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ANALYSIS OF A BAND-SAW

by
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Analysis of a Band-Saw

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ABSTRACT

Analysis of a Band-Saw

by
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The solution for the transverse oscillations of a band-saw is determined. Two different models are treated: a fourth-order model and a second-order model. The response characteristics for both models are determined using Laplace transformation. To obtain the inverse Laplace Transform for the fourth-order model, it was necessary to find the frequencies by applying the Extended Lanczos Method in order to overcome the problem for computer overflow.

The limits of stability for both models are studied by plotting the eigenvalues against changing parameter values. Conditions for the onset of divergence and flutter instabilities, which need to be taken into account in designing a band-saw, are given. For increased axial tension, the critical velocities are shown to increase for both models. This serves as a means of increasing the stable region.

The less accurate second-order model yields solutions with relative ease. The accuracy of this solution is evaluated by comparing with the fourth order model. The results of the second-order came very close to those of the fourth-order model for high values of tension.

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This thesis is dedicated to
my wife Figen Can Ozsoylu
and to my parents Selma Ozsoylu and Professor Dr. Sinasi Ozsoylu

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Figure	Page

CHAPTER 1

INTRODUCTION

The axially moving beam appears frequently as an element in the mathematical modeling of various devices such as high-speed magnetic tapes, aerial cable tramways, band-saws, pipes with fluid flow, power transmission chains and belts. The study of transverse vibrations and the characterization of such systems is required for effective design.

This work will focus on the transverse vibration characteristics of a band-saw to illustrate the application of the Laplace Transformation to such problems and to demonstrate the advantages of this approach. The common model for a band-saw is the axially moving Bernoulli-Euler Beam under tension. The problem formulation and the solution for the transverse oscillations of this model will be made clear in the process of this study. Below, we will review the literature on this subject, and we will re-examine the mathematical model as well as the method of obtaining the solution, particularly with respect to the boundary conditions and external forces acting on the band-saw.

Simpson [2] studied the transverse modes and frequencies of beams, without the axial tension, axially translating between fixed-end supports. He presented numerical and graphical results to show the effects of the velocity of the beam on the natural frequencies. Mote [3] created a mathematical model of a band-saw treating the axially moving beam with axial tension and simple supports. His work included numerical and graphical results showing the change in the characteristics of the system for increasing values of the velocity. Recently, Mote and Wickert [1] studied a similar problem where modal analysis and the Green's Function method was used to find the natural frequencies, the modes and closed-form solutions of the fourth-order and second-order models. The steady state response of an axially moving strip subjected to lateral load was studied by Chonan [5], but his study pertains to motion in the plane of the strip; the present study as well as the other references cited are concerned with the motion perpendicular to the plane of the strip.

We will review the physical characteristics of the problem in order to justify our mathematical models. It turns out that a beam with one end fixed and the other simply supported needs to be studied. In addition, we need to examine the case in which the forcing function consists of a periodic displacement of one of the supports. The previous studies did not treat the case with the fixed-simple boundary conditions, nor the case of non-homogenous boundary conditions. For these and other reasons that will become clear later, we will use Laplace Transformation to obtain the solutions, and we will explain the advantages of this method in the treatment of this and similar problems. We will employ the Extended Lanczos' Method (Herman [4]) to obtain the eigen-frequencies and explain the reasons for this step. We will compare the results of the fourth-order model (Bernoulli-Euler Beam) with the much simpler second-order model (axially moving string in tension) to observe the differences in results. This will provide a basis for choosing between the second-order and fourth-order models.

Results are presented for a range of parameter values which were chosen from physical data on equipment and operating conditions recommended by the equipment manufacturer. The results include resonant frequencies and limits on the velocity of the band-saw based on divergence and flutter instabilities.

CHAPTER 2

FOURTH-ORDER MODEL

2.1 Discussion of the Problem

The main components of a band-saw assembly are shown in Figure 1. Basically, a mechanism provides tension in the steel blade (band) which is wrapped around the driving and the idler pulleys. For the cutting to take place, the driving pulley transmits motion to the blade by friction, where the friction force depends on the tension in the blade. Then, the work piece is fed to the cutting edge where additional supports (Marked C in Figure 1) are provided to guide and increase the rigidity of the blade. The location of the driving pulley, which corresponds to the left pulley in Figure 1, will be such that it will pull the band away from the work piece.

