\sin(\alpha\theta_1)\dot{\theta}_1^2\,] \tag{28b}$$

[equation (28b) is obtained by differentiating equation (28a) with respect to time when S_1 is sinusoidal]

STEP (6) Solution of the dynamic equations:

There are sixteen unknowns, including both forces and torques and seventeen equations. Although one equation is redundant, all seventeen equations have been used to avoid sine or cosine functions in the denominator as much as possible, because small sine or cosine terms in the denominator sometimes results in inaccurate division. The details of the solution procedure are as follows:

From equation (62e),

$$ {}^{2}F_{2k} = -({}^{1}F_{1k}+f_{1zp}) \tag{69a}$$

Multiplying equation (63c) by parameter a_2, adding equation (63f) and then keeping the known terms on the right hand side,

$$ {}^{3}F_{3j} = -(f_{2yp}+\frac{f_{2zd}}{a_2}) \tag{69b}$$

Multiplying equation (63e) by parameter a_2, subtracting that from equation (63d) and then keeping the known terms on the right hand side,

$$ {}^{3}F_{3k} = \frac{f_{2yd}}{a_2} - f_{2zp} \tag{69c}$$

From equation (64b),

$$ {}^{4}T_{4i} = -f_{3xd} \tag{69d}$$

From equation (64d),

$$ {}^{4}T_{4j} = -f_{3yd} \tag{69e}$$

From equation (64f),

$$ {}^{4}T_{4k} = -f_{3zd} \tag{69f}$$

Multiplying equation (62c) by parameter a_1, adding that to equation (62f) and then keeping the known terms on the right hand side,

$$ {}^{2}F_{2j} = -f_{1yp} - \frac{f_{1zd} + T_{1k}}{a_1} \tag{69g}$$

From equation (63f),

$$ {}^{2}F_{2i} = \frac{{}^{2}F_{2j}\, a_2\, c\theta_2 - f_{2zd}}{a_2 s\theta_2} \tag{69h}$$

The denominator of the right hand side of this equation involves the sine function. This, in fact, resulted in some abnormal outcomes when values of angle θ_2 were very close to zero, while running the program. So, the results obtained from this equation need to be modified. The modification is done by a numerical method as follows.

Multiplying equation (62f) by the term $s\theta_1$, multiplying equation (62a) by the term $a_1 c\theta_1$, adding both the results and then keeping the known terms on the right hand side,

$$^1F_{1i} = -(^2F_{2i}+f_{1xp})c\theta_1-(^1T_{1k}+f_{1zd})\frac{s\theta_1}{a_1} \tag{69i}$$

Multiplying equation (62f) by the term $c\theta_1$, multiplying (62a) by the term $a_1 s\theta_1$, subtracting one from another and then keeping the known terms on the right hand side,

$$^1F_{1j} = -(^2F_{2i}+f_{1xp})s\theta_1+(^1T_{1k}+f_{1zd})\frac{c\theta_1}{a_1} \tag{69j}$$

Both the equations (69i) and (69j) contain the previously-found value of force $^2F_{2i}$ from equation (69h). Moreover the value for $^2F_{2i}$ may not be correct because of the reason described before. If that value is incorrect, equation (62a) involving with the terms $^1F_{1i}$, $^1F_{1j}$ and $^2F_{2i}$ will not be satisfied by the values obtained from equations (69h), (69i) and (69j). The computer program is written in such a way that it checks whether equation (62a) is satisfied by those values or not. If that equation is satisfied, the program goes to the next equation for the next unknown. If not satisfied, then a better value of the force $^2F_{2i}$ is found from the following equation (69k), using previously found values of the terms $^1F_{1i}$ and $^1F_{1j}$ from the eqations (69i) and (69j) respectively. The program then goes to equations (69i) and (69j) respectively to improve the values of the terms $^1F_{1i}$ and $^1F_{1j}$, using the better value of the term $^2F_{2i}$ obtained from equation (69k). This modification operation is continued until equation (62a) is approximately satisfied (since it is a numerical technique, it is not expected to satisfy exactly). Equation (69k) is obtained from (62a).

$$^2F_{2i} = -^1F_{1i}c\theta_1-^1F_{1j}s\theta_1-f_{1xp} \tag{69k}$$

From equation (63a),

$$ {}^{3}F_{3i} = -{}^{2}F_{2i}c\theta_2 c\eta_2 - {}^{2}F_{2j}s\theta_2 c\eta_2 + {}^{2}F_{2k}s\eta_2 - f_{2xp} \tag{69l} $$

From equation (64e),

$$ {}^{4}F_{4k} = {}^{3}F_{3j}c\theta_3 - {}^{3}F_{3i}s\theta_3 - f_{3zp} \tag{69m} $$

From equation (64c),

$$ {}^{4}F_{4j} = -({}^{3}F_{3i}c\theta_3 s\eta_3 + {}^{3}F_{3j}s\theta_3 s\eta_3 + {}^{3}F_{3k}c\eta_3 + f_{3yp}) \tag{69n} $$

From equation (64a),

$$ {}^{4}F_{4i} = -({}^{3}F_{3i}c\theta_3 c\eta_3 + {}^{3}F_{3j}s\theta_3 c\eta_3 - {}^{3}F_{3k}s\eta_3 + f_{3xp}) \tag{69o} $$

Multiplying equation (62b) by the term $s\theta_1$, multiplying equation (62d) by the term $c\theta_1$, adding them together and then keeping the known terms on the right hand side,

$$ {}^{1}T_{1j} = {}^{1}F_{1i}S_1 - f_{1xd}s\theta_1 - ({}^{1}F_{1k}a_1 + f_{1yd})c\theta_1 \tag{69p} $$

Multiplying equation (62b) by the term $c\theta_1$, multiplying equation (62d) by the term $s\theta_1$, adding them together and then keeping the known terms on the right hand side,

$$ {}^{1}T_{1i} = -{}^{1}F_{1j}S_1 - f_{1xd}c\theta_1 + ({}^{1}F_{1k}a_1 + f_{1yd})s\theta_1 \tag{69q} $$

Equations (69a) through (69q) involve acceleration terms which are obtained by differentiating equations (38) through (42) with respect to time. Hence the five acceleration equations are as follows:

$$ \ddot{\theta}_2 = \frac{\begin{array}{c} a_2 s\theta_3[\ddot{S}_1 c\theta_3 s\eta_2 c\eta_2 - \dot{S}_1\dot{\theta}_3 s\theta_3 s\eta_2 c\eta_2 + \dot{S}_1\dot{\eta}_2 c\theta_3 c^2\eta_2 - \dot{S}_1\dot{\eta}_2 c\theta_3 s^2\eta_2 \\ -a_1(\ddot{\theta}_1 s\theta_2 c\theta_3 + \dot{\theta}_1\dot{\theta}_2 c\theta_2 c\theta_3 - \dot{\theta}_1\dot{\theta}_3 s\theta_2 s\theta_3 + \ddot{\theta}_1 c\theta_2 s\theta_3 c\eta_2 - \dot{\theta}_1\dot{\theta}_2 s\theta_2 s\theta_3 c\eta_2 \\ +\dot{\theta}_1\dot{\theta}_3 c\theta_2 c\theta_3 c\eta_2 - \dot{\theta}_1\dot{\eta}_2 c\theta_2 s\theta_3 s\eta_2) - a_2\ddot{\theta}_1 s\theta_3 - a_2\dot{\theta}_1\dot{\theta}_3 c\theta_3] - \\ a_2\dot{\theta}_3 c\theta_3[\dot{S}_1 c\theta_3 s\eta_2 c\eta_2 - a_1(\dot{\theta}_1 s\theta_2 c\theta_3 + \dot{\theta}_1 c\theta_2 s\theta_3 c\eta_2) - a_2\dot{\theta}_1 s\theta_3] \end{array}}{(a_2 s\theta_3)^2} \tag{38a} $$

$$\ddot{\theta}_3 = \frac{\begin{gathered} a_2 s\theta_3[a_1(\ddot{\theta}_1 c\theta_2 s\theta_3 - \dot{\theta}_1\dot{\theta}_2 s\theta_2 s\theta_3 + \dot{\theta}_1\dot{\theta}_3 c\theta_2 c\theta_3 + \\ \ddot{\theta}_1 s\theta_2 c\theta_3 c\eta_2 + \dot{\theta}_1\dot{\theta}_2 c\theta_2 c\theta_3 c\eta_2 - \dot{\theta}_1\dot{\theta}_3 s\theta_2 s\theta_3 c\eta_2 - \\ \dot{\theta}_1\dot{\eta}_2 s\theta_2 c\theta_3 s\eta_2) - \ddot{S}_1 c\theta_3 s\eta_2 + \dot{S}_1\dot{\theta}_3 s\theta_3 s\eta_2 - \dot{S}_1\dot{\eta}_2 c\theta_3 c\eta_2] - \\ a_2\dot{\theta}_3 c\theta_3[a_1(\dot{\theta}_1 c\theta_2 s\theta_3 + \dot{\theta}_1 s\theta_2 c\theta_3 c\eta_2) - \dot{S}_1 c\theta_3 s\eta_2] \end{gathered}}{(a_2 s\theta_3)^2} \tag{39a}$$

$$\ddot{\eta}_2 = \frac{\begin{gathered} a_2 s\theta_2 s\theta_3[-\dot{\eta}_2 c\eta_2[a_1(\dot{\theta}_1 c\theta_2 s\theta_3 + \dot{\theta}_1 s\theta_2 c\theta_3 c\eta_2) - \\ \dot{S}_1 s\eta_2 c\theta_3] - s\eta_2[a_1(\ddot{\theta}_1 c\theta_2 s\theta_3 - \dot{\theta}_1\dot{\theta}_2 s\theta_2 s\theta_3 + \dot{\theta}_1\dot{\theta}_3 c\theta_2 c\theta_3 + \ddot{\theta}_1 s\theta_2 c\theta_3 c\eta_2 \\ + \dot{\theta}_1\dot{\theta}_2 c\theta_2 c\theta_3 c\eta_2 - \dot{\theta}_1\dot{\theta}_3 s\theta_2 s\theta_3 c\eta_2 - \dot{\theta}_1\dot{\eta}_2 s\theta_2 c\theta_3 s\eta_2) \\ - \ddot{S}_1 s\eta_2 c\theta_3 - \dot{S}_1\dot{\eta}_2 c\eta_2 c\theta_3 + \dot{S}_1\dot{\theta}_3 s\eta_2 s\theta_3]] + a_2 s\eta_2[a_1(\dot{\theta}_1 c\theta_2 s\theta_3 + \dot{\theta}_1 s\theta_2 c\theta_3 c\eta_2) \\ - \dot{S}_1 s\eta_2 c\theta_3](\dot{\theta}_2 c\theta_2 s\theta_3 + \dot{\theta}_3 s\theta_2 c\theta_3) \end{gathered}}{(a_2 s\theta_2 s\theta_3)^2} \tag{40a}$$

$$\ddot{\eta}_3 = \frac{\begin{gathered} a_2 s\theta_2 s\theta_3[(-\dot{\theta}_2 s\theta_2 s\eta_2 + \dot{\eta}_2 c\theta_2 c\eta_2)[\dot{\theta}_1 a_1(s\theta_2 c\theta_3 c\eta_2 + c\theta_2 s\theta_3) \\ - \dot{S}_1 s\eta_2 c\theta_3] + c\theta_2 s\eta_2[\ddot{\theta}_1 a_1(s\theta_2 c\theta_3 c\eta_2 + c\theta_2 s\theta_3) + \dot{\theta}_1 a_1 \\ (\dot{\theta}_2 c\theta_2 c\theta_3 c\eta_2 - \dot{\theta}_3 s\theta_2 s\theta_3 c\eta_2 - \dot{\eta}_2 s\theta_2 c\theta_3 s\eta_2 + \dot{\theta}_3 c\theta_2 c\theta_3 - \dot{\theta}_2 s\theta_2 s\theta_3) \\ - \ddot{S}_1 s\eta_2 c\theta_3 - \dot{S}_1\dot{\eta}_2 c\eta_2 c\theta_3 + \dot{S}_1\dot{\theta}_3 s\eta_2 s\theta_3]] - a_2 c\theta_2 s\eta_2 \\ [\dot{\theta}_1 a_1(s\theta_2 c\theta_3 c\eta_2 + c\theta_2 s\theta_3) - \dot{S}_1 s\eta_2 c\theta_3](\dot{\theta}_2 c\theta_2 s\theta_3 + \dot{\theta}_3 s\theta_2 c\theta_3) \end{gathered}}{(a_2 s\theta_2 s\theta_3)^2} \tag{41a}$$

$$\ddot{S}_4 = \frac{\begin{gathered} s\theta_3[\ddot{S}_1 s\eta_2 + \dot{S}_1\dot{\eta}_2 c\eta_2 - a_1(\ddot{\theta}_1 s\theta_2 c\eta_2 + \dot{\theta}_1\dot{\theta}_2 c\theta_2 c\eta_2 - \dot{\theta}_1\dot{\eta}_2 s\theta_2 s\eta_2)] \\ - (\dot{S}_1 s\eta_2 - \dot{\theta}_1 a_1 s\theta_2 c\eta_2)\dot{\theta}_3 c\theta_3 \end{gathered}}{(s\theta_3)^2} \tag{42a}$$

ILLUSTRATIVE EXAMPLES, RESULTS AND CONCLUSIONS

3.1 Example

In order to run the program and check the feasibility of the method a number of physical examples are considered, one of which has been illustrated here. The known link-joint parameters, dimensions, other known parameters and some basic formulas used in the computer program are as follows:

Crank length $a_1 = 12$ inches,

Connecting rod length $a_2 = 50$ inches,

Offset $a_4 = 0, 10, 20, 30$ inches,

Offset $\alpha_4 = 232, 270, 310$ degrees,

Axial displacement of the crank $S_1 = 0, 3, 6, 9$ inches,

Rotation of crank $\theta_1 =$ from 0 to 360 degrees,

The cross-section of crank is a circle of radius $r_1 = 1.5$ inches,

The cross-section of connecting rod is a circle of radius $r_2 = 1$ inch,

The slider is a cube of each side $r_3 = 3$ inches,

Specific gravity of the material (mild steel is used for all the links) $d = 0.28$,

Acceleration due to gravity $g = 386.4$ inches per square second,

Crank RPM = 50,

Rotational acceleration of the crank $\ddot{\theta}_1$ is taken to be zero assuming that it is rotating at constant RPM,

Torque acting in the crank ${}^{1}T_{1k} = 100$ pound-inch,

Parameter $\alpha = 0$, since the axial displacement of the crank is considered to be constant. Parameter α will have some positive value if sinusoidal axial displacement of crank is considered.

Location of center of the crank $g_1 = \dfrac{a_1}{2}$, Location of center of mass of the connecting rod $g_2 = \frac{a_2}{2}$,

Location of center of mass of the slider $g_3 = 0$,

Mass of the crank $m_1 = \frac{\pi r_1^2 a_1 d}{g}$,

Mass of the connecting rod $m_2 = \frac{\pi r_2^2 a_2 d}{g}$,

Mass of the slider $m_3 = \frac{r_3^3 d}{g}$,

Radii of gyration are formulated as written below:

$$K_{1x} = \frac{r_1}{\sqrt{2}},$$

$$K_{1y} = K_{1z} = \left[\frac{r_1^2}{4} + \frac{a_1^2}{3}\right]^{\frac{1}{2}},$$

$$K_{2x} = \frac{r_2}{\sqrt{2}},$$

$$K_{2y} = K_{2z} = \left[\frac{r_2^2}{4} + \frac{a_2^2}{3}\right]^{\frac{1}{2}},$$

$$K_{3x} = K_{3y} = K_{3z} = \frac{r_3}{\sqrt{6}},$$

3.2 Discussions of results:

The results obtained as computer output are the slider displacements, dynamic forces and torques acting at each of the four joints for each position of the crank from 0 through 360 degrees. The program was first tested for the planar case and the results found to be same as that obtained by the complex algebra method, a widely used procedure for analyzing planar mechanisms. This verifies the feasibility of the method. Then the program was run for different spatial cases. All the results have been graphed against input crank angle as shown in figs. 5-38. In figs. 5, 8, 11, 14, 16, 18, 21, 24, 27, 30, 33, 36, slider displacements, different joint forces and torques are shown for variable offset distance a_4 (varying from 10" to 30"), constant crank axial offset S_1 (=3.5") and ideal offset angle α_4 (=270 degrees, i.e. the crank shaft is perpendicular to the cylinder). In figs. 6, 9, 12, 15, 17, 19, 22, 25, 28, 31, 34, 37, slider displacements, different joint forces and torques are shown for variable crank axial offset S_1 (varing from 3" to 9"), constant offset distance a_4 (=30") and ideal offset angle α_4 (=270 degree). In figs. 7, 10, 13, 20, 23, 26, 29, 32, 35, 38, slider displacements, different joint forces and no torques are shown for variable offset angle α_4 (varing from 232 degrees to 310 degrees), constant offset distance a_4 (=20") and no crank axial offset (ideal). In each of the graphs different spatial cases have been compared with the ideal case.

Although all the dynamic equations ((68a) through (68g) and (68i) through (68q)) are free from sine or cosine terms in the denominator, the expressions for acceleration (equation (38a) through (42a)) are not free from sine terms which become very very close to zero at some particular positions of the crank, i.e. near 100 degrees and 300 degrees. Using the computer program it has been observed that near those particular values for input crank angle, the joint angle θ_2 goes very close to zero degrees and thus the calculated values of the acceleration terms, at those particular positions, go abnormally high on account of roundoff errors in the computer calculations, which is clearly inaccurate. Therefore, in order to draw graphs, those particular values have been discarded, otherwise those points would have taken most of the space and the valid details of the graph would have been diminished.

All the torque components at joint *4*, i.e. ${}^4T_{4i}$, ${}^4T_{4j}$, and ${}^4T_{4k}$ and the force components along joint axis (k) at joint *2*, i.e. ${}^2F_{2k}$ are found to be zero throughout one complete revolution of the crank at any offset. Interestingly, by manual manipulation of the equation those components are also found to be zero. Another force component along the j-axis of joint *2* , i.e. ${}^2F_{2j}$ is constant (-8.33 lbs) throughout one complete rotation of the crank at any offset. By manual manipulation that force component is found to be negative of the ratio of input torque (100 lb-in) and the crank length (12 inches). Two torque components at joint *1*, i.e. ${}^{1}T_{1i}$ and ${}^{1}T_{1j}$ reduce to zero, when there is no crank axial offset S_1. By manual manipulation it is found that ${}^{1}T_{1i} = -\,{}^{1}F_{1j}S_1$ and ${}^{1}T_{1j} = {}^{1}F_{1i}S_1$.

For the particular link lengths as mentioned before, when offset a_4 is zero, the program fails to converge if the term S_1 is greater than 1.5 inches. When offset is increased to 10 inches or greater, the program fails to converge if the term S_1 is greater than 3.5 inches and when offset a_4 is 30 inches, the program converges as long as the term S_1 is less than or equal to 9 inches. From this observation it can be inferred that in order for the mechanism to work, with the increase of crank axial shift, offset a_4 should be increased. In the case of mechanism failure, it will either lose closure, i.e. one link or more will get detached from another at the joint or the slider will go to the opposite side. When offset a_4 is 20 inches and crank axial displacement S_1 is zero, the program converges as long as offset α_4 is between 232 and 310 degrees.

3.3 Conclusions

Slider displacement curves are almost semi-sinusoidal type of graphs which seem to be quite reasonable. When crank is placed at higher level than the plane on which slider is moving by providing positive offset distance a_4, the stroke of the engine is reduced. So in that case, while consuming less power the same RPM of the engine can be obtained. But with the increase of offset distance a_4, joint forces and torques also increase. So proper design is necessary for the safe operation of the engine, considering that factor. In most of the joint force and torque graphs (except for ${}^{3}F_{3j}$ with no spike, ${}^{3}F_{3k}$ with one spike and ${}^{4}F_{4k}$ with one spike) two spikes are seen, one near 100 degrees and other near 300 degrees. Hence it is very obvious that with the increase of offset the chance of fatique failure of the links at high speed is very large. In the presence of offset distance S_1, two extra torque components ${}^1T_{1i}$ and ${}^1T_{1j}$ are acting at joint *1*. So extra precaution should be taken by designer, so that engine does not fail, and the manufacturer should pay special attention to minimize this tolerance error.

Since the present work investigates dynamic forces and torques at each joint of a slider crank with all kinds of manufacturing tolerances, the results provide information needed for the sizing and dynamic balancing of a high speed slider crank in the design stage. This analysis can be used to determine the effect and significance of the inertia forces acting on each link due to its own mass and mass moment of inertia. It is expected that this investigation will enable designers to have better insight into the designing and testing of high speed spatial mechanisms and robot manipulators.

The main disadvantage of this method is that if we can not make good initial estimates for joint displacements, (which is normally done by using the direct formulas for kinematic displacements, derived either by using plane geometry rules as shown before in section 2.4.2 or by using the complex algebra method or by the dual-number method, considering that there is no offset at all, i.e. the planar case) the method will not converge. Usually the process converges in 3 to 6 iterations if estimates are within 20 degrees, which is quite possible by using the direct formula for the planar case. Another great disadvantage is that matrix A may become singular. Fortunately it never happened while running the program. Since this method is well adapted for digital computation, any complex system can be analyzed using this analytical tool. The main advantages of this method are its generality and that all calculations can be performed on digital computer. Other advantages are that this method is concise, flexible, economical and its analytical approach is independent of visualization.

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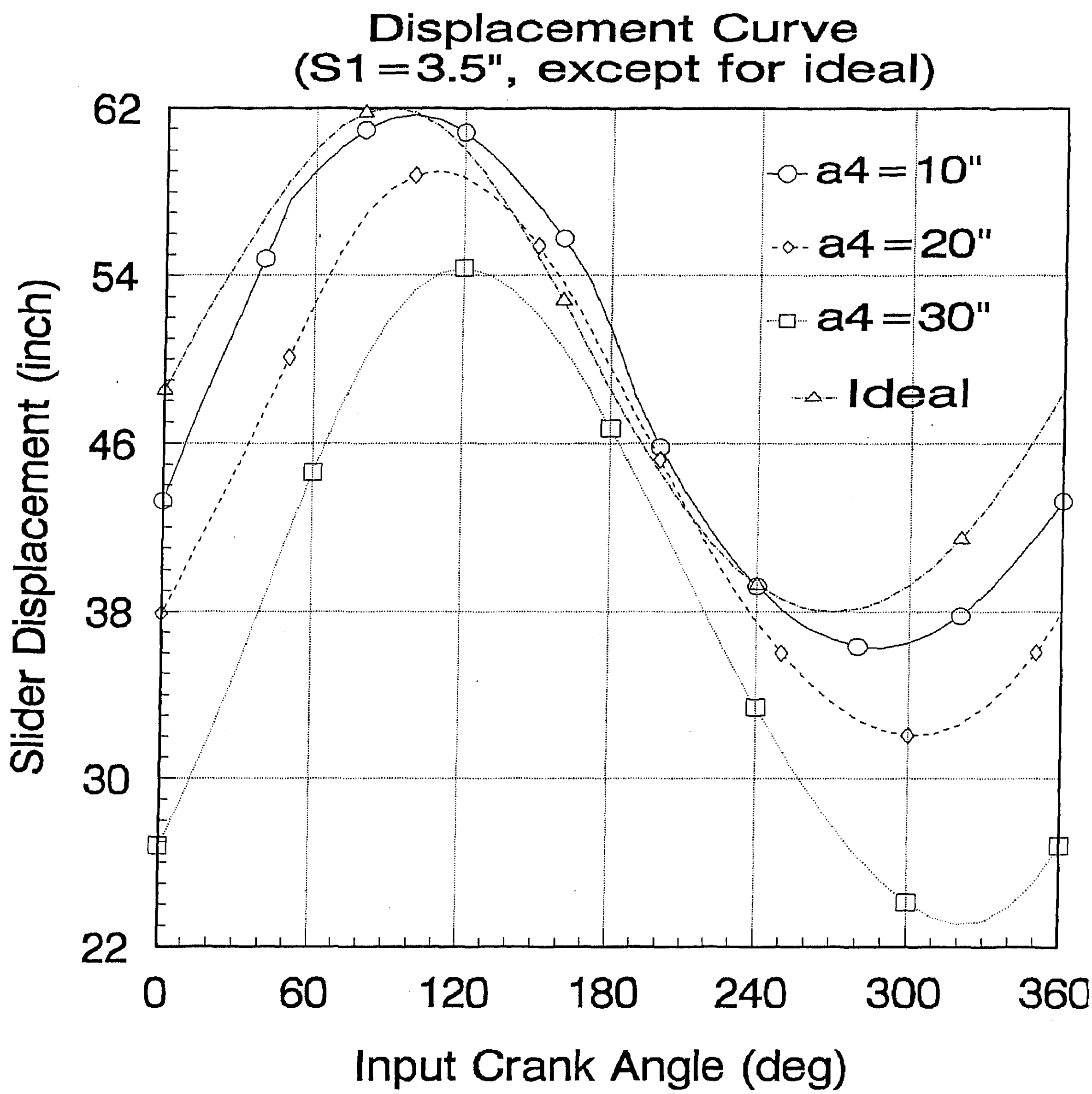


Displacement Curve
(S1=3.5", except for ideal)
Slider Displacement (inch)
62
54
46
38
30
22
0
60
120
180
240
300
360
Input Crank Angle (deg)
a4=10"
a4=20"
a4=30"
Ideal

Figure 5

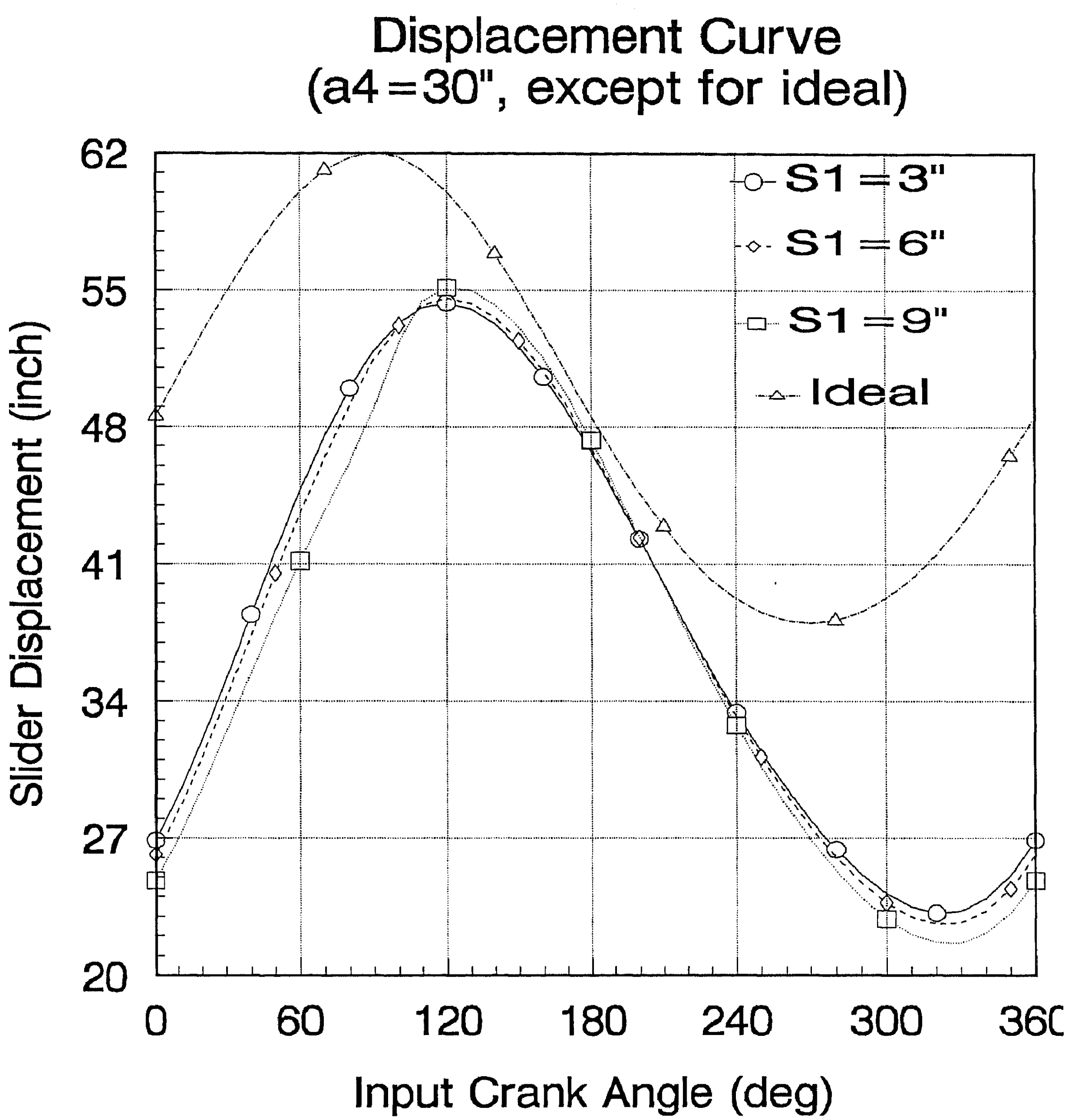

Displacement Curve
(a4=30", except for ideal)
Slider Displacement (inch)
Input Crank Angle (deg)
S1=3"
S1=6"
S1=9"
Ideal
62
55
48
41
34
27
20
0
60
120
180
240
300
360

Figure 6

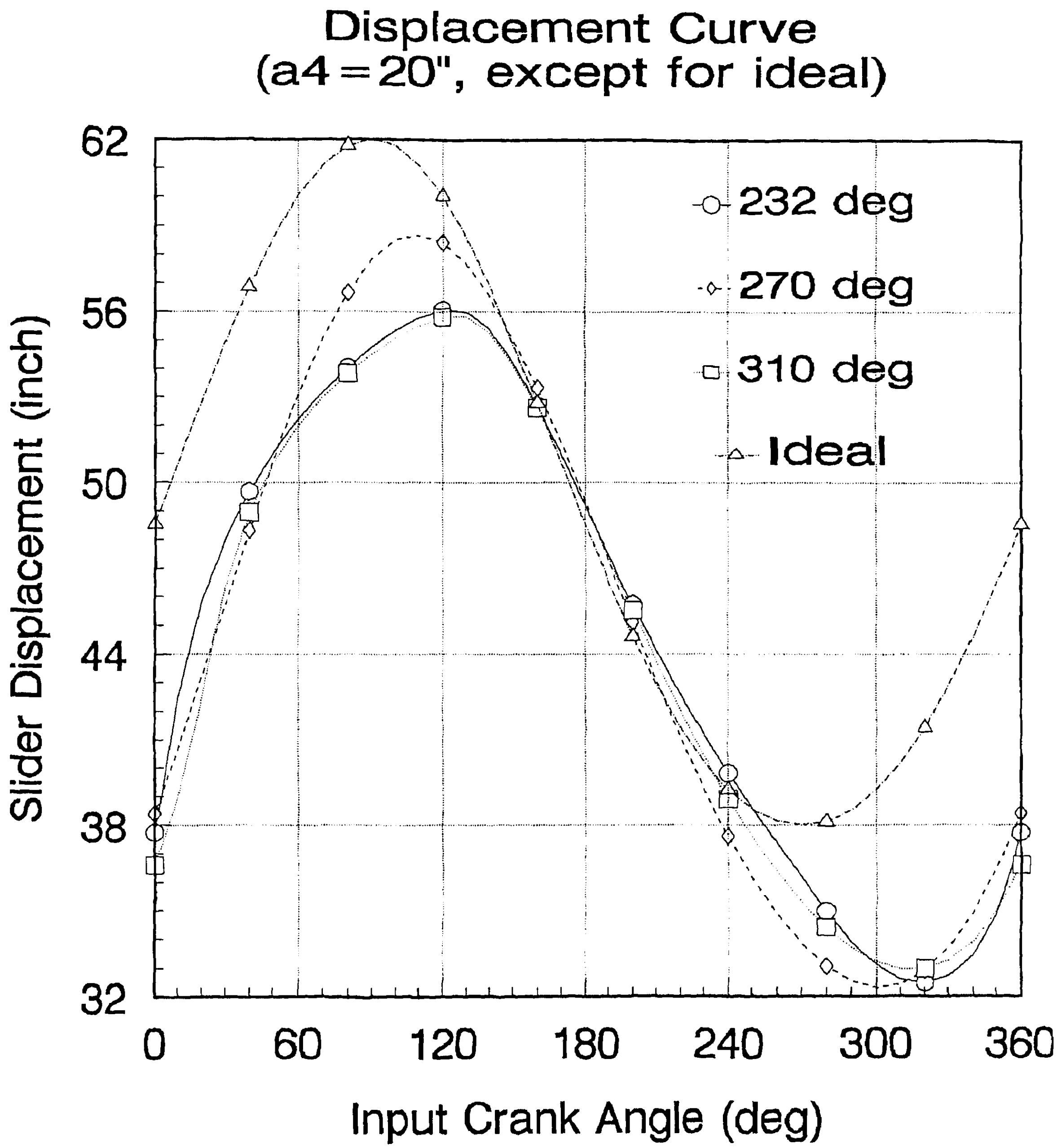
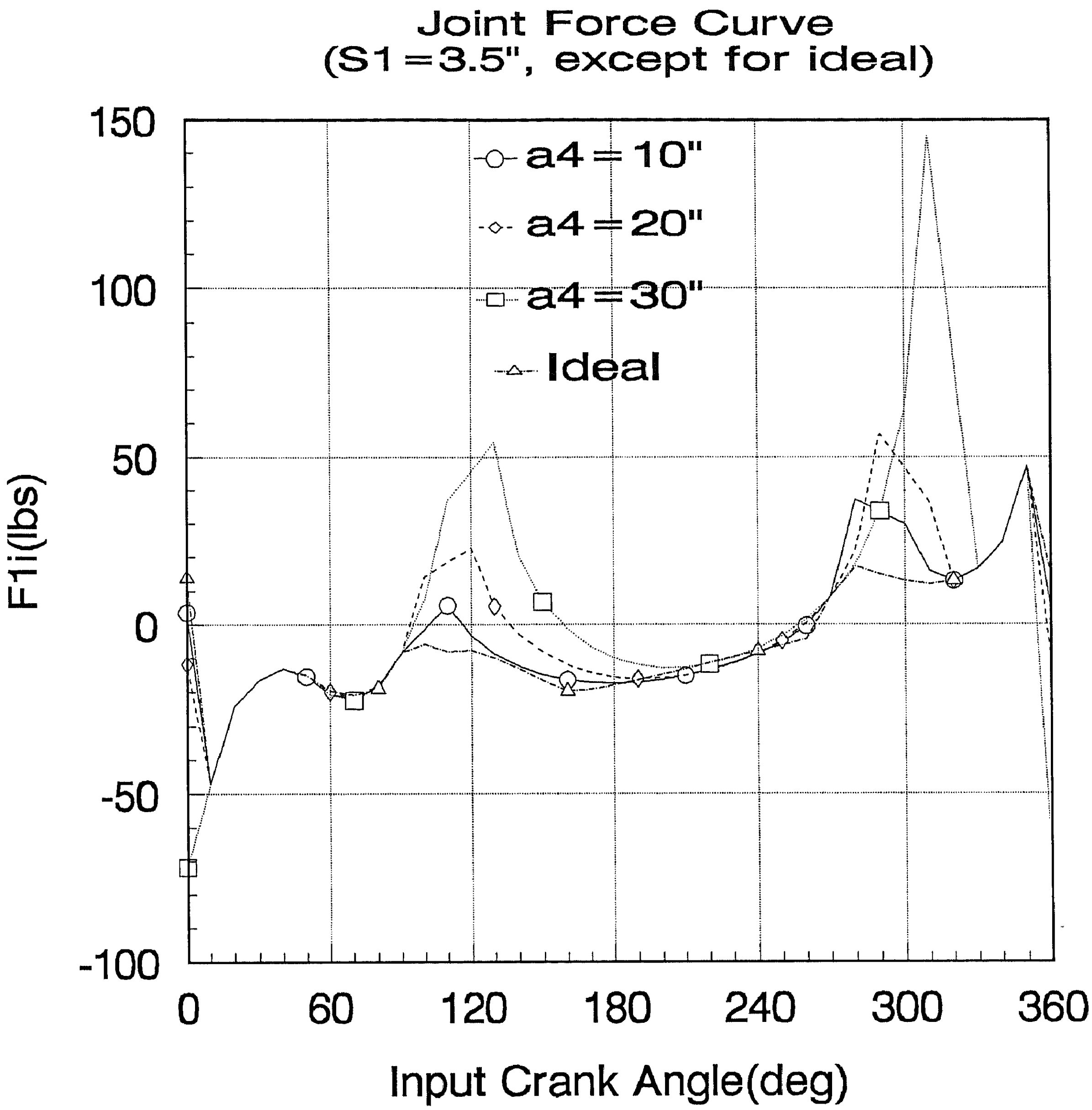

Displacement Curve
(a4=20", except for ideal)
62
56
50
44
38
32
Slider Displacement (inch)
0
60
120
180
240
300
360
Input Crank Angle (deg)
232 deg
270 deg
310 deg
Ideal

Figure 7



Joint Force Curve
(S1=3.5", except for ideal)
a4=10"
a4=20"
a4=30"
Ideal
150
100
50
0
-50
-100
F1i(lbs)
0
60
120
180
240
300
360
Input Crank Angle(deg)