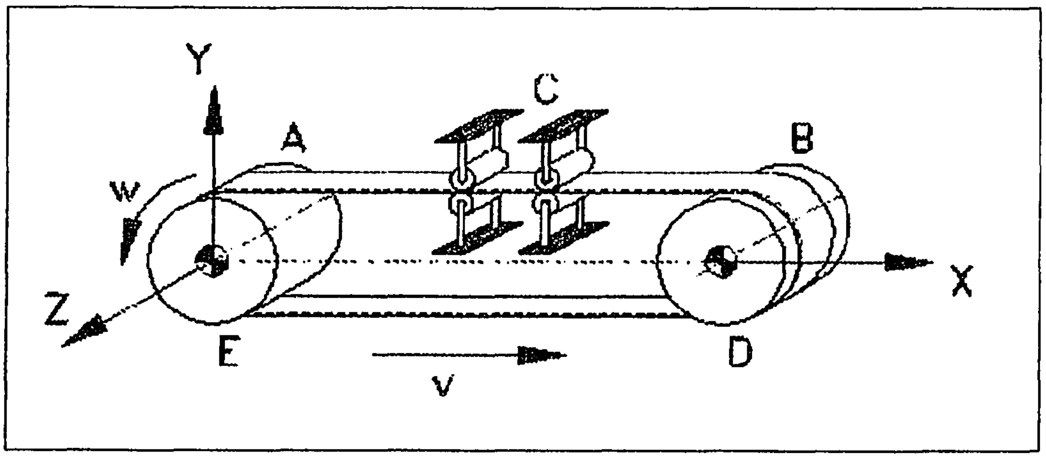

Figure 1 Schematic Band-Saw Assembly

To create a mathematical model of the band-saw, the assembly will be analyzed in three portions (Figure 1), AC, BC and ED. Each segment of the blade will be modeled as an axially moving Bernoulli-Euler beam under tension. Based on the coordinate frame defined in Figure 1, the transverse direction for all models is along the y-axis, and the width of the blade is along the z-axis. The boundary conditions for these models will be: simple support at the end adjacent to a pulley, and fixed support at the end adjacent to the cutting region (Marked C in Figure 1). Therefore, segments AC and BC will have simple-fixed supports; and segment ED will have simple-simple supports as their boundary conditions.

There are two reasons to designate the boundary conditions at a pulley to be simple supports. The first reason is that the moment caused by the transverse deflection will be negligible relative to the tension forces applied (the blade thickness is very small compared to the other dimensions in the assembly), and the second reason is that the band will follow the pulley's transverse deflection, whether the pulley is moving or stationary. These two characteristics are best approximated by a simple support.

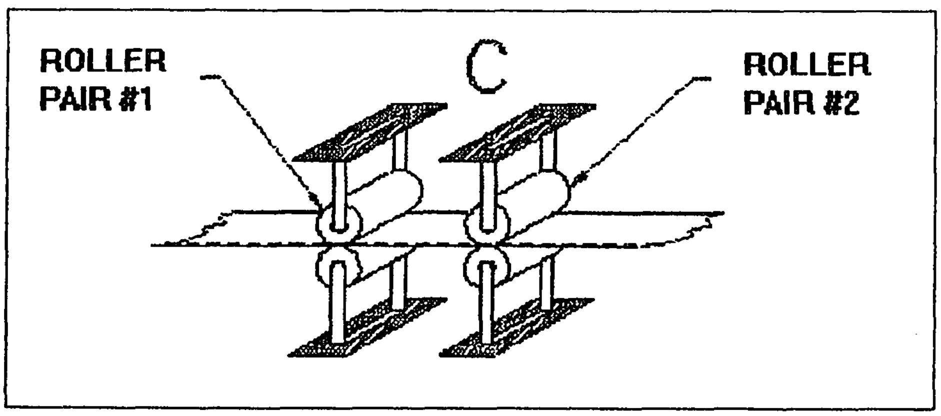

Figure 2 . The Supports at Point C in Figure 1.

The support assembly at the cutting area is shown in Figure 2. Because the distance between the roller pairs is small at all times, the support assembly will strongly resist bending and act as a fixed support for the portions of the band-saw assembly that are of interest. This behavior is enhanced when there is a work-piece between the support members (roller pairs) during the cutting procedure.

The beam width will be considered to be uniform and constant for the model, even though there are teeth at the cutting edge (Figure 3), because of the small size of the teeth relative to the width of the blade. Similarly, the thickness will be considered uniform throughout the length of the beam. The thickness, relative to the other dimensions, is small, therefore the rotary inertia and the transverse shearing deformation terms will be neglected.

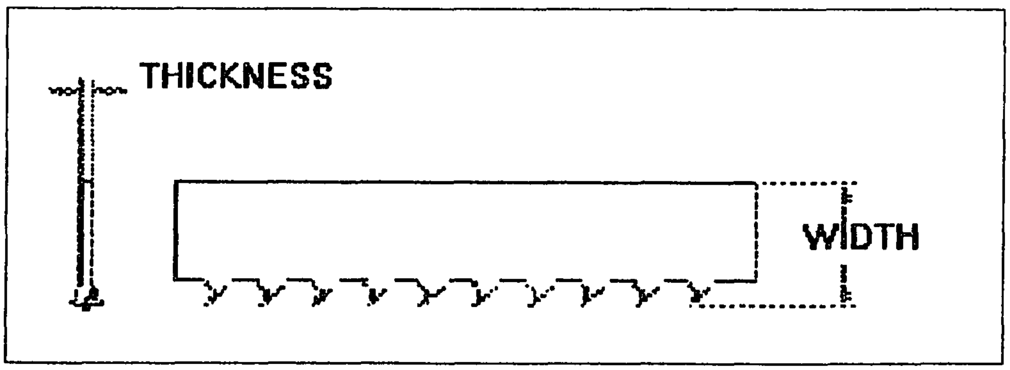

Figure 3 Band-Saw Blade

As previously stated, the pulleys apply tension to the blade. One of the reasons for tension in the blade is to provide the necessary friction force between the driving-pulley and the blade to overcome the cutting force. In addition, as we will find, the tension in the blade increases stability of operation. However, increased tension increases the fluctuating tensile stresses in the blade, reducing its life.