Figure 8

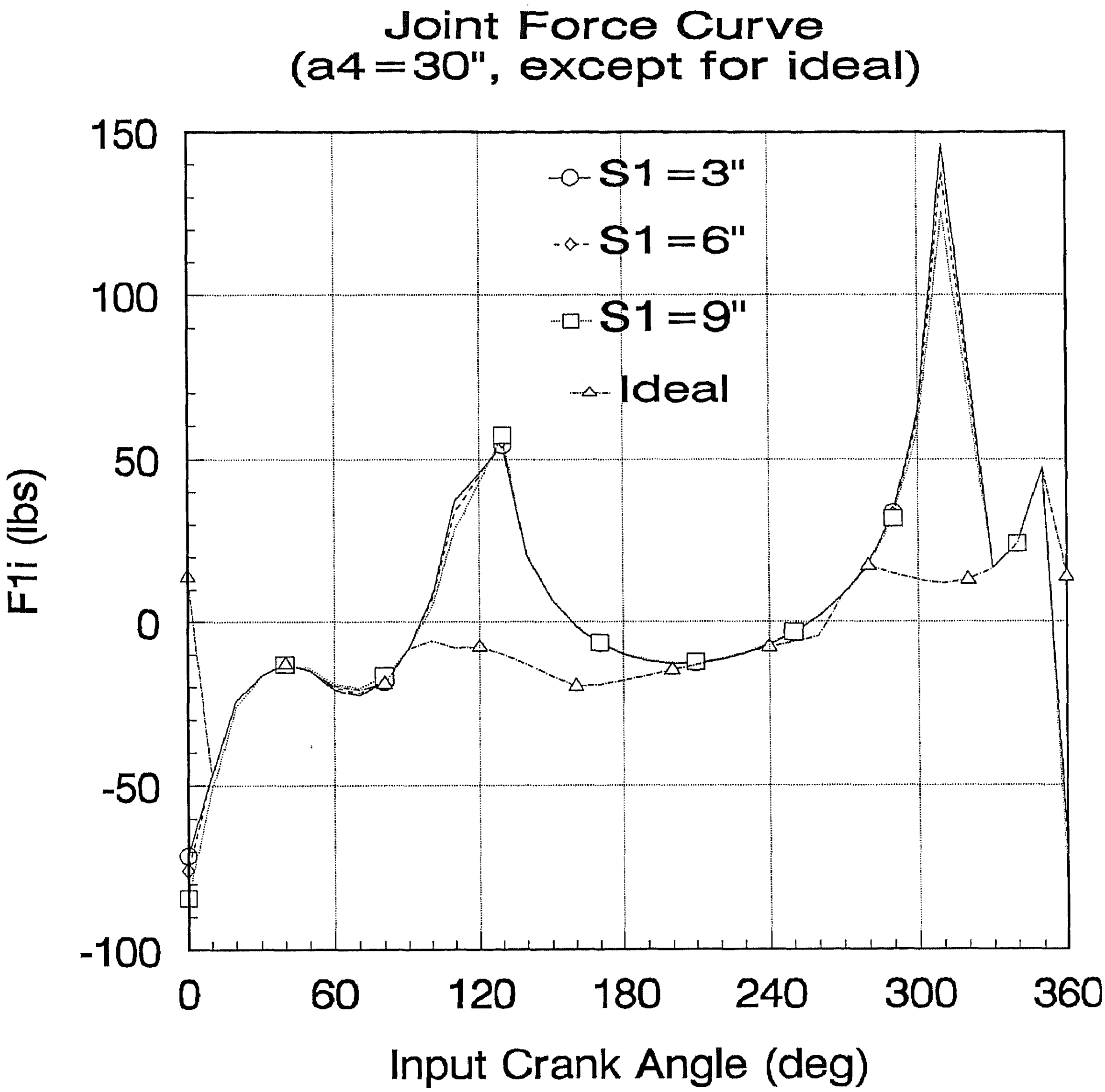
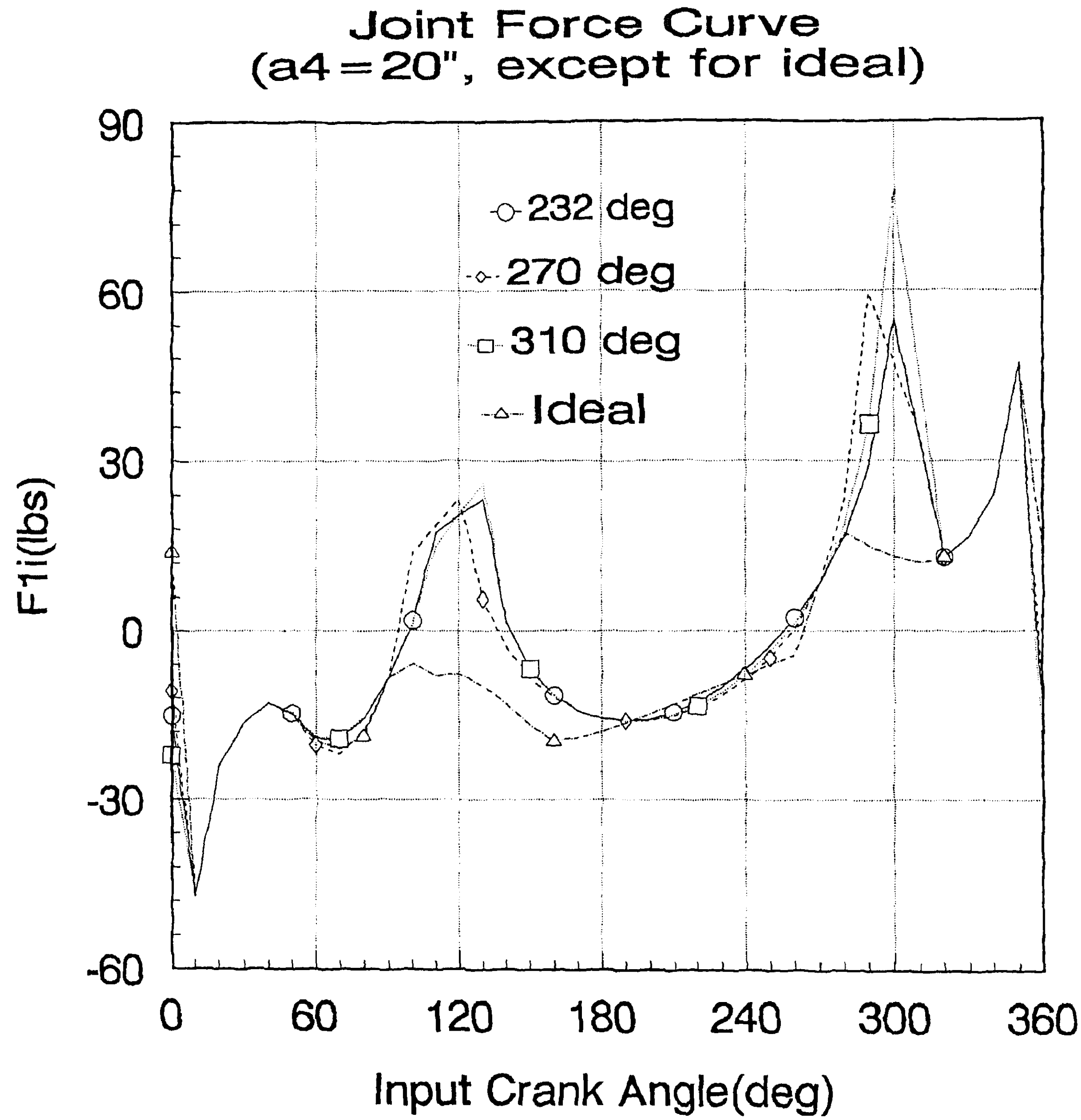
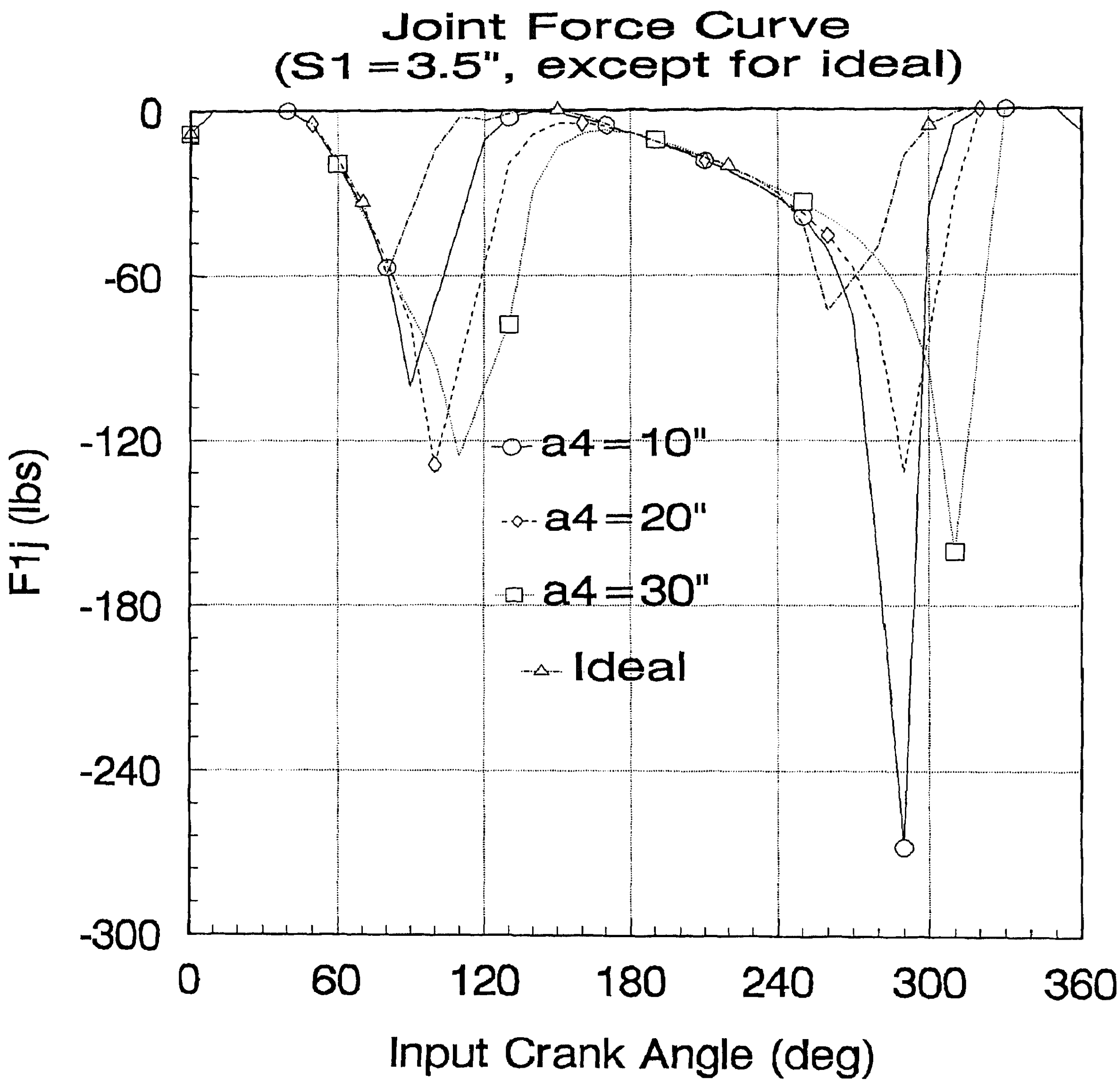

Joint Force Curve
(a4=30", except for ideal)
S1=3"
S1=6"
S1=9"
Ideal
150
100
50
0
-50
-100
F1i (lbs)
0
60
120
180
240
300
360
Input Crank Angle (deg)

Figure 9



Joint Force Curve
(a4=20", except for ideal)
232 deg
270 deg
310 deg
Ideal
F1i(lbs)
90
60
30
0
-30
-60
0
60
120
180
240
300
360
Input Crank Angle(deg)

Figure 10



Joint Force Curve
(S1 =3.5", except for ideal)
0
-60
-120
-180
-240
-300
F1j (lbs)
0
60
120
180
240
300
360
Input Crank Angle (deg)
a4=10"
a4=20"
a4=30"
Ideal

Figure 11

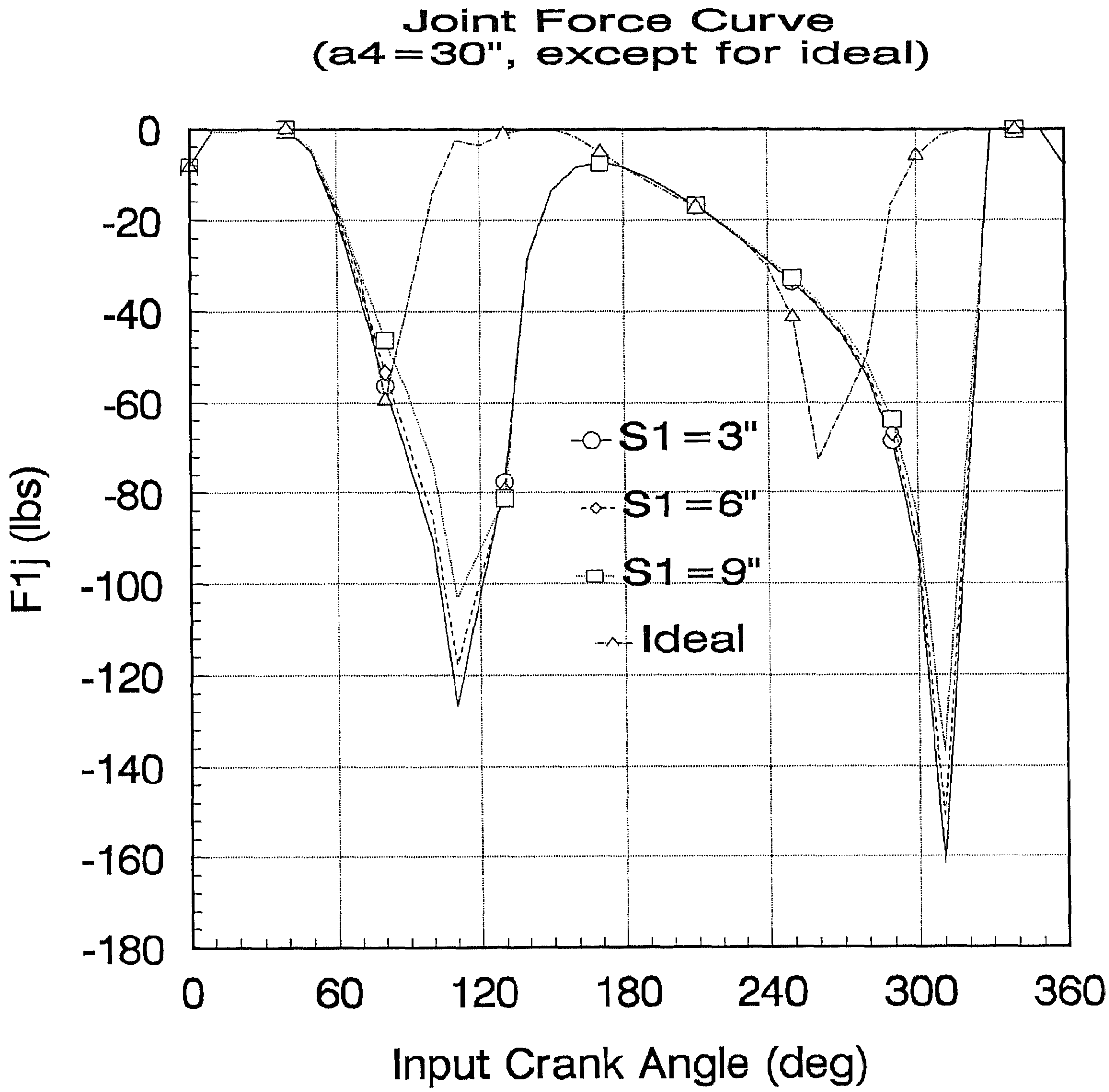

Joint Force Curve
(a4=30", except for ideal)
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
F1j (lbs)
0
60
120
180
240
300
360
Input Crank Angle (deg)
S1=3"
S1=6"
S1=9"
Ideal

Figure 12

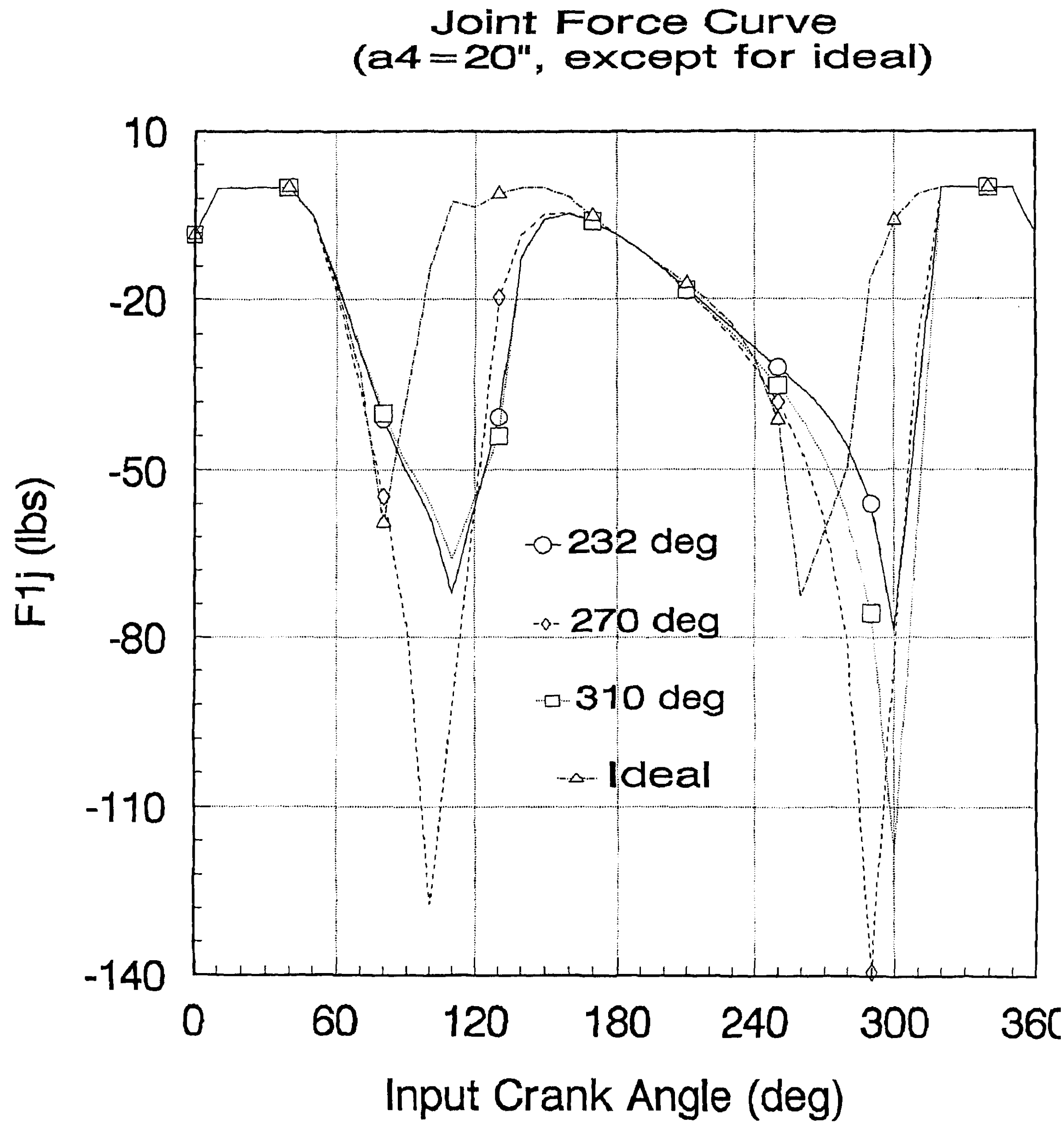

Joint Force Curve
(a4=20", except for ideal)
10
-20
-50
-80
-110
-140
F1j (lbs)
0
60
120
180
240
300
360
Input Crank Angle (deg)
232 deg
270 deg
310 deg
Ideal