Throughout the cutting procedure, the change in the axial velocity of the blade will be considered negligible. Especially when the work-piece is cut manually, uneven feed would lead to speed fluctuation; but from a practical point of view, the machinist is expected to intervene to avoid such machine behavior. Besides, because of the transmission, such erratic feeding might cause unwanted slippage between the driving pulley and the band.

Damping will not be included in the model. Such non-conservative forces are very important in similar problems, such as magnetic tapes, but the governing forces in band-saws are too high for the damping to be significant.

2.2 Formulation of the Fourth Order Model

2.2.1 Simple-Simple Boundary Conditions

Figure 4 shows the axially moving Bernoulli-Euler beam with simple-simple boundary conditions. The blade excitation stems from the oscillations or the eccentricity of the driving pulley, and will have a frequency (w) equal to the angular velocity of the pulley. The equation of transverse motion of an axially moving Bernoulli-Euler beam (Mote[3]) is

$$(EI)u_{xxxx} + \delta\{2vu_{xt} + u_{tt} - v^2 u_{xx}\} - Pu_{xx} = 0 \qquad (2.1)$$

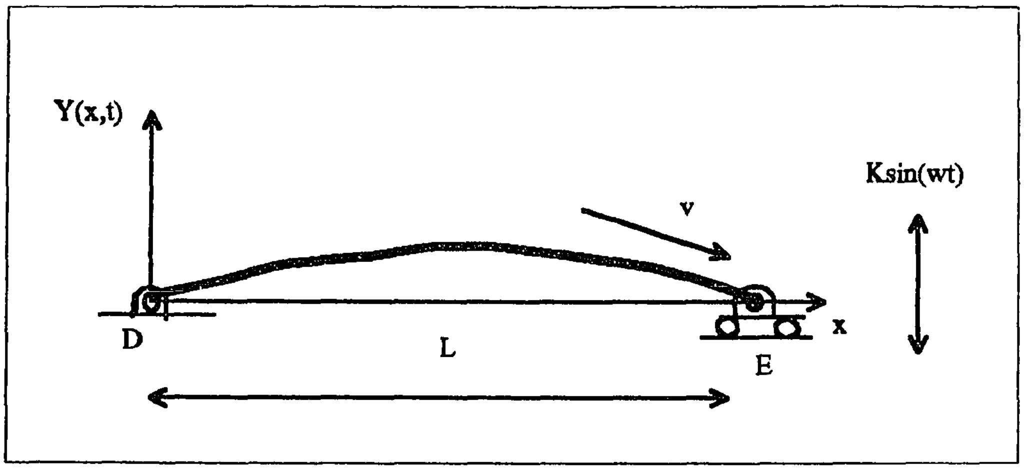

Figure 4 The Axially Moving Bernoulli-Euler Beam Under Tension

The boundary conditions are

$$u(x=0,t) = u_{xx}(x=0,t) = u_{xx}(x=L,t) = 0; \tag{2.2}$$

$$u(x=L,t) = K\sin(wt)$$

Where E : Elastic modulus
I : Area moment of inertia
δ : Density per unit length
U : Transverse displacement
U_x : Derivative with respect to x
v : Velocity of the beam in the axial direction
K : Transverse displacement of the support
w : Frequency of the excitation
L : The length between supports
P : Axial tensile force

In non-dimensionalized form, Equations (2.1) and (2.2) will be

$$Y_{XXXX} + \tfrac{1}{EI}\{\delta v^2 L^2 - PL^2\} Y_{XX} + v2L(\tfrac{\delta}{EI})^{\frac{1}{2}} Y_{XT} + Y_{TT} = 0 \tag{2.3}$$

$$Y(X=0,T) = Y_{XX}(X=0,T) = Y_{XX}(X=1,T) = 0 \tag{2.4}$$

$$Y(X=1,T) = K\sin(\beta T)$$

where

$$Y(X,T) = \frac{U}{L}; \quad X = \frac{x}{L}; \quad T = t\sqrt{\frac{EI}{\delta L^4}}; \quad \beta = w\sqrt{\frac{\delta L^4}{EI}} \tag{2.5}$$

For convenience this equation can be further simplified by substituting the parameters $b_2 = \frac{1}{EI}\{\delta v^2 L^2 - PL^2\}$, $b_1 = v2L(\frac{\delta}{EI})^{\frac{1}{2}}$:

$$Y_{XXXX} + b_2 Y_{XX} + b_1 Y_{XT} + Y_{TT} = 0 \tag{2.6}$$

The governing differential operator of Equation (2.6) is non-selfadjoint. In addition, the boundary conditions are non-homogenous. These two conditions make it impossible to satisfy the orthogonality conditions that are associated with the classical eigenvalue solution technique (Appendix A). To overcome these difficulties, Laplace transforms of Equations (2.5) and (2.6) are taken with respect to T, where the initial conditions are taken to be:

$$Y_X(X,T=0) = 0; \quad Y(X,T=0) = 0; \quad Y_T(X,T=0) = 0 \tag{2.7}$$

Then Equation (2.5) will become

$$Y_{XXXX}(X,s) + b_2 Y_{XX}(X,s) + b_1 s Y_X(X,s) + s^2 Y(X,s) = 0 \tag{2.8}$$

with the boundary conditions

$$Y(X=0,s) = Y_{XX}(X=0,s) = Y_{XX}(X=1,s) = 0 \tag{2.9}$$

$$Y(X=1,s) = \frac{K\beta}{s^2+\beta^2}$$

A solution of the form $Y(X,s) = Ae^{\lambda x}$ is assumed for Equation (2.8). When the solution is substituted, the characteristic equation (that is a fourth order polynomial in λ) will be found.