Figure 13

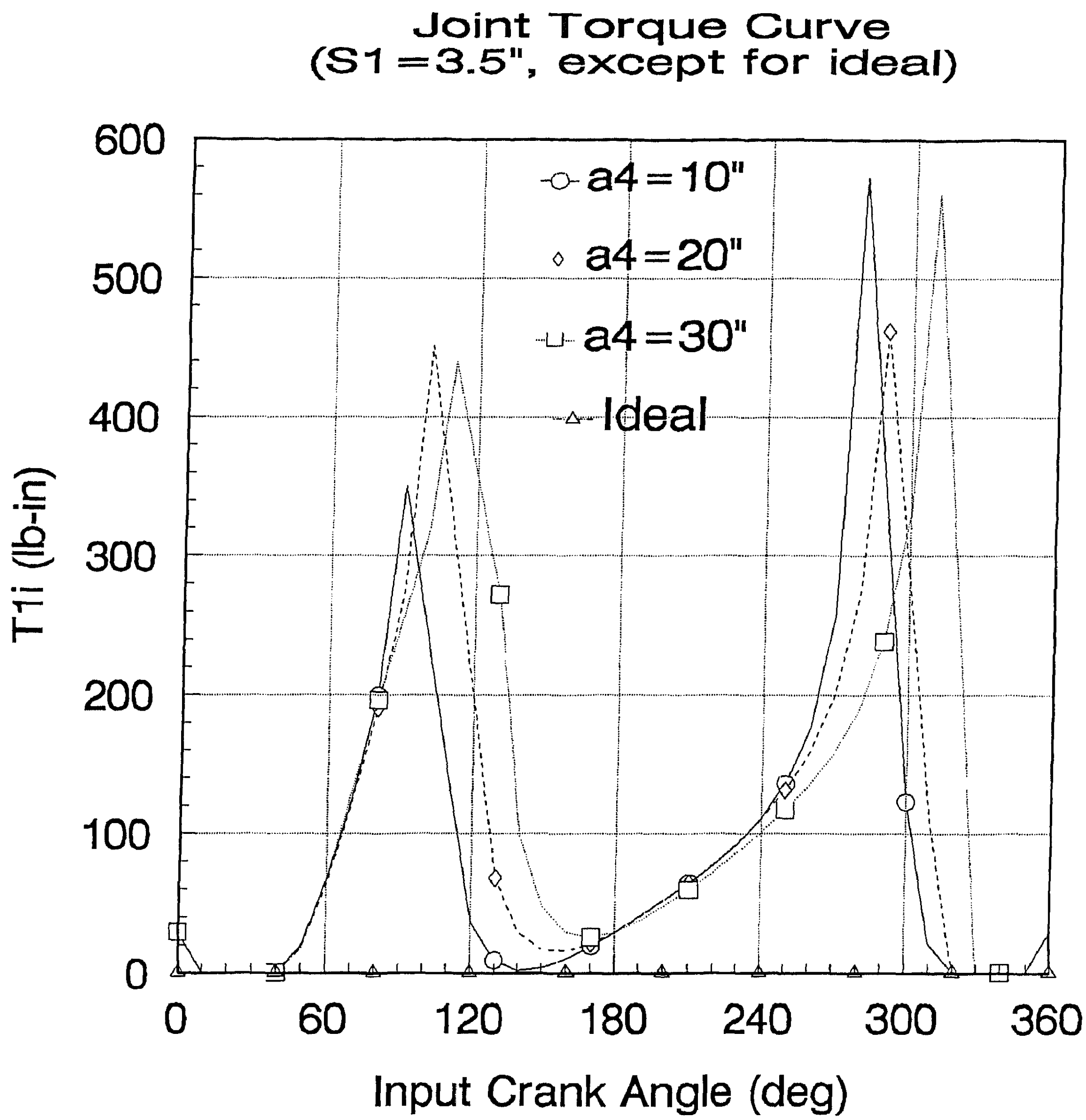
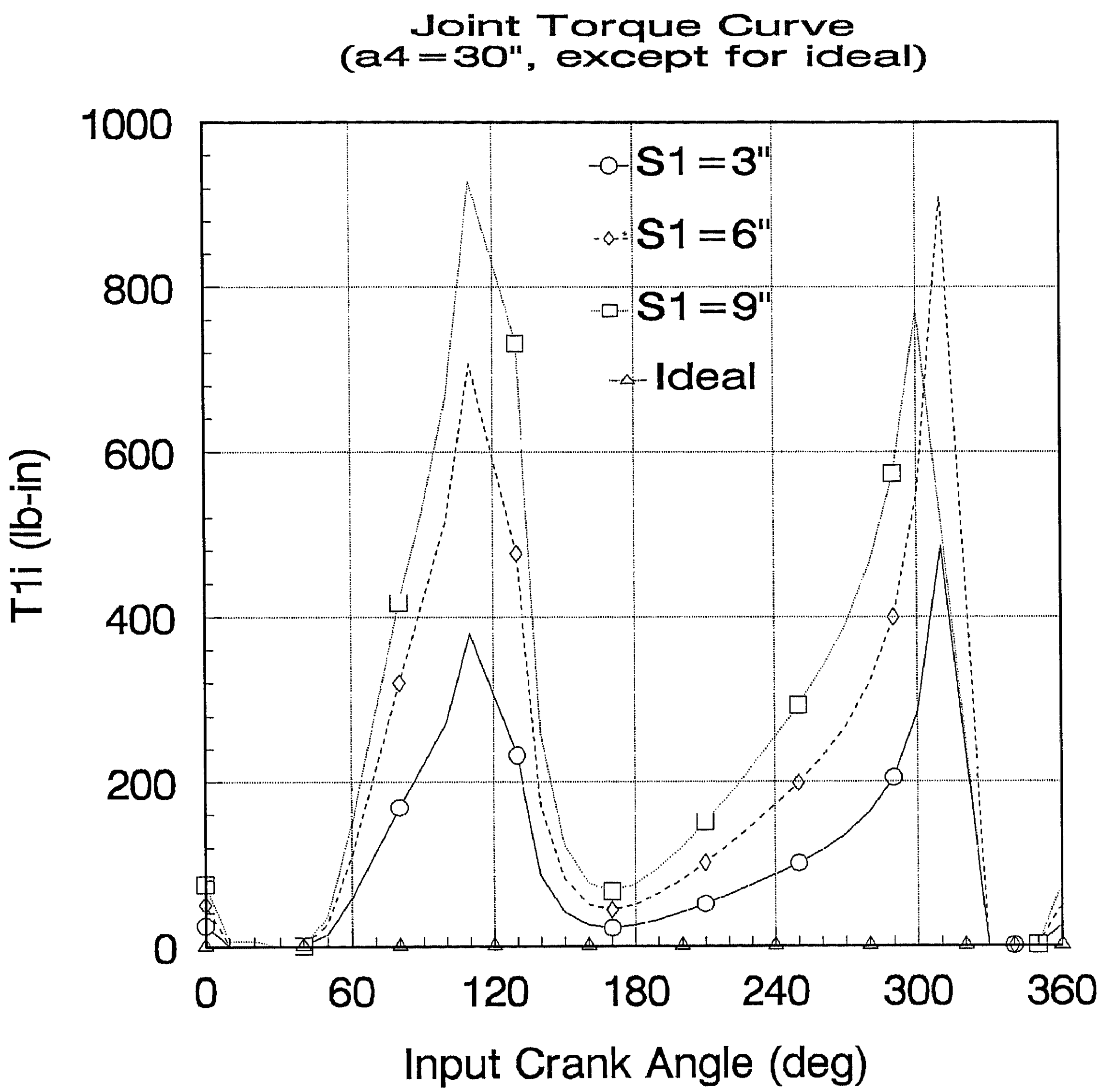

Joint Torque Curve
(S1=3.5", except for ideal)
a4=10"
a4=20"
a4=30"
Ideal
T1i (lb-in)
600
500
400
300
200
100
0
0
60
120
180
240
300
360
Input Crank Angle (deg)

Figure 14



Joint Torque Curve
(a4=30", except for ideal)
S1=3"
S1=6"
S1=9"
Ideal
T1i (lb-in)
0
200
400
600
800
1000
0
60
120
180
240
300
360
Input Crank Angle (deg)

Figure 15

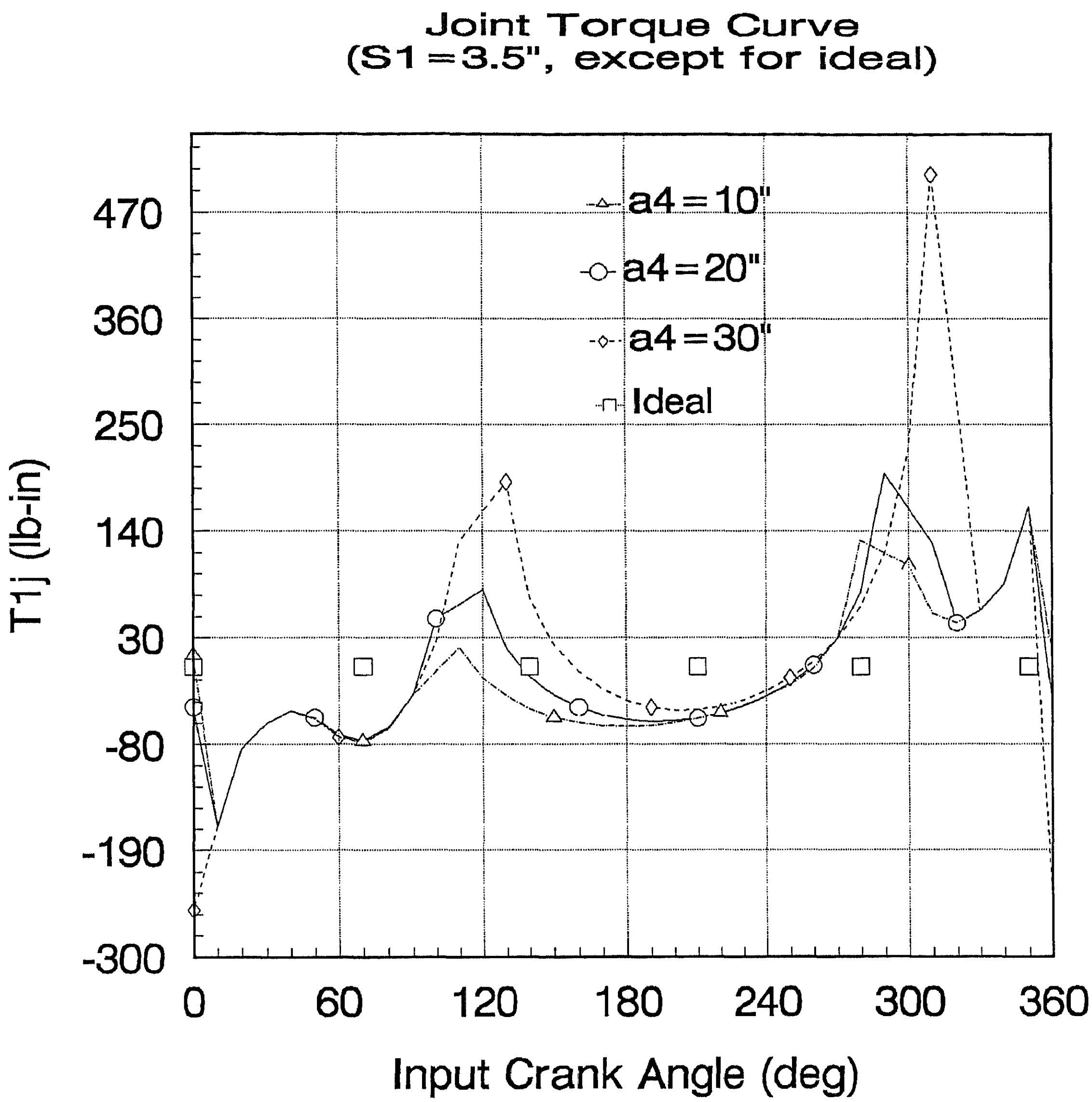

Joint Torque Curve
(S1=3.5", except for ideal)
a4=10"
a4=20"
a4=30"
Ideal
470
360
250
140
30
-80
-190
-300
T1j (lb-in)
0
60
120
180
240
300
360
Input Crank Angle (deg)

Figure 16

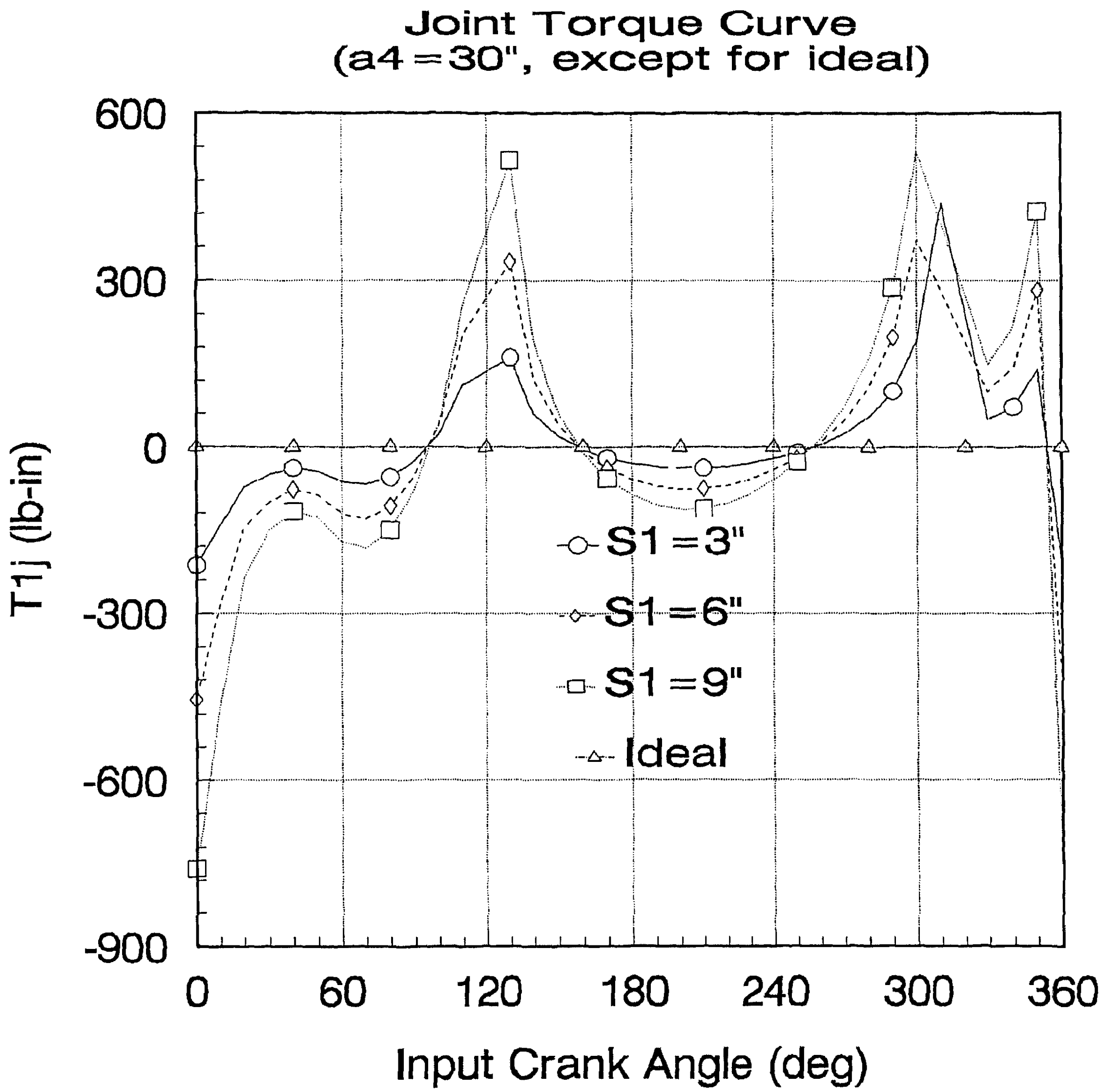

Joint Torque Curve
(a4=30", except for ideal)
600
300
0
-300
-600
-900
T1j (lb-in)
0
60
120
180
240
300
360
Input Crank Angle (deg)
S1=3"
S1=6"
S1=9"
Ideal

Figure 17

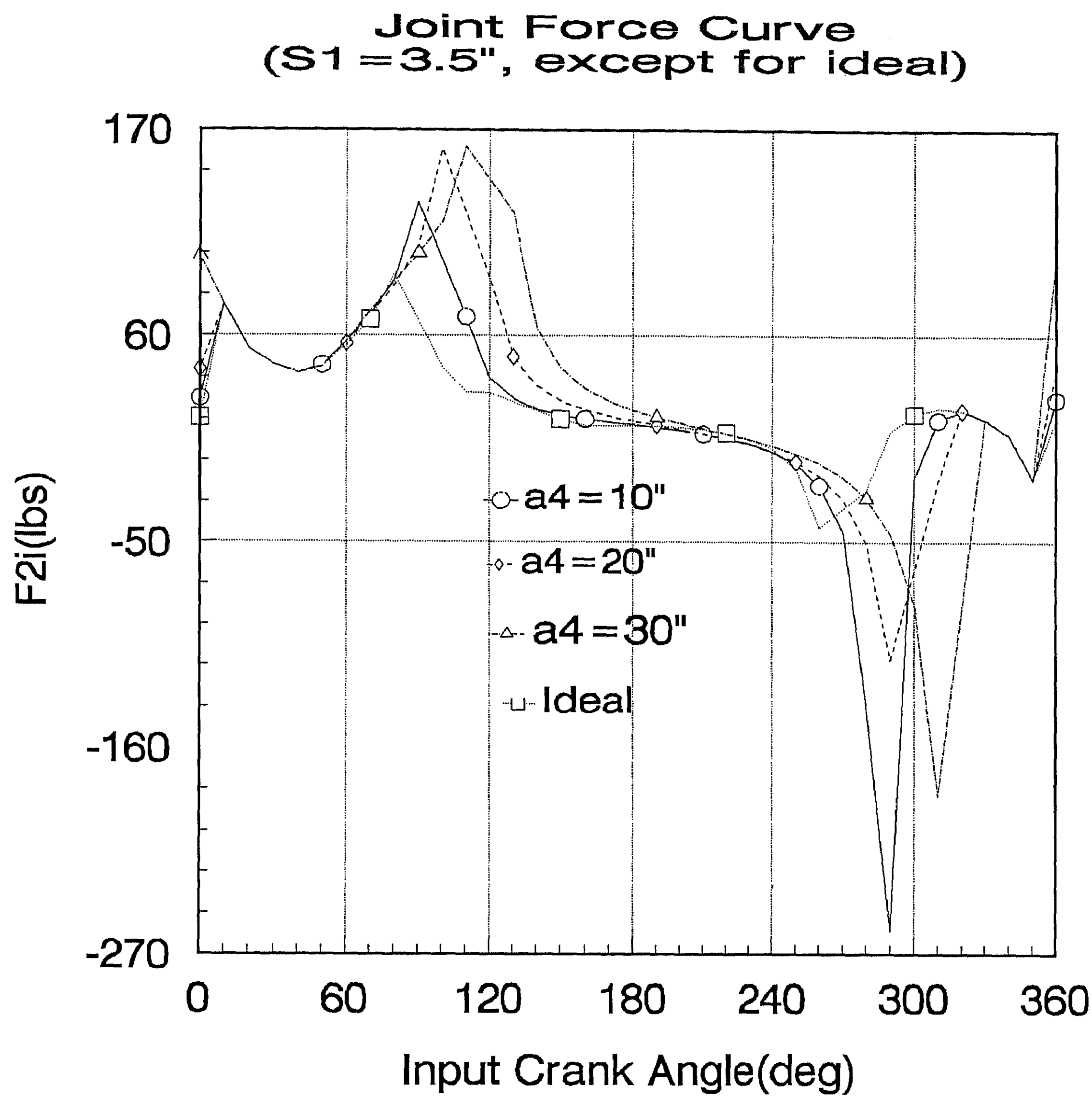

Joint Force Curve
(S1 =3.5", except for ideal)
170
60
-50
-160
-270
0
60
120
180
240
300
360
F2i(lbs)
Input Crank Angle(deg)
a4=10"
a4=20"
a4=30"
Ideal