$$\lambda^4 + b_2\lambda^2 + b_1 s\lambda + s^2 = 0 \tag{2.10}$$

To find the form of the roots, Equation (2.10) will be written as a product of two quadratic equations.

$$(\lambda^2 - a.\lambda + r_1)(\lambda^2 + a.\lambda + r_2) = \lambda^4 + b_2\lambda^2 + b_1 s\lambda + s^2 = 0 \tag{2.11}$$

Equating the coefficients will give the following equations;

$$b_2 + a^2 = r_1 + r_2$$

$$\frac{b_1 s}{a} = r_1 - r_2 \qquad\qquad r_1 r_2 = s^2 \tag{2.12a}$$

$$r_1 = \frac{1}{2}\{b_2 + a^2 + \frac{b_1 s}{a}\}; \quad r_2 = \frac{1}{2}\{b_2 + a^2 - \frac{b_1 s}{a}\}$$

Then the roots will be as follows:

$$\lambda_1 = \frac{a+\sqrt{a^2-4r_2}}{2} \quad \lambda_2 = \frac{a-\sqrt{a^2-4r_1}}{2} \quad \lambda_3 = \frac{-a+\sqrt{a^2-4r_1}}{2} \quad \lambda_4 = \frac{-a-\sqrt{a^2-4r_2}}{2} \tag{2.12b}$$

Assuming all roots to be distinct, the solution will be as follows:

$$Y(X,s) = A_1 e^{\lambda_1 X} + A_2 e^{\lambda_2 X} + A_3 e^{\lambda_3 X} + A_4 e^{\lambda_4 X} \tag{2.13}$$

To find the unknown parameters A_1, A_2, A_3, A_4 , the boundary conditions are applied to Equation (2.13). The resultant four equations in four unknowns will be written in a matrix form, where the unknowns can be obtained using matrix algebra.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{\lambda_1} & e^{\lambda_2} & e^{\lambda_3} & e^{\lambda_4} \\ \lambda_1^2 \cdot e^{\lambda_1} & \lambda_2^2 \cdot e^{\lambda_2} & \lambda_3^2 \cdot e^{\lambda_3} & \lambda_4^2 \cdot e^{\lambda_4} \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} 0 \\ K \cdot \dfrac{\beta}{\beta^2 + s^2} \\ 0 \\ 0 \end{bmatrix} \tag{2.14}$$

The unknown coefficients are found to be:

$$A_1 = \frac{f_1(s)}{g(s)}$$

$$f_1(s) = K\beta\{\lambda_2^2(\lambda_4^2 - \lambda_3^2)e^{\lambda_2} + \lambda_3^2(\lambda_2^2 - \lambda_4^2)e^{\lambda_3} - \lambda_4^2(\lambda_2^2 - \lambda_3^2)e^{\lambda_4}\} \tag{2.15a}$$

$$A_2 = \frac{f_2(s)}{g(s)}$$

$$f_2(s) = K\beta\{\lambda_1^2(\lambda_3^2 - \lambda_4^2)e^{\lambda_1} - \lambda_3^2(\lambda_1^2 - \lambda_4^2)e^{\lambda_3} - \lambda_4^2(\lambda_3^2 - \lambda_1^2)e^{\lambda_4}\} \tag{2.15b}$$

$$A_3 = \frac{f_3(s)}{g(s)}$$

$$f_3(s) = K\beta\{\lambda_1^2(\lambda_4^2 - \lambda_2^2)e^{\lambda_1} + \lambda_2^2(\lambda_1^2 - \lambda_4^2)e^{\lambda_2} - \lambda_4^2(\lambda_1^2 - \lambda_2^2)e^{\lambda_4}\} \tag{2.15c}$$

$$A_4 = \frac{f_4(s)}{g(s)}$$

$$f_4(s) = K\beta\{\lambda_1^2(\lambda_2^2 - \lambda_3^2)e^{\lambda_1} - \lambda_2^2(\lambda_1^2 - \lambda_3^2)e^{\lambda_2} + \lambda_3^2(\lambda_1^2 - \lambda_2^2)e^{\lambda_3}\} \tag{2.15d}$$

$$g(s)=(s^2+\beta^2)\{(\lambda_1^2-\lambda_2^2)(\lambda_4^2-\lambda_3^2)(e^{\lambda_1+\lambda_2}+e^{\lambda_3+\lambda_4})+$$

$$(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)(e^{\lambda_1+\lambda_3}+e^{\lambda_2+\lambda_4})+(\lambda_1^2-\lambda_4^2)(\lambda_3^2-\lambda_2^2)(e^{\lambda_1+\lambda_4}+e^{\lambda_2+\lambda_3})\}\qquad(2.15e)$$