Figure 18

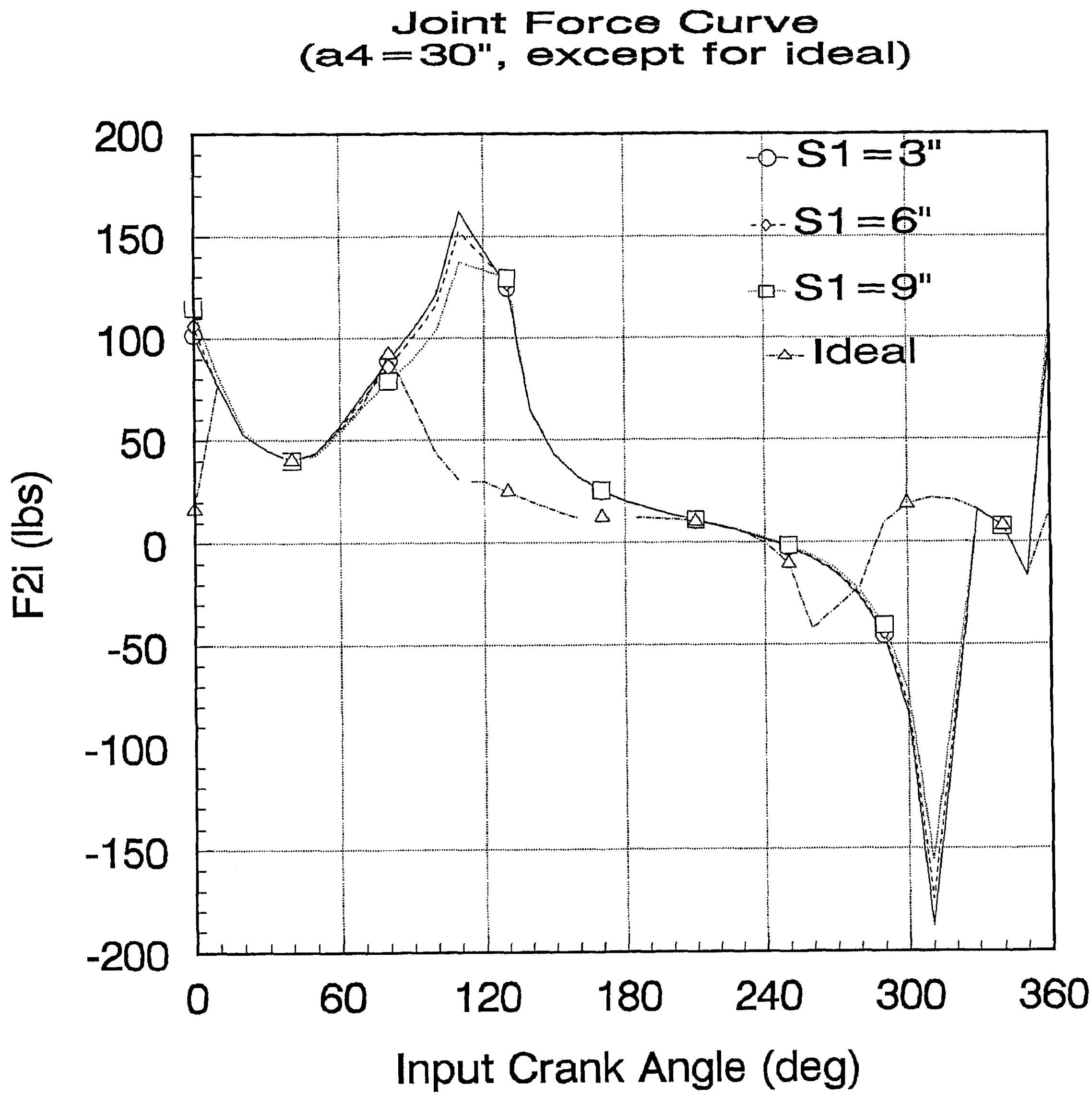

Joint Force Curve
(a4=30", except for ideal)
S1=3"
S1=6"
S1=9"
Ideal
F2i (lbs)
200
150
100
50
0
-50
-100
-150
-200
0
60
120
180
240
300
360
Input Crank Angle (deg)

Figure 19

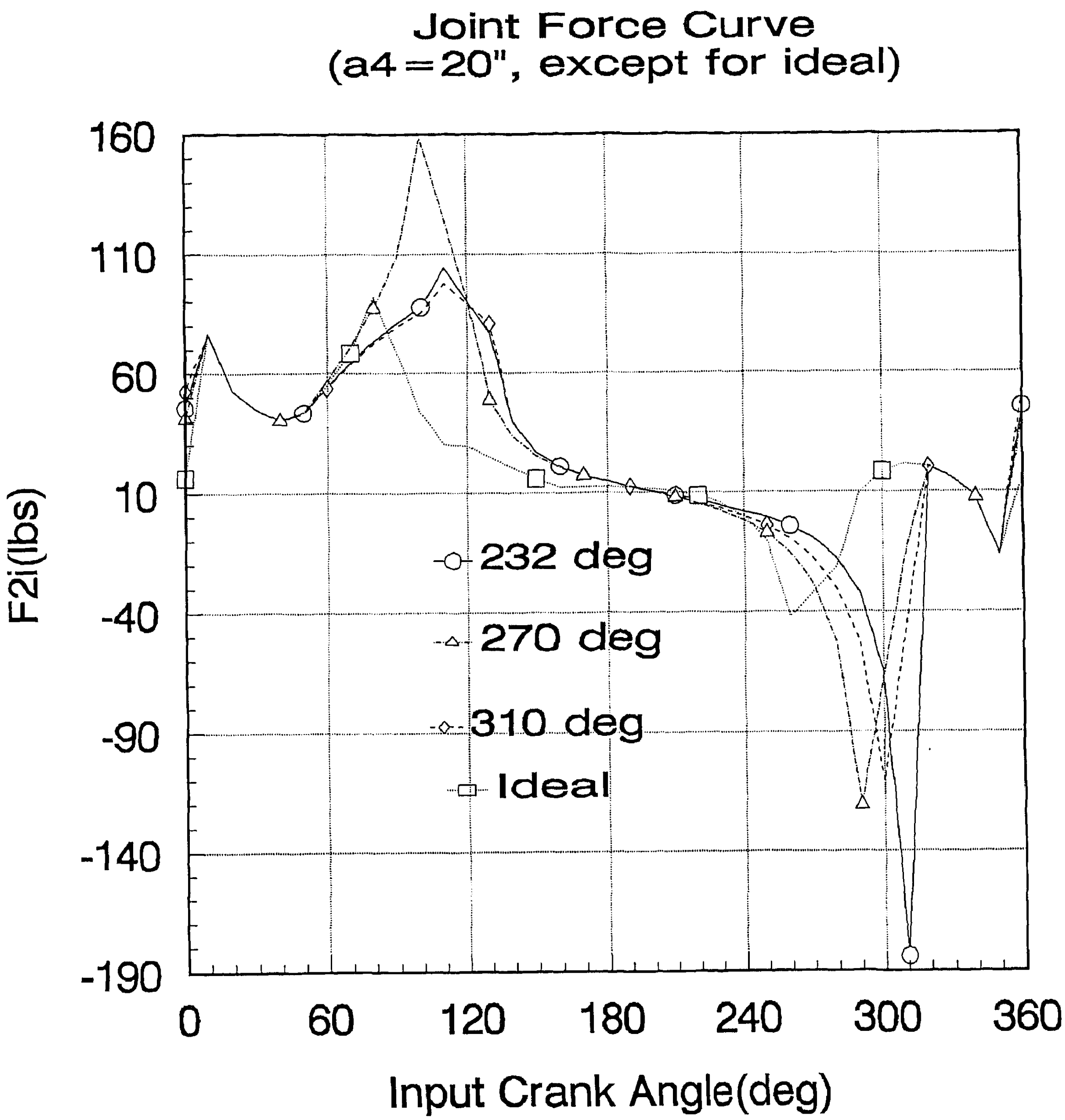

Joint Force Curve
(a4=20", except for ideal)
160
110
60
10
-40
-90
-140
-190
F2i(lbs)
0
60
120
180
240
300
360
Input Crank Angle(deg)
232 deg
270 deg
310 deg
Ideal

Figure 20

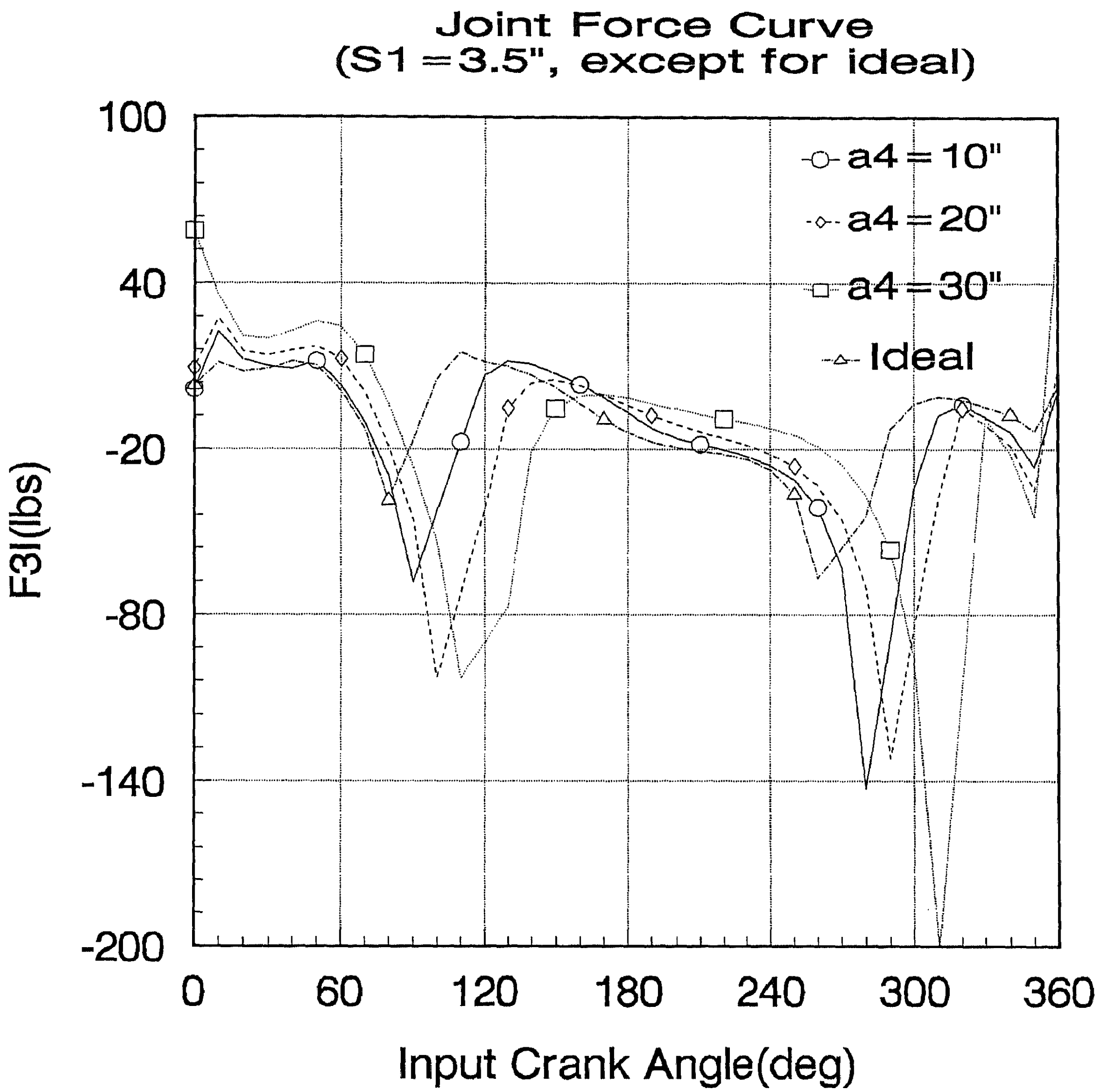

Joint Force Curve
(S1 = 3.5", except for ideal)
100
40
-20
-80
-140
-200
0
60
120
180
240
300
360
F3I(lbs)
Input Crank Angle(deg)
a4 = 10"
a4 = 20"
a4 = 30"
Ideal

Figure 21

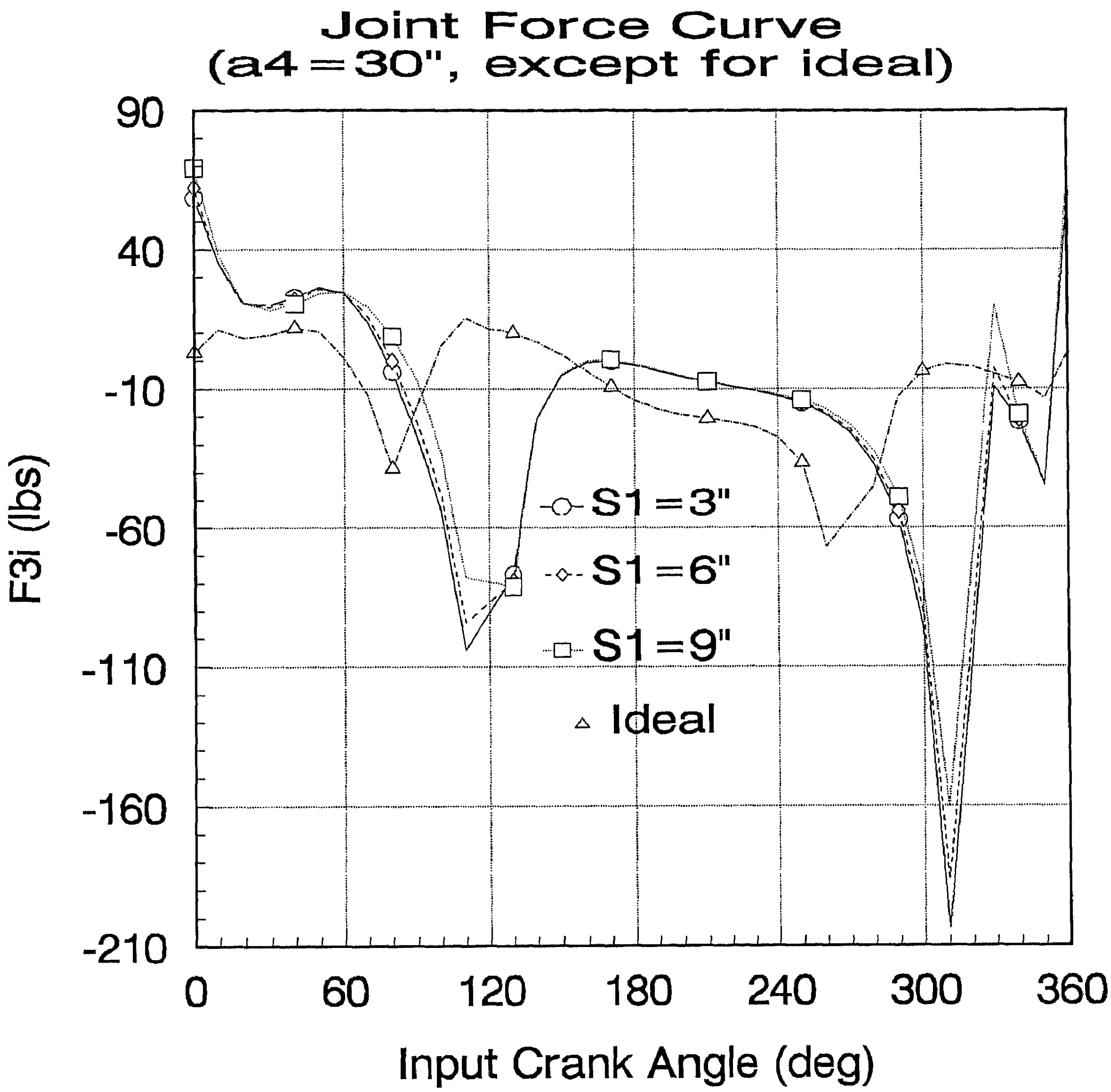
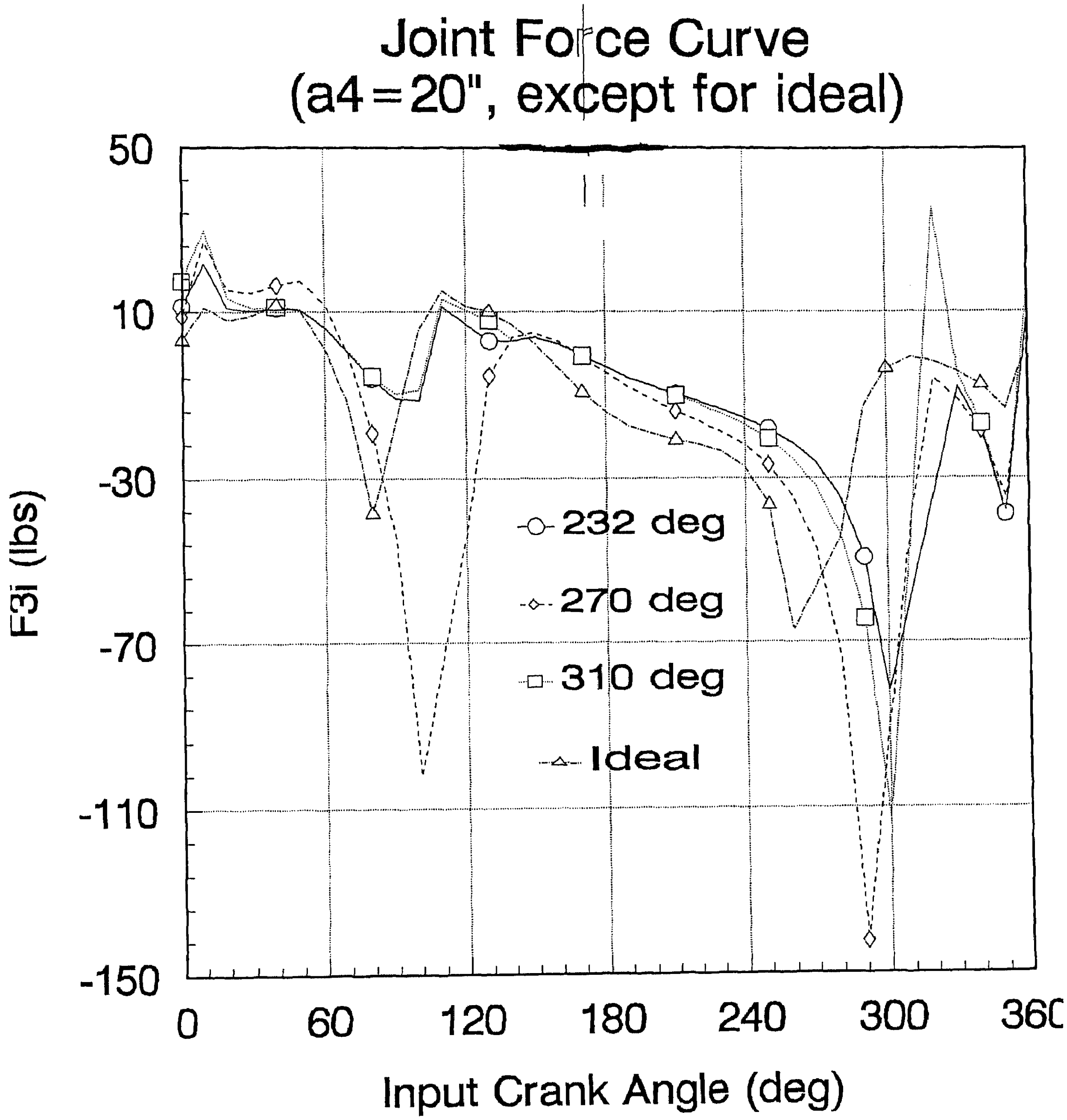

Joint Force Curve
(a4 = 30", except for ideal)
90
40
-10
-60
-110
-160
-210
F3i (lbs)
0
60
120
180
240
300
360
Input Crank Angle (deg)
S1 = 3"
S1 = 6"
S1 = 9"
Ideal

Figure 22



Joint Force Curve
(a4=20", except for ideal)
F3i (lbs)
50
10
-30
-70
-110
-150
0
60
120
180
240
300
360
Input Crank Angle (deg)
232 deg
270 deg
310 deg
Ideal