Then Equation (2.13) can be rewritten in terms of the new functions introduced as:

$$Y(X,s)=\frac{f_1(s)}{g(s)}e^{\lambda_1 X}+\frac{f_2(s)}{g(s)}e^{\lambda_2 X}+\frac{f_3(s)}{g(s)}e^{\lambda_3 X}+\frac{f_4(s)}{g(s)}e^{\lambda_4 X}\qquad(2.16)$$

Equation (2.16) represents the solution to the problem in the X-s plane. To find the solution in the X-T plane, it is necessary to find the inverse of Equation (2.16). This is obtained by summing the residues of the poles of Y(X,s) in the complex s-plane, [6]; equivalently we need to find the zeros of g(s). From Equation (2.15e), the only apparent zeros are $s=\pm i\beta$. Efforts (Appendix C) to find the rest of the zeros of g(s) were unsuccessful, because the range of values of g(s), in the region of interest, exceeds the magnitudes that the computer can handle. To circumvent this problem, Equation (2.8) is treated as an eigenvalue problem with s as the eigenvalue parameter. This is equivalent to finding the zeros of g(s) besides $s=\pm i\beta$. The method employed in obtaining the eigenvalues is the Extended Lanczos' Method presented by Herman [4]. Then the inversion of Y(X,s) will be:

$$Y(X,T)=\sum_r\{\frac{f_1(s_r)}{g'(s_r)}e^{\lambda_1 X}e^{s_r t}\}+\sum_r\{\frac{f_2(s_r)}{g'(s_r)}e^{\lambda_2 X}e^{s_r t}\}+\sum_r\{\frac{f_3(s_r)}{g'(s_r)}e^{\lambda_3 X}e^{s_r t}\}+$$

$$\sum_r\{\frac{f_4(s_r)}{g'(s_r)}e^{\lambda_4 X}e^{s_r t}\}\qquad(2.17)$$

Where s_r are the poles of the of Y(X,s) and, g'(s) is the derivative of g(s) with respect to s.

To find the derivative of the denominator, g(s) will be treated as a function λ and s where $\lambda=\lambda(s)$. Applying the chain rule:

$$\frac{dg}{ds} = \left\{\sum_{i=1}^{4} \frac{\partial g}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial s}\right\} + \frac{\partial g}{\partial s} \tag{2.18}$$

where $\frac{\partial}{\partial \lambda}$ and $\frac{\partial}{\partial s}$ represent partial differentiation with respect to λ and s respectively.

$\frac{\partial \lambda}{\partial s}$ are found from the characteristic equation (Equation (2.10)) :

$$\frac{\partial \lambda_i}{\partial s} = -\frac{b_1\lambda_i+2s}{4\lambda_i^3+2b_2\lambda_i+b_1 s} \quad ; \quad i=1, 2, 3, 4 \tag{2.19}$$

And

$$\begin{aligned}\frac{\partial g}{\partial \lambda_1} = (s^2+\beta^2)\{&(\lambda_3^2-\lambda_4^2)\left[\lambda_1^2+2\lambda_1-\lambda_2^2\right]e^{\lambda_1+\lambda_2}+\\ &(\lambda_4^2-\lambda_2^2)\left[\lambda_1^2+2\lambda_1-\lambda_3^2\right]e^{\lambda_1+\lambda_3}+(\lambda_2^2-\lambda_3^2)\left[\lambda_1^2+2\lambda_1-\lambda_4^2\right]e^{\lambda_1+\lambda_4}+\\ &2\lambda_1[(\lambda_2^2-\lambda_3^2)e^{\lambda_2+\lambda_3}-(\lambda_2^2-\lambda_4^2)e^{\lambda_2+\lambda_4}+(\lambda_3^2-\lambda_4^2)e^{\lambda_3+\lambda_4}]\}\end{aligned} \tag{2.20a}$$

$$\begin{aligned}\frac{\partial g}{\partial \lambda_2} = (s^2+\beta^2)\{&(\lambda_3^2-\lambda_4^2)\left[\lambda_1^2+2\lambda_2-\lambda_2^2\right]e^{\lambda_1+\lambda_2}+\\ &(\lambda_1^2-\lambda_4^2)\left[\lambda_2^2+2\lambda_2-\lambda_3^2\right]e^{\lambda_2+\lambda_3}+(\lambda_3^2-\lambda_1^2)\left[\lambda_2^2+2\lambda_2-\lambda_4^2\right]e^{\lambda_2+\lambda_4}+\\ &2\lambda_2[(\lambda_3^2-\lambda_1^2)e^{\lambda_1+\lambda_3}+(\lambda_1^2-\lambda_4^2)e^{\lambda_1+\lambda_4}+-(\lambda_3^2-\lambda_4^2)e^{\lambda_3+\lambda_4}]\}\end{aligned} \tag{2.20b}$$

$$\frac{\partial g}{\partial \lambda_3} = (s^2+\beta^2)\{(\lambda_2^2-\lambda_4^2)\left[\lambda_3^2+2\lambda_3-\lambda_1^2\right]e^{\lambda_1+\lambda_3}+$$

$$(\lambda_1^2-\lambda_4^2)\left[\lambda_2^2-2\lambda_3-\lambda_3^2\right]e^{\lambda_2+\lambda_3}+(\lambda_1^2-\lambda_2^2)\left[\lambda_3^2+2\lambda_3-\lambda_4^2\right]e^{\lambda_3+\lambda_4}+$$