Figure 23

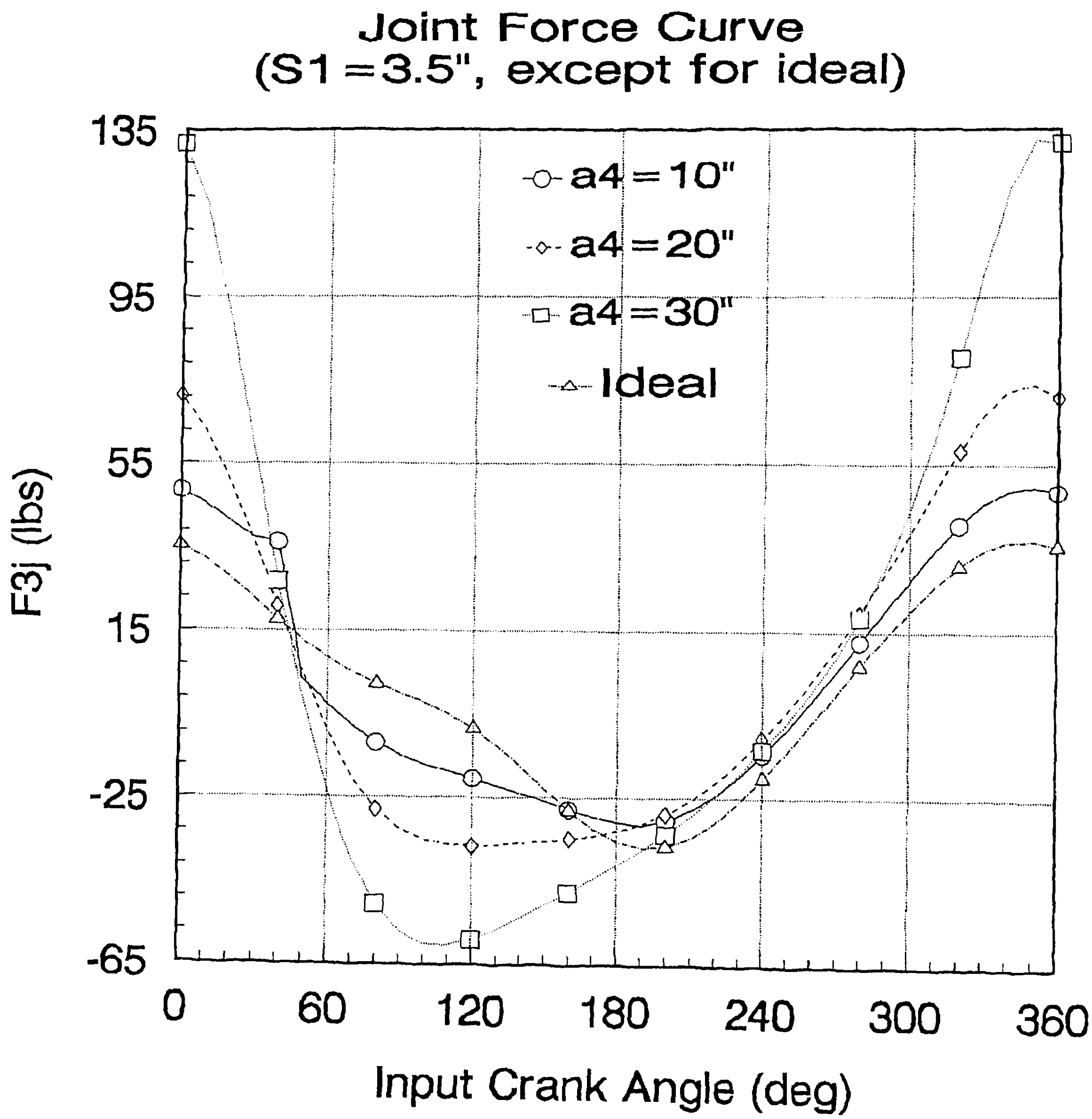
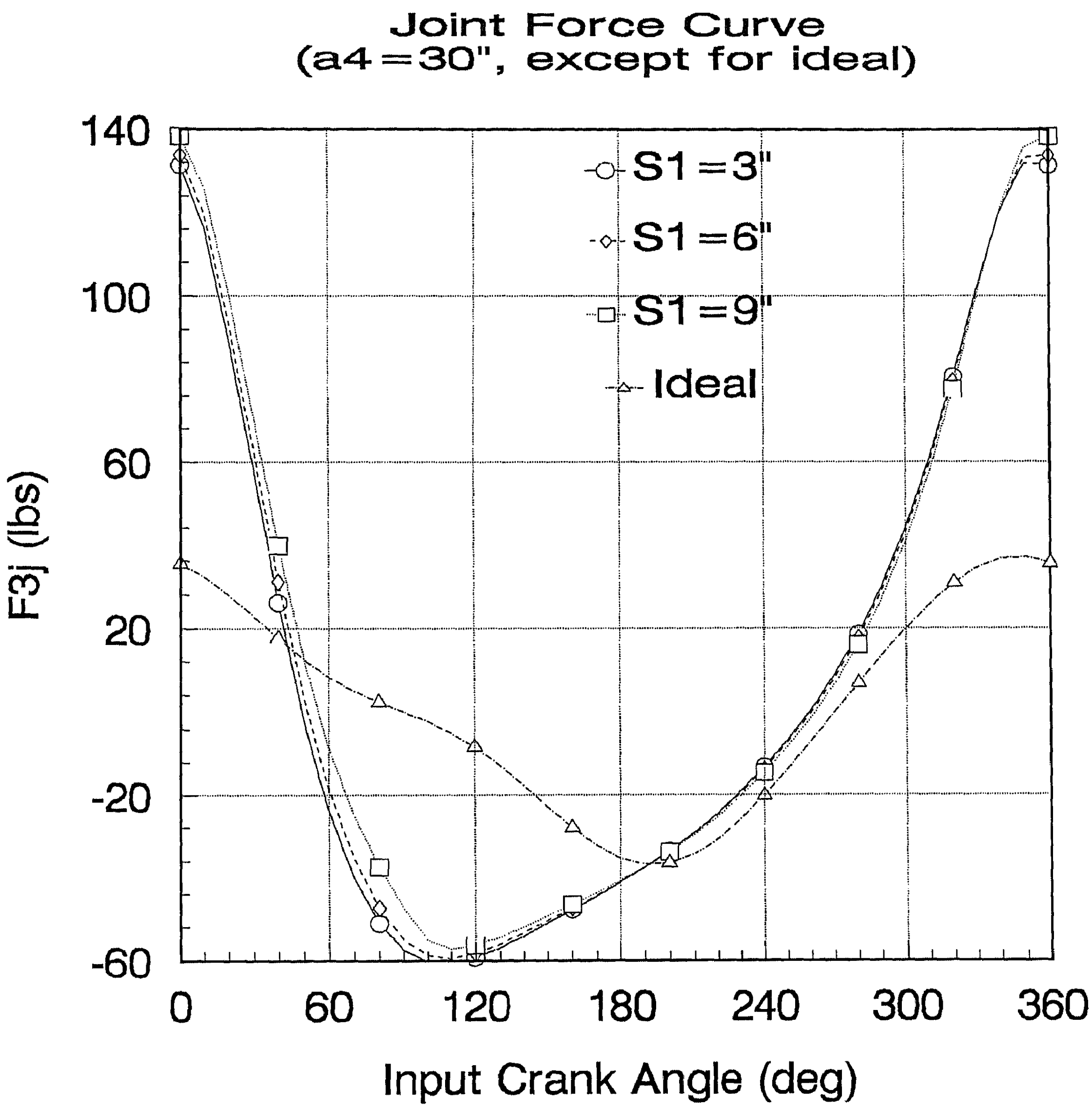

Joint Force Curve
(S1 =3.5", except for ideal)
a4=10"
a4=20"
a4=30"
Ideal
135
95
55
15
-25
-65
F3j (lbs)
0
60
120
180
240
300
360
Input Crank Angle (deg)

Figure 24



Joint Force Curve
(a4=30", except for ideal)
S1=3"
S1=6"
S1=9"
Ideal
F3j (lbs)
140
100
60
20
-20
-60
0
60
120
180
240
300
360
Input Crank Angle (deg)

Figure 25

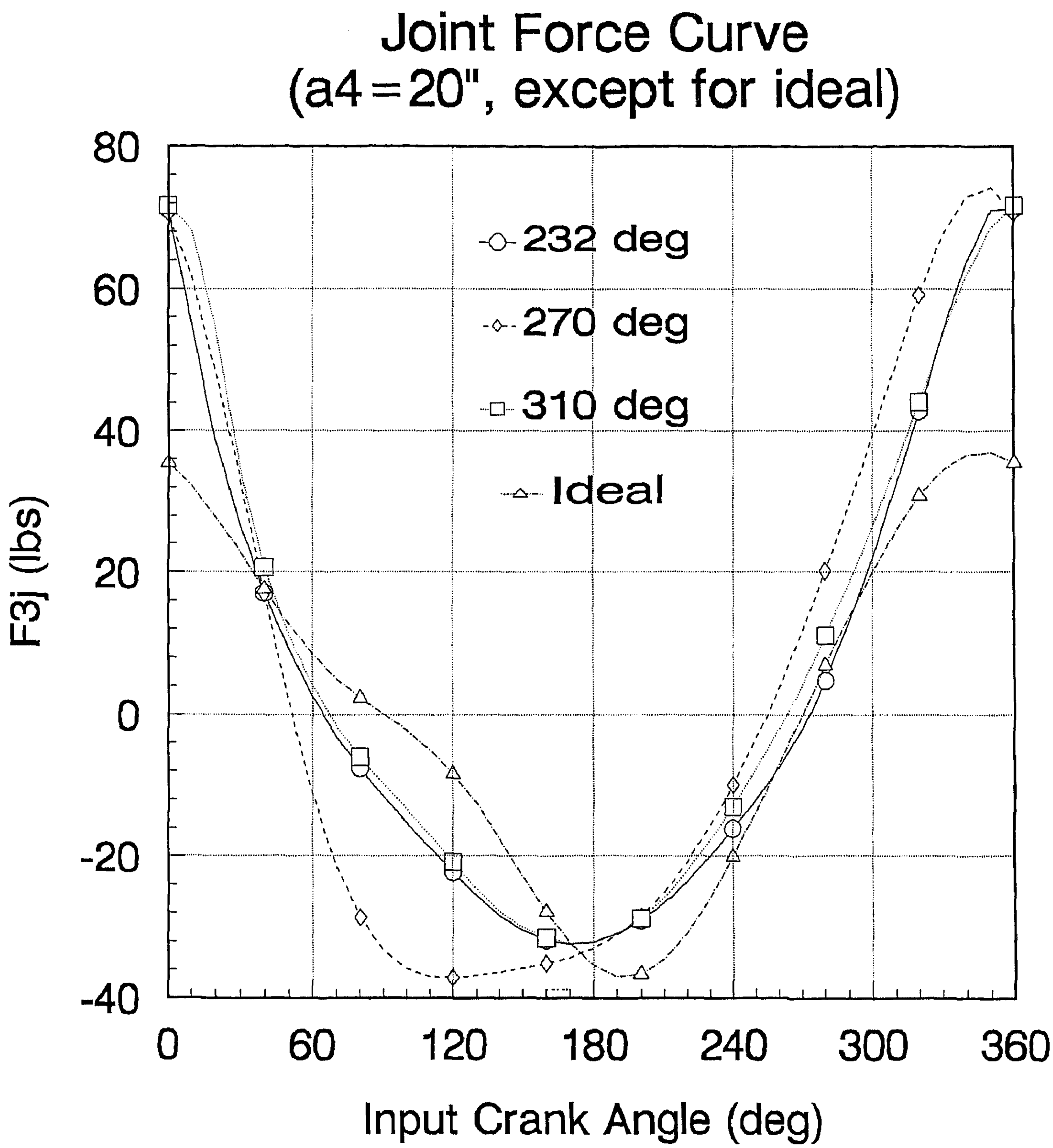

Joint Force Curve
(a4=20", except for ideal)
232 deg
270 deg
310 deg
Ideal
F3j (lbs)
80
60
40
20
0
-20
-40
0
60
120
180
240
300
360
Input Crank Angle (deg)

Figure 26

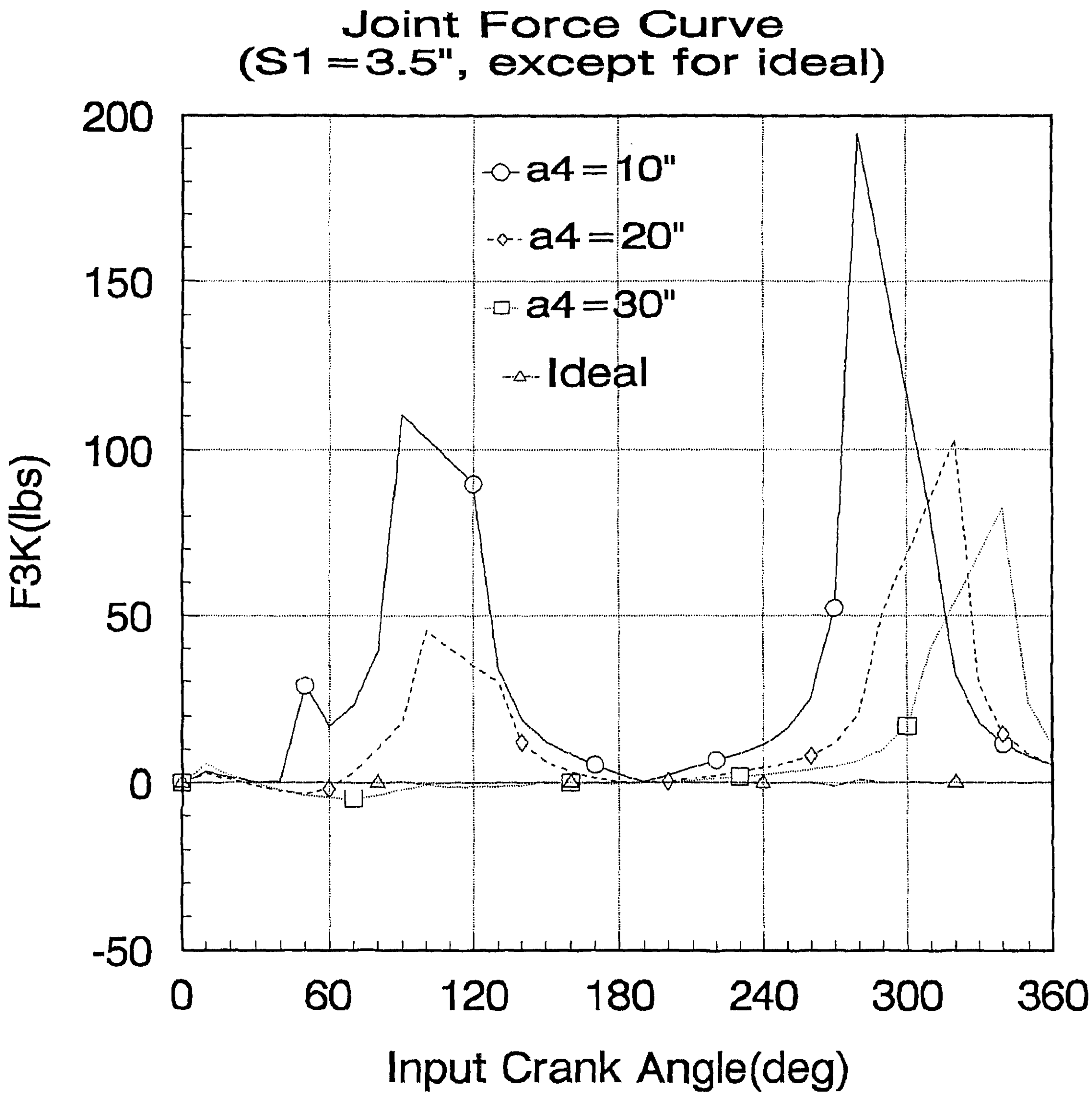

Joint Force Curve
(S1=3.5", except for ideal)
a4=10"
a4=20"
a4=30"
Ideal
200
150
100
50
0
-50
F3K(lbs)
0
60
120
180
240
300
360
Input Crank Angle(deg)

Figure 27

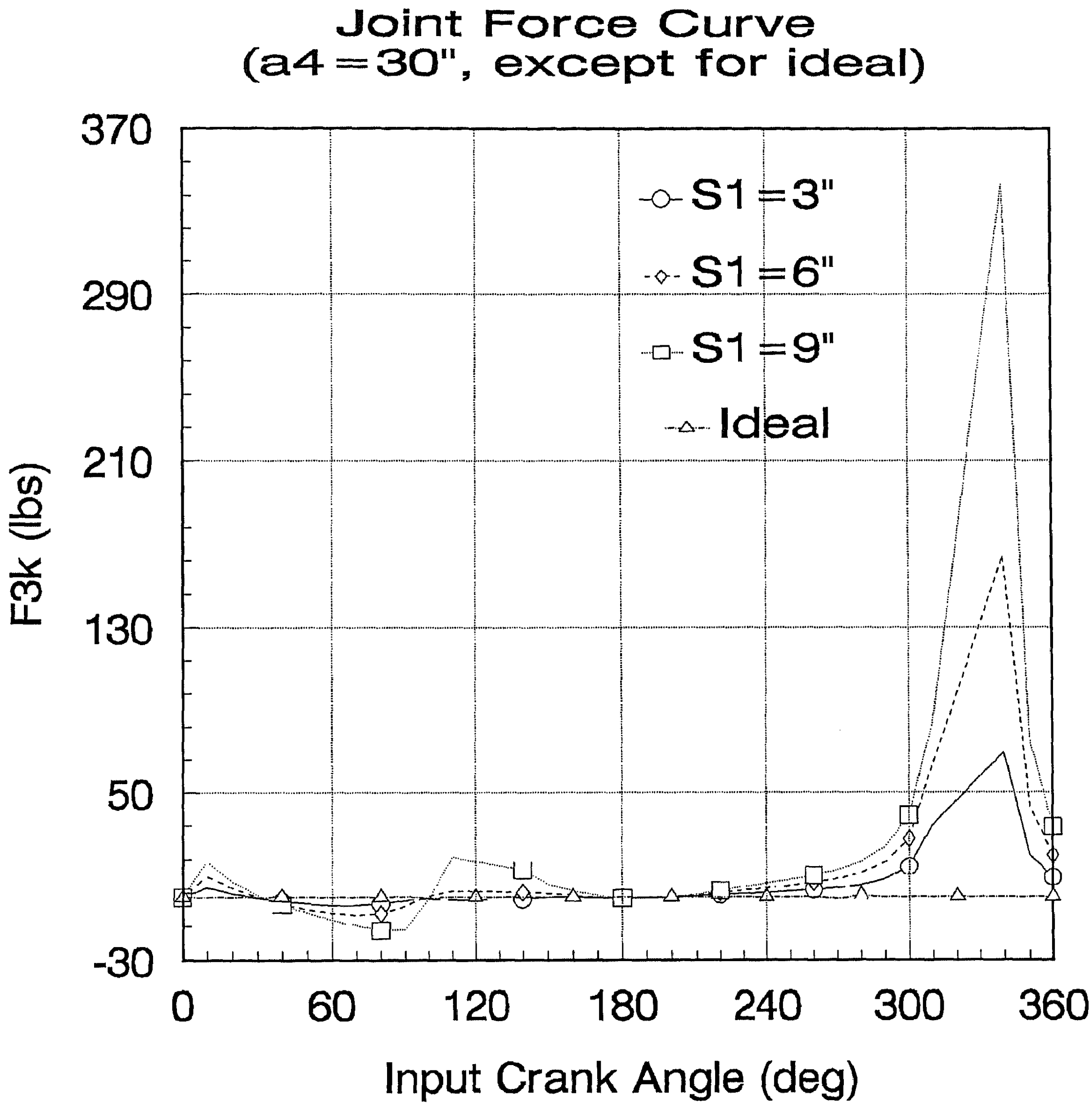

Joint Force Curve
(a4 = 30", except for ideal)
S1 = 3"
S1 = 6"
S1 = 9"
Ideal
F3k (lbs)
370
290
210
130
50
-30
0
60
120
180
240
300
360
Input Crank Angle (deg)