$$2\lambda_3[(\lambda_1^2-\lambda_2^2)e^{\lambda_1+\lambda_2}-(\lambda_1^2-\lambda_4^2)e^{\lambda_1+\lambda_4}+(\lambda_2^2-\lambda_4^2)e^{\lambda_2+\lambda_4}]\} \qquad (2.20c)$$

$$\frac{\partial g}{\partial \lambda_4} = (s^2+\beta^2)\{(\lambda_2^2-\lambda_3^2)\left[\lambda_1^2-2\lambda_4-\lambda_4^2\right]e^{\lambda_1+\lambda_4}+$$

$$(\lambda_3^2-\lambda_1^2)\left[\lambda_2^2-2\lambda_4-\lambda_4^2\right]e^{\lambda_4+\lambda_2}+(\lambda_1^2-\lambda_2^2)\left[\lambda_3^2-2\lambda_4-\lambda_4^2\right]e^{\lambda_3+\lambda_4}+$$

$$2\lambda_4[(\lambda_2^2-\lambda_1^2)e^{\lambda_2+\lambda_1}+(\lambda_1^2-\lambda_3^2)e^{\lambda_1+\lambda_3}+(\lambda_3^2-\lambda_2^2)e^{\lambda_2+\lambda_3}]\} \qquad (2.20d)$$

$$\frac{\partial g(s)}{\partial s} = 2s\{(\lambda_1^2-\lambda_2^2)(\lambda_4^2-\lambda_3^2)(e^{\lambda_1+\lambda_2}+e^{\lambda_3+\lambda_4})+ \qquad (2.20e)$$

$$(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)(e^{\lambda_1+\lambda_3}+e^{\lambda_2+\lambda_4})+(\lambda_1^2-\lambda_4^2)(\lambda_3^2-\lambda_2^2)(e^{\lambda_1+\lambda_4}+e^{\lambda_2+\lambda_3})\}$$

Then Equation (2.14) can be rewritten as:

$$Y(X,T) = \sum_r\{\frac{f_1(s_r)}{\sum_{i=1}^{4}\frac{\partial g(s_r)}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial s}+\frac{\partial g}{\partial s}}e^{\lambda_1 X}e^{s_r.t}\}+\sum_r\{\frac{f_2(s_r)}{\sum_{i=1}^{4}\frac{\partial g(s_r)}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial s}+\frac{\partial g}{\partial s}}e^{\lambda_2 X}e^{s_r.t}\}+$$

$$\sum_r\{\frac{f_3(s_r)}{\sum_{i=1}^{4}\frac{\partial g(s_r)}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial s}+\frac{\partial g}{\partial s}}e^{\lambda_3 X}e^{s_r.t}\}+\sum_r\{\frac{f_4(s_r)}{\sum_{i=1}^{4}\frac{\partial g(s_r)}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial s}+\frac{\partial g}{\partial s}}e^{\lambda_4 X}e^{s_r.t}\} \qquad (2.21)$$

For convenience, Equation (2.21) can be further simplified as:

$$Y(X,T) = \sum_{i=1}^{4}\sum_r C_i(s_r)e^{\lambda_i X}e^{s_r T} \qquad (2.22)$$

Previously, the s_r were defined to be the roots of g(s) (Equation (2.15e)). Recall that the Extended Lanczos' Method [4] was adapted to this problem to find the roots besides $s = \pm i\beta$, because the method yields a polynimial in s, and allows the computation of the roots in a way that circumvents the problem of computer overflow. Within the stable region, s_r will take complex values and will appear in conjugate pairs. Similarly, both C_i and λ_i's appear in complex conjugate pairs, having the following properties.

$$\tilde{s} = s_{r_{Real}} - i s_{r_{Imaginary}}$$

$$\lambda_i(\tilde{s}_r) = \tilde{\lambda}_i(s_r) = \lambda_{i_{Real}}(s_r) - i\lambda_{i_{Imaginary}}(s_r) \tag{2.23}$$

$$C_i(\tilde{s}_r) = \tilde{C}_i(s_r) = C_{i_{Real}}(s_r) - iC_{i_{Imaginary}}(s_r)$$

Then the final form of the solution will become a real valued function:

$$\begin{aligned} Y(X,T) = \sum_{r=1}^{\infty}\sum_{i=1}^{4}\{ & 2C_{i_{Real}} e^{(\lambda_{i_{Real}}X + s_{r_{Real}}T)} \cos(\lambda_{i_{imaginary}}X + s_{r_{imaginary}}T) - \\ & 2C_{i_{imaginary}} e^{(\lambda_{i_{Real}}X + s_{r_{Real}}T)} \sin(\lambda_{i_{imaginary}}X + s_{r_{imaginary}}T)\} \end{aligned} \tag{2.24}$$

where

$$C_i = C_{i_{Real}} + iC_{i_{Imaginary}}$$

$$\lambda_i = \lambda_{i_{Real}} + i\lambda_{i_{Imaginary}} \tag{2.25}$$

$$s_r = s_{r_{Real}} + i s_{r_{Imaginary}}$$

The resonance frequencies correspond to the excitation frequencies at which the magnitude of the oscillations become large when the system is at steady state. Mathematically, these frequencies are the excitation frequencies (β) that make the denominator of the steady state response zero (g'(s)=0).