Figure 28

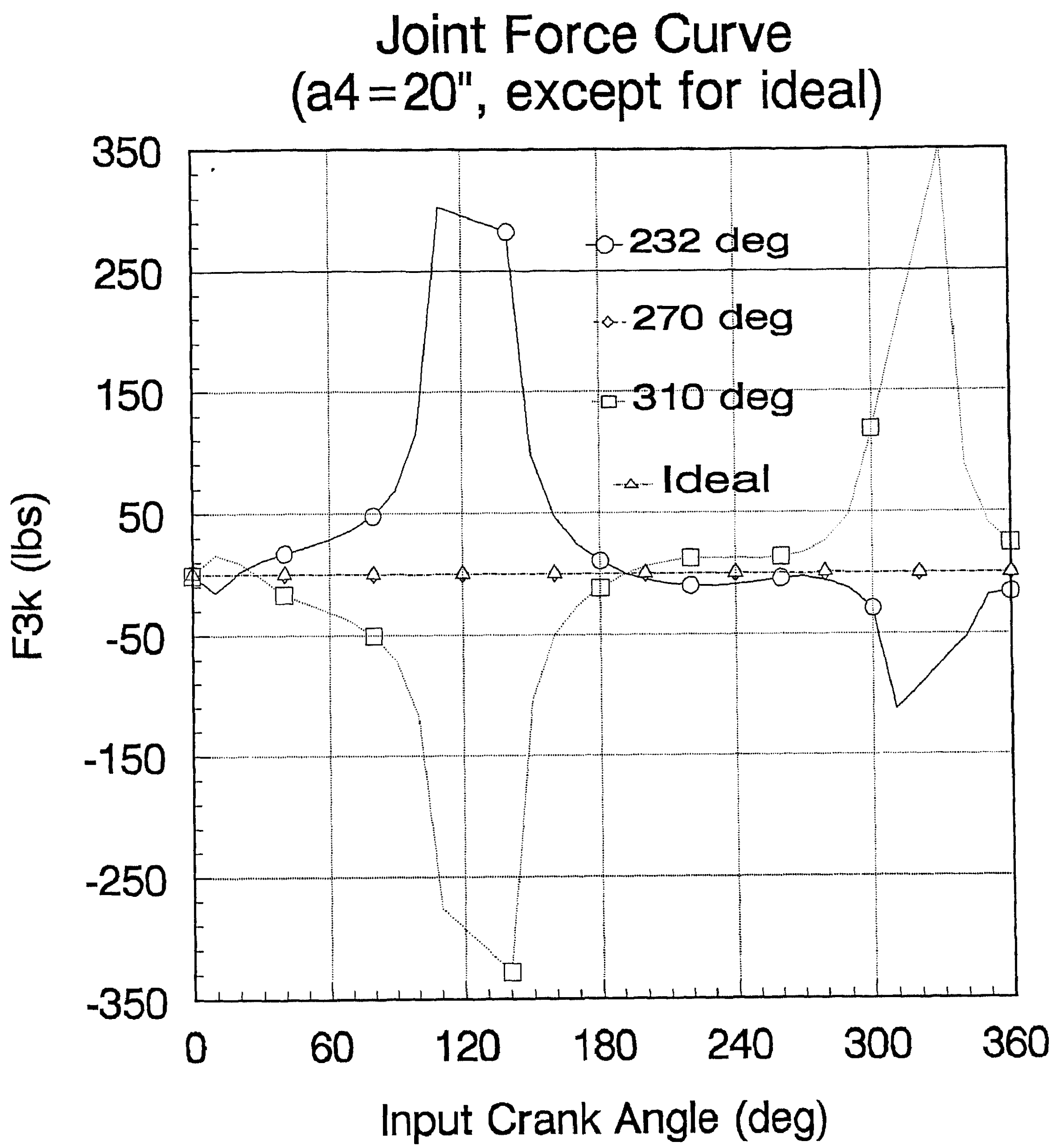

Joint Force Curve
(a4=20", except for ideal)
232 deg
270 deg
310 deg
Ideal
F3k (lbs)
350
250
150
50
-50
-150
-250
-350
0
60
120
180
240
300
360
Input Crank Angle (deg)

Figure 29

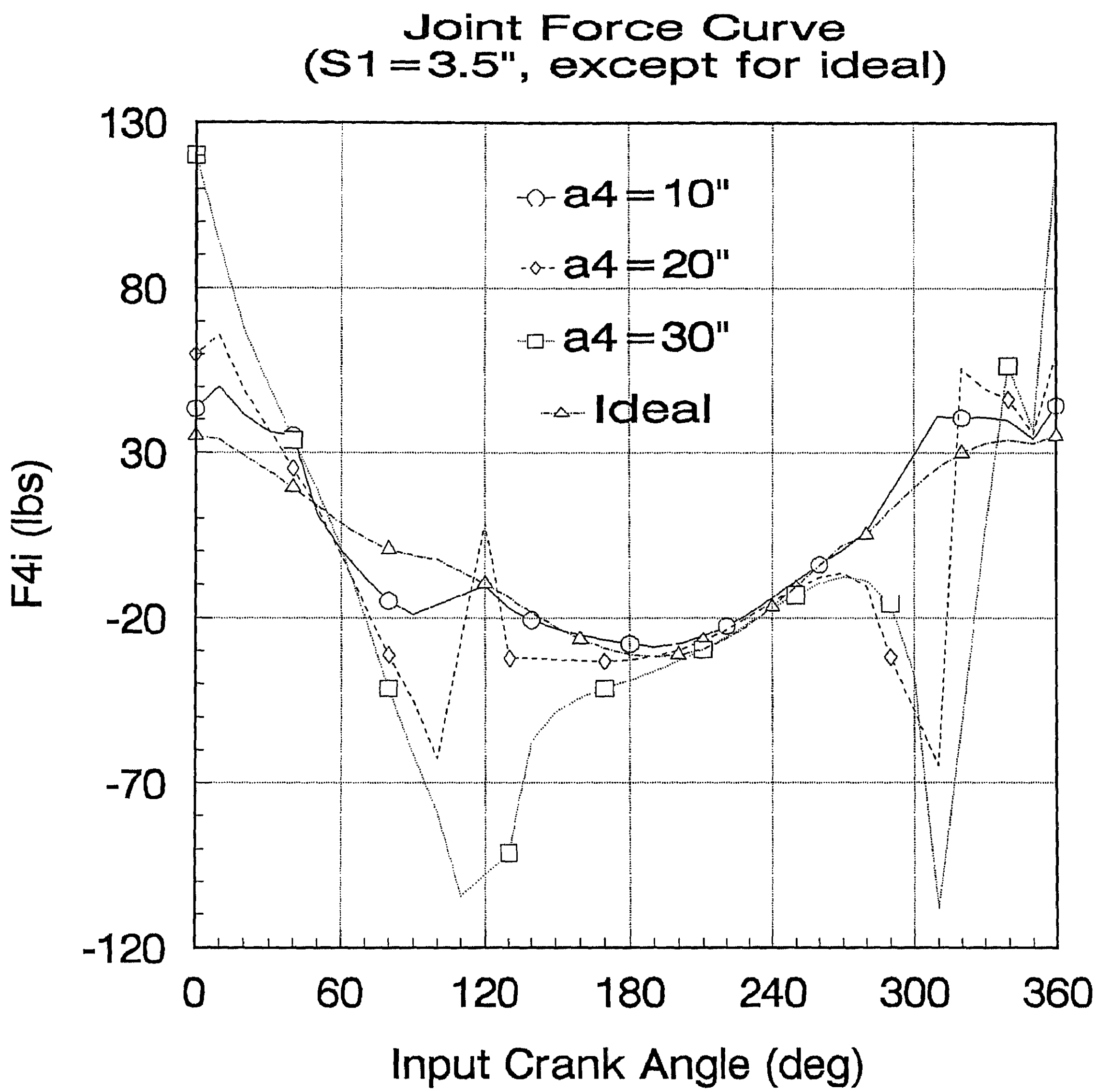

Joint Force Curve
(S1=3.5", except for ideal)
a4=10"
a4=20"
a4=30"
Ideal
F4i (lbs)
130
80
30
-20
-70
-120
0
60
120
180
240
300
360
Input Crank Angle (deg)

Figure 30

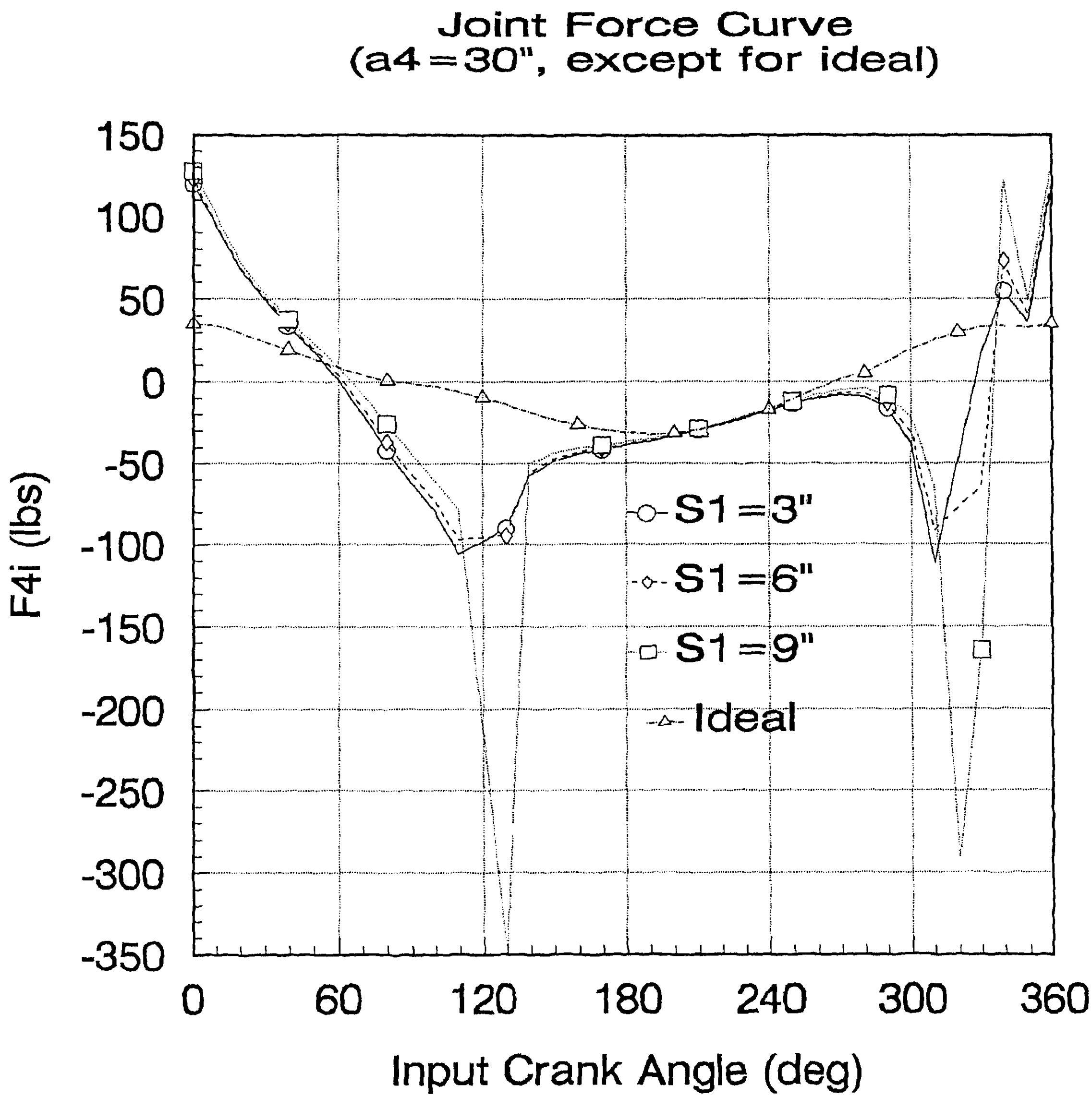
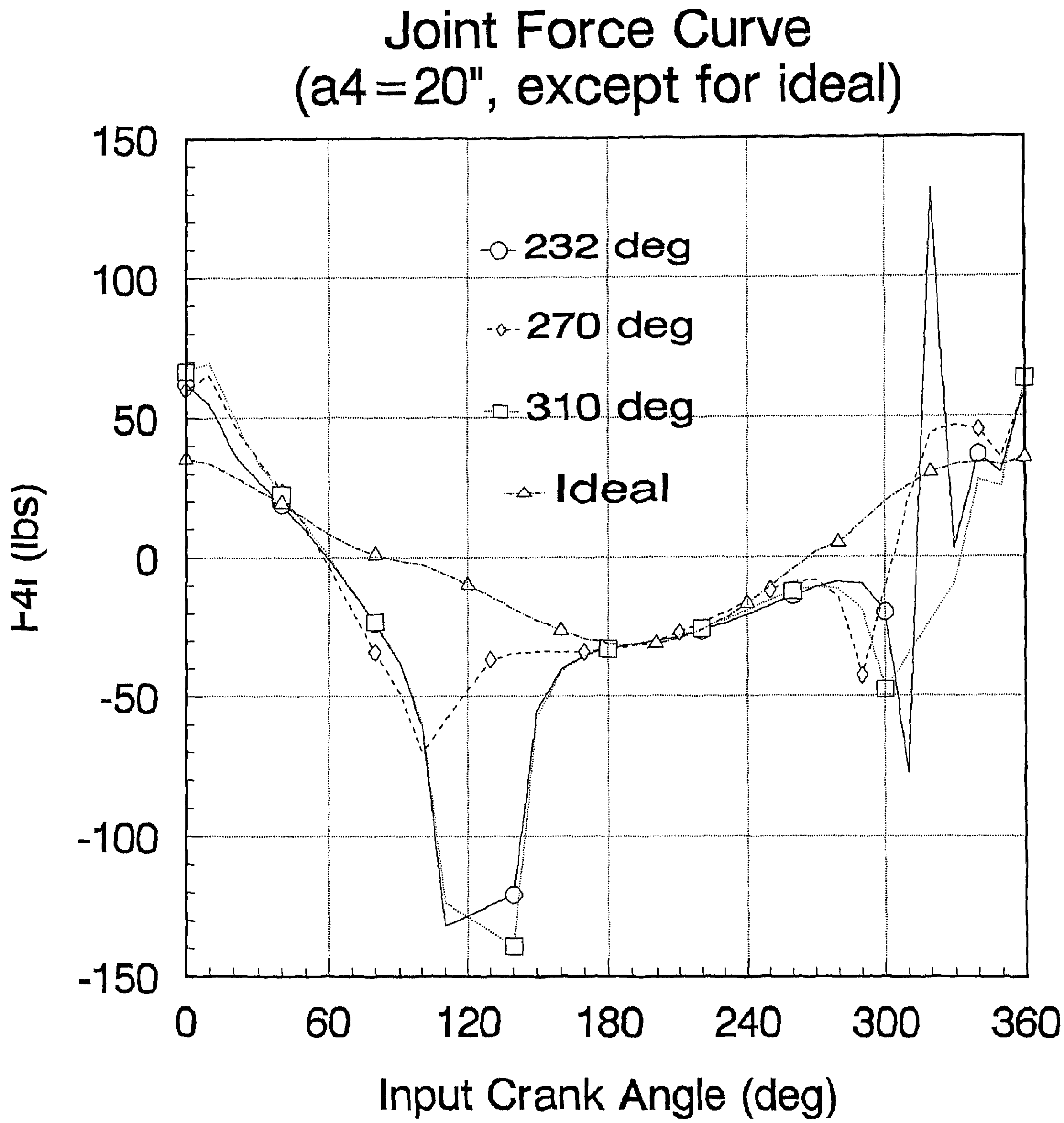
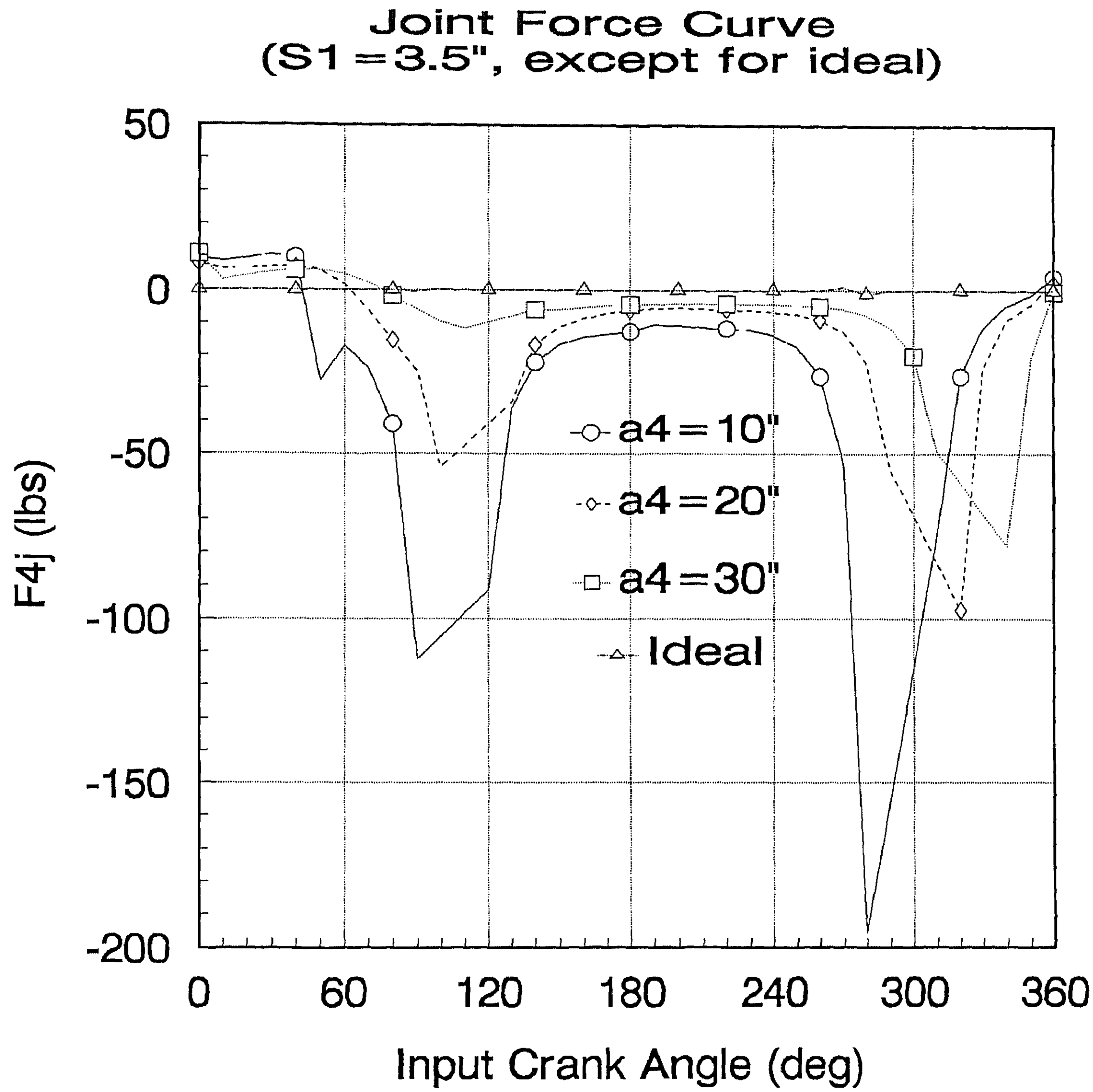

Joint Force Curve
(a4=30", except for ideal)
150
100
50
0
-50
-100
-150
-200
-250
-300
-350
F4i (lbs)
0
60
120
180
240
300
360
Input Crank Angle (deg)
S1=3"
S1=6"
S1=9"
Ideal

Figure 31



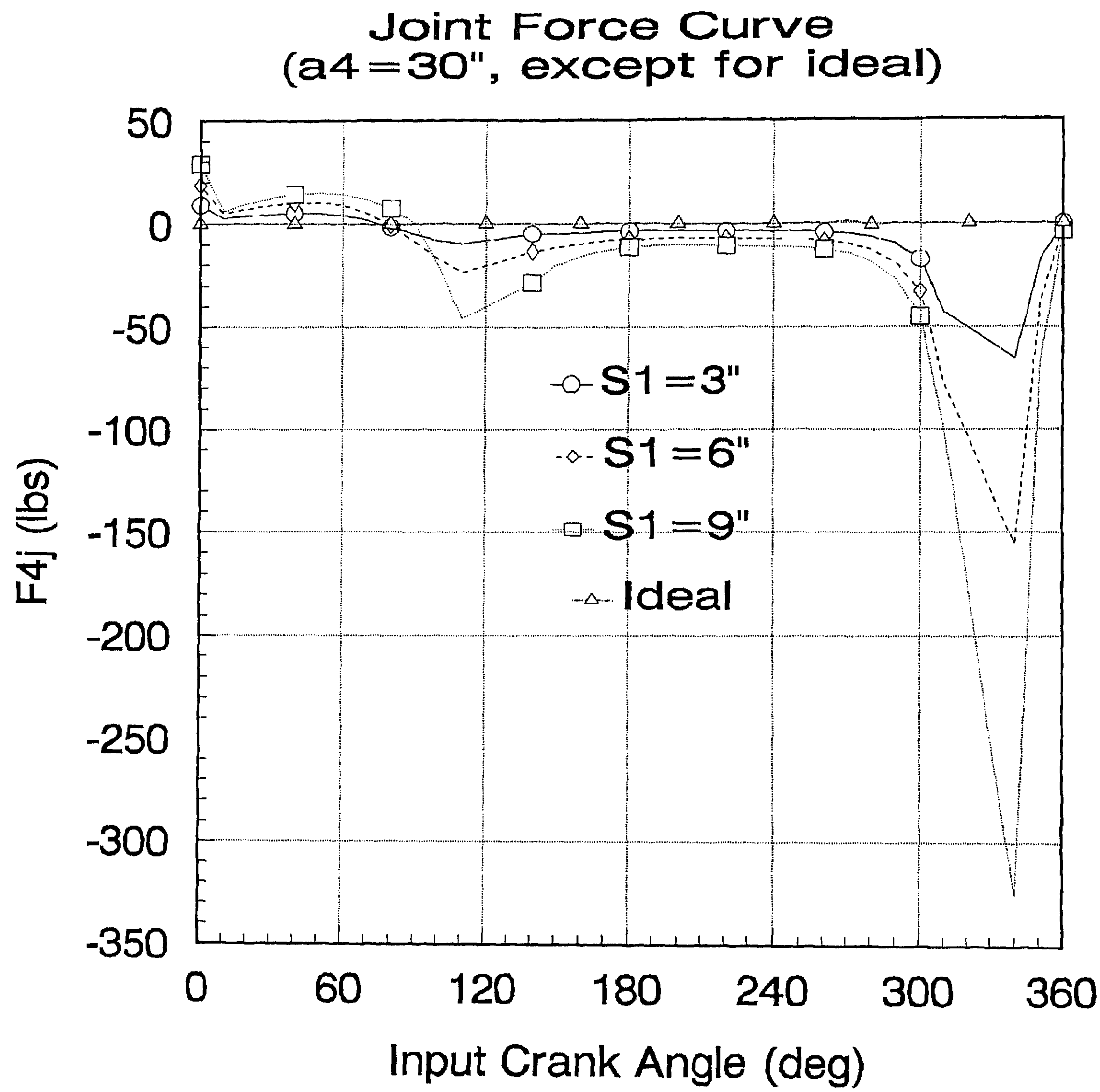
Joint Force Curve
(a4=20", except for ideal)
232 deg
270 deg
310 deg
Ideal
150
100
50
0
-50
-100
-150
F4l (lbs)
0
60
120
180
240
300
360
Input Crank Angle (deg)

Figure 32



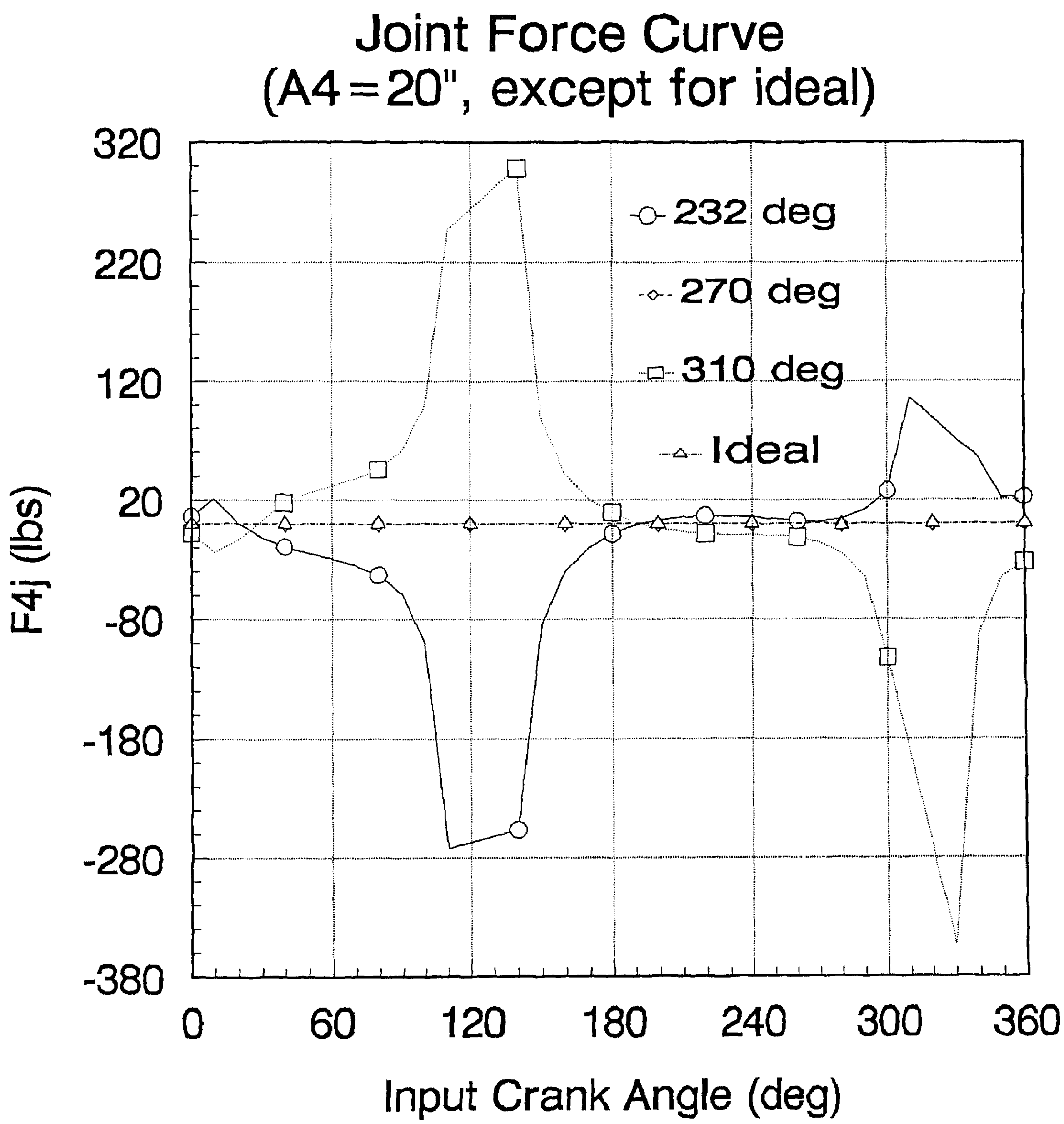
Joint Force Curve
(S1 =3.5", except for ideal)
50
0
-50
-100
-150
-200
F4j (lbs)
0
60
120
180
240
300
360
Input Crank Angle (deg)
a4=10"
a4=20"
a4=30"
Ideal

Figure33



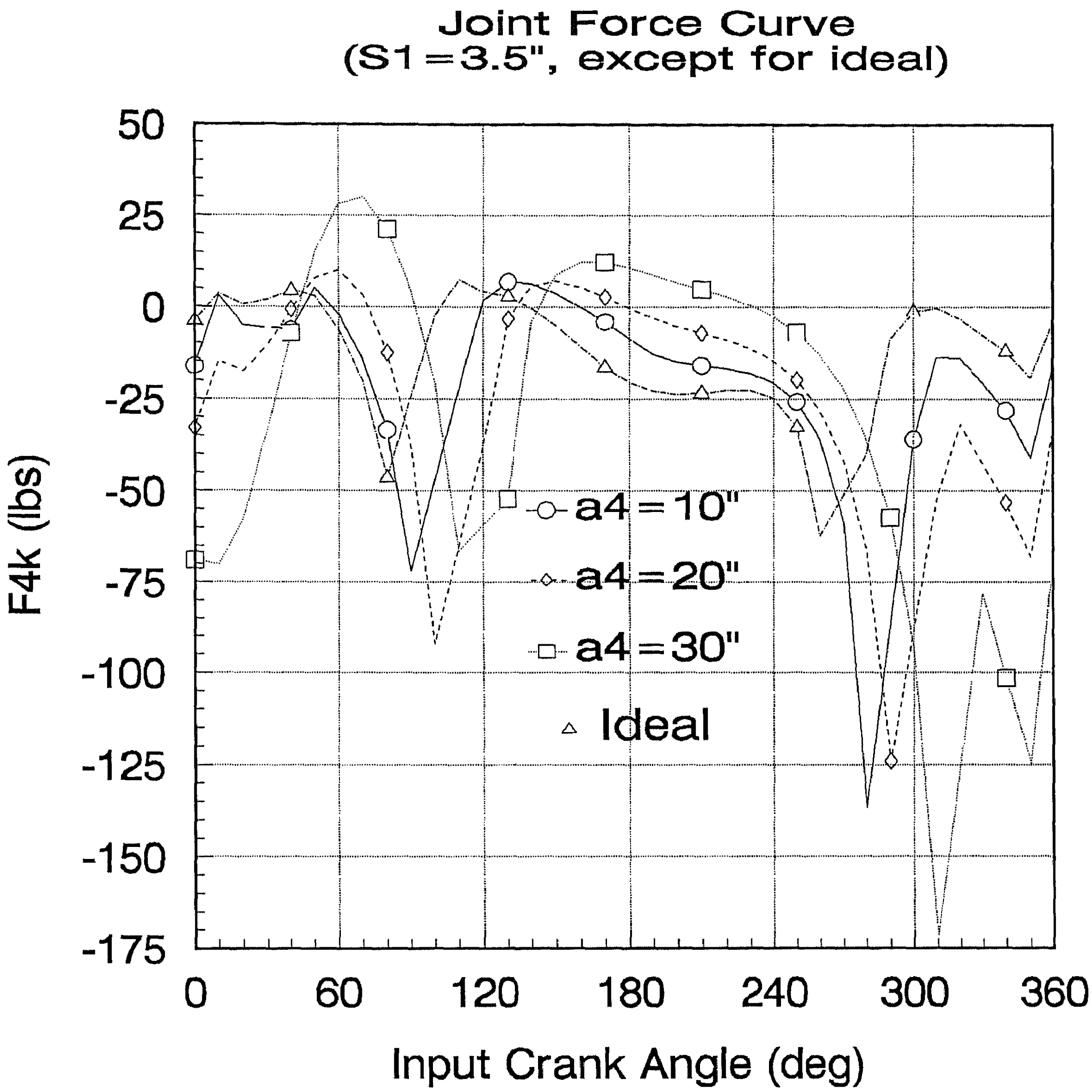
Joint Force Curve
(a4=30", except for ideal)
50
0
-50
-100
-150
-200
-250
-300
-350
F4j (lbs)
0
60
120
180
240
300
360
Input Crank Angle (deg)
S1=3"
S1=6"
S1=9"
Ideal

Figure 34



Joint Force Curve
(A4=20", except for ideal)
232 deg
270 deg
310 deg
Ideal
320
220
120
20
-80
-180
-280
-380
F4j (lbs)
0
60
120
180
240
300
360
Input Crank Angle (deg)

Figure 35



Joint Force Curve
(S1 = 3.5", except for ideal)
50
25
0
-25
-50
-75
-100
-125
-150
-175
F4k (lbs)
0
60
120
180
240
300
360
Input Crank Angle (deg)
a4 = 10"
a4 = 20"
a4 = 30"
Ideal

Figure 36

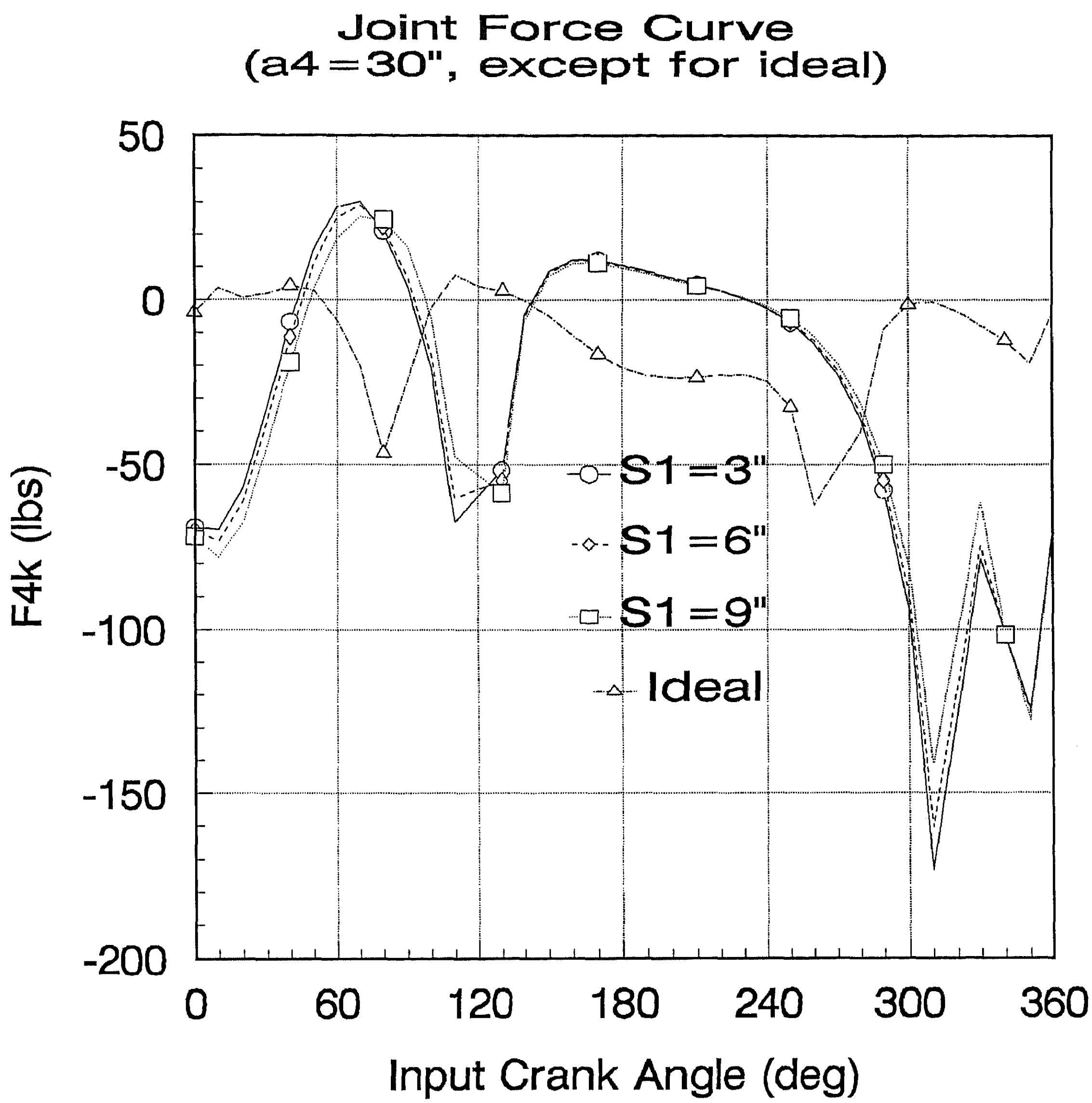

Joint Force Curve
(a4=30", except for ideal)
50
0
-50
-100
-150
-200
F4k (lbs)
0
60
120
180
240
300
360
Input Crank Angle (deg)
S1=3"
S1=6"
S1=9"
Ideal

Figure 37

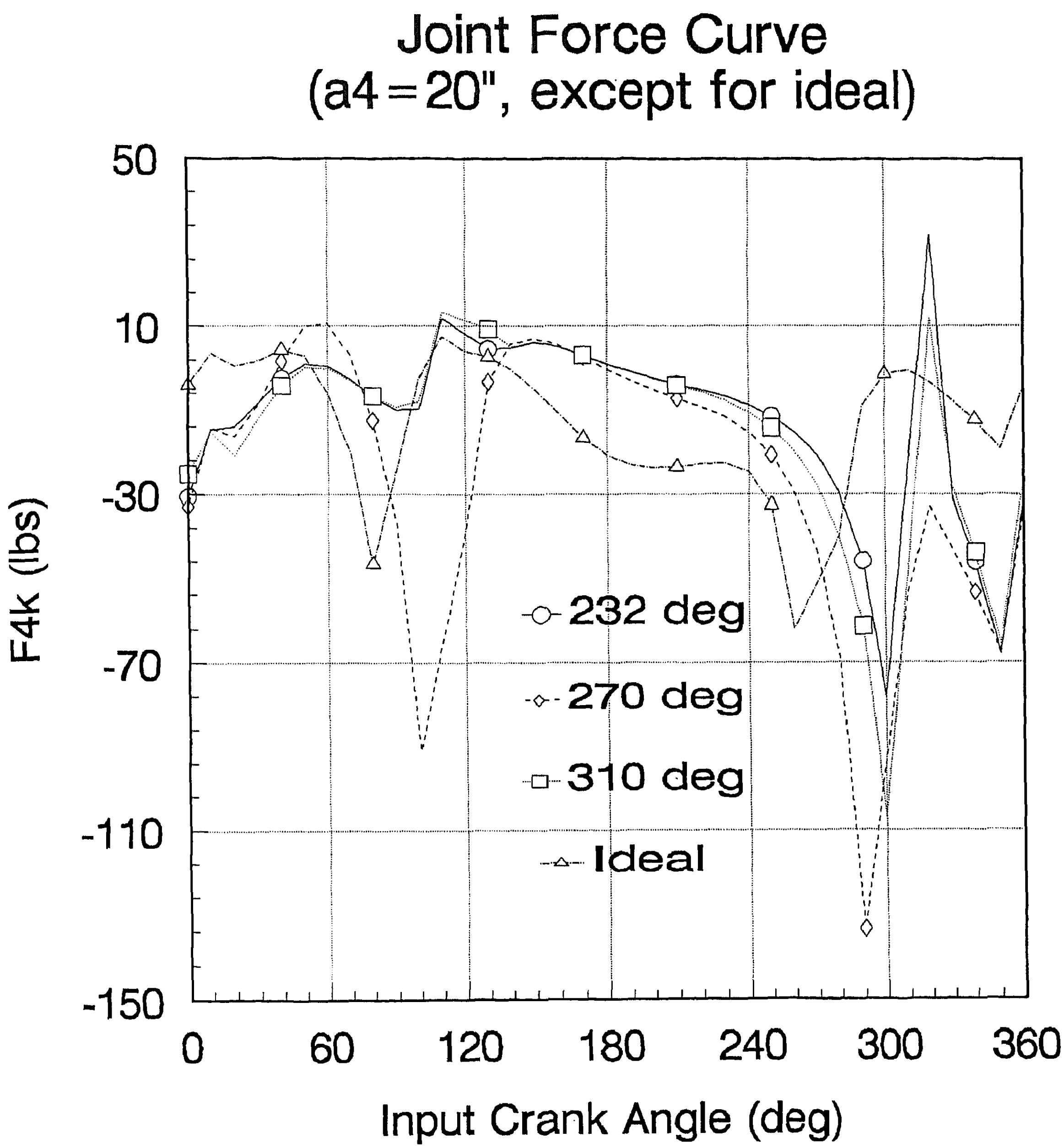

Joint Force Curve
(a4=20", except for ideal)
50
10
-30
-70
-110
-150
0
60
120
180
240
300
360
F4k (lbs)
Input Crank Angle (deg)
232 deg
270 deg
310 deg
Ideal

Figure 38