$$g'(s_r) = \sum_{i=1}^{4} \frac{\partial g(s_r)}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial s} + \frac{\partial g(s_r)}{\partial s} = 0 \qquad (2.26)$$

The expression for g(s) written in a simplified form for convenience.

$$g(s) = (s^2 + \beta^2)g_1(s) \qquad (2.27)$$

$$g'(s) = 2sg_1(s) + (s^2 + \beta^2)g_1'(s) \qquad (2.28)$$

Then, for any s_r corresponding to a root of g_1 (s), Equation (2.26) will be

$$g'(s_r) = (s_r^2 + \beta^2)g_1'(s_r) \qquad (2.29)$$

When β is equal to $\pm is_r$, Equation (2.29) becomes zero, but β can only be real (excitation frequency), therefore, g'(s) will be minimized as function of β for fixed s_r when

$$Re\{s_r^2 + \beta^2) = 0; \qquad \Rightarrow \qquad \beta^2 = s^2_{r_{Imaginary}} - s^2_{r_{Real}} \qquad (2.30)$$

On the other hand, when $s = \pm i\beta$ Equation (2.28) will become

$$g'(\pm i\beta) = \pm i2\beta g_1(\pm i\beta)$$

If the final solution (Equation (2.17)) is considered , β will cancel from this expression as it will appear to multiply both the denominator and the numerator. Then, the resonance will occur at the value of β, closest to the roots of g_1 (s), which will correspond to the magnitude of the imaginary parts of the system's eigenvalues. These frequencies will be the resonance frequencies that the operating conditions should avoid. Even though the resonance frequencies are determined, the system may become unstable before it attains steady state. The instabilities associated with the system are discussed in Chapter 5.

2.2.2 Simple-Fixed Support

Figure 5 shows the axially moving Bernoulli-Euler beam under tension with fixed-simple supports. The only difference in form between the simple-fixed and the simple-simple support case is that one of the boundary conditions is a fixed support. Therefore, Equation (2.6) is still valid but the boundary conditions need to be redefined.

$$\frac{\partial^2 Y(X=0,T)}{\partial X^2} = Y(X=1,T) = \frac{\partial Y(X=1,T)}{\partial X} = 0 \tag{2.31}$$

$$Y(X=0,T) = K\sin(\beta T)$$

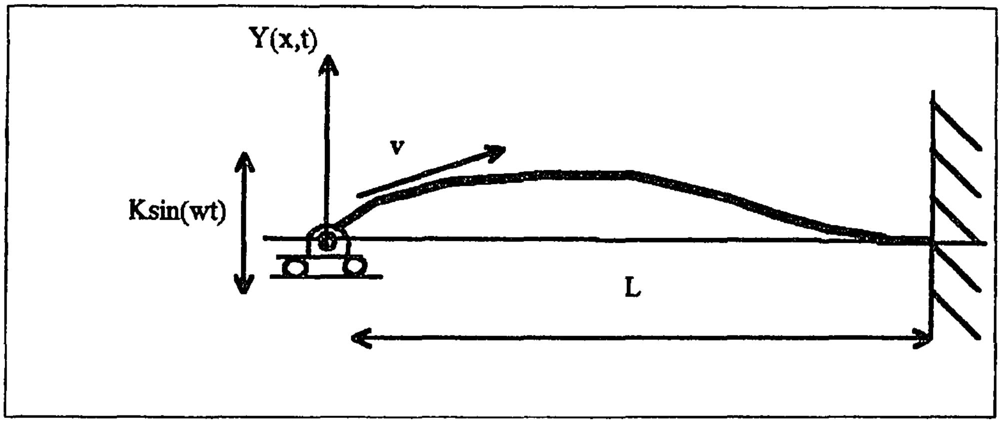

Figure 5 The Fixed-Simple Support Beam Model For segments AC and BC in Figure 1

For similar reasons discussed for the simple-simple support case, the Laplace transform of Equation (2.6) and (2.31) will be taken with respect to T, where the initial conditions are as in Equation (2.7).

$$Y_{XXXX}(X,s) + b_2 Y_{XX}(X,s) + b_1 s Y_X(X,s) + s^2 Y(X,s) = 0 \qquad (2.32a)$$

$$Y_{XX}(X=0,s) = Y(X=1,s) = Y_X(X=1,s) = 0; \qquad (2.32b)$$

$$Y(X=0,s) = \frac{K\beta}{s^2+\beta^2} \qquad (2.32c)$$

A solution of the form $Y(X,s) = Ae^{\lambda X}$ is assumed to get the characteristic equation (a fourth order polynomial in λ). The characteristic equation is similar to Equation (2.10); therefore the roots of the characteristic equation as well as the form of the solution will be similar to Equations (2.12) and (2.13) respectively. Then

$$Y(X,s) = \sum_{i=1}^{4} A_i e^{\lambda_i X} \qquad (2.33)$$

The boundary conditions will be applied to Equation (2.33) and the set of equations obtained will be written in matrix form, from which the unknowns can be determined using matrix algebra.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ e^{\lambda_1} & e^{\lambda_2} & e^{\lambda_3} & e^{\lambda_4} \\ \lambda_1 \cdot e^{\lambda_1} & \lambda_2 \cdot e^{\lambda_2} & \lambda_3 \cdot e^{\lambda_3} & \lambda_4 \cdot e^{\lambda_4} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} K \cdot \frac{\beta}{s^2+\beta^2} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.34}$$

The unknown coefficients will be;

$$A_1 = \frac{f_1(s)}{g(s)}; \qquad A_2 = \frac{f_2(s)}{g(s)}; \qquad A_3 = \frac{f_3(s)}{g(s)}; \qquad A_4 = \frac{f_4(s)}{g(s)}; \tag{2.35}$$

and

$$f_1(s) = K\beta\{-\lambda_4^2(\lambda_2 - \lambda_3)e^{\lambda_2+\lambda_3} + \lambda_3^2(\lambda_2 - \lambda_4)e^{\lambda_2+\lambda_4} + \lambda_2^2(\lambda_4 - \lambda_3)e^{\lambda_3+\lambda_4}\} \tag{2.36a}$$

$$f_2(s) = K\beta\{\lambda_4^2(\lambda_1 - \lambda_3)e^{\lambda_1+\lambda_3} - \lambda_3^2(\lambda_1 - \lambda_4)e^{\lambda_1+\lambda_4} + \lambda_1^2(\lambda_3 - \lambda_4)e^{\lambda_3+\lambda_4}\} \tag{2.36b}$$

$$f_3(s) = K\beta\{-\lambda_4^2(\lambda_1 - \lambda_2)e^{\lambda_1+\lambda_2} + \lambda_2^2(\lambda_1 - \lambda_4)e^{\lambda_1+\lambda_4} + \lambda_1^2(\lambda_4 - \lambda_2)e^{\lambda_2+\lambda_4}\} \tag{2.36c}$$

$$f_4(s) = K\beta\{\lambda_3^2(\lambda_1 - \lambda_2)e^{\lambda_2+\lambda_1} - \lambda_2^2(\lambda_1 - \lambda_3)e^{\lambda_1+\lambda_3} + \lambda_1^2(\lambda_2 - \lambda_3)e^{\lambda_3+\lambda_2}\} \tag{2.36d}$$

$$g(s) = (s^2 + \beta^2)\{\{\lambda_1\left(\lambda_3^2 - \lambda_4^2\right) + \lambda_2\left(\lambda_4^2 - \lambda_3^2\right)\}e^{\lambda_1+\lambda_2} +$$

$$\{\lambda_1\left(\lambda_4^2 - \lambda_2^2\right) + \lambda_3\left(\lambda_2^2 - \lambda_4^2\right)\}e^{\lambda_1+\lambda_3}$$

$$\{\lambda_1^2(\lambda_2 - \lambda_3) + \lambda_4^2(\lambda_3 - \lambda_2)\}e^{\lambda_2+\lambda_3} + \{\lambda_1^2(\lambda_4 - \lambda_2) + \lambda_3^2(\lambda_2 - \lambda_4)\}e^{\lambda_2+\lambda_4} +$$

$$\{\lambda_1^2(\lambda_3 - \lambda_4) + \lambda_2^2(\lambda_4 - \lambda_3)\}e^{\lambda_3+\lambda_4}\} \tag{2.36e}$$

The final form of the solution will be as follows:

$$Y(X,T) = \sum_r\{\frac{f_1(s_r)}{g'(s_r)}e^{\lambda_1 X}e^{s_r t}\} + \sum_r\{\frac{f_2(s_r)}{g'(s_r)}e^{\lambda_2 X}e^{s_r t}\} + \sum_r\{\frac{f_3(s_r)}{g'(s_r)}e^{\lambda_3 X}e^{s_r t}\} +$$

$$\sum_r\{\frac{f_4(s_r)}{g'(s_r)}e^{\lambda_4 X}e^{s_r t}\} \tag{2.37}$$

Where g'(s) is the derivative of g(s) with respect to s. Treating g(s) as afunction of $\lambda(s)$ and s, and applying the chain rule to find g'(s) will give:

$$g'(s) = \frac{dg}{ds} = \sum_{i=1}^{4} \frac{\partial g}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial s} + \frac{\partial g}{\partial s} \tag{2.38}$$

To find $\frac{\partial \lambda_i}{\partial s}$ the characteristic equation (Equation (2.10)) will be used;

$$\frac{\partial \lambda_i}{\partial s} = \frac{b_1 \lambda_i + 2s}{4\lambda_i^3 + 2b_2\lambda_i + b_1 s} \tag{2.39}$$

The rest of the terms are as follows:

$$\frac{\partial g(s)}{\partial \lambda_1} = (s^2+\beta^2)\{(\lambda_1-\lambda_2+1)(\lambda_3^2-\lambda_4^2)e^{\lambda_1+\lambda_2} + (\lambda_1-\lambda_3+1)(\lambda_4^2-\lambda_2^2)e^{\lambda_1+\lambda_3} +$$

$$(\lambda_1-\lambda_4+1)(\lambda_2^2-\lambda_3^2)e^{\lambda_1+\lambda_4} + 2\lambda_1(\lambda_2-\lambda_3)e^{\lambda_2+\lambda_3} - 2\lambda_1(\lambda_2-\lambda_4)e^{\lambda_2+\lambda_4} +$$

$$2\lambda_1(\lambda_3-\lambda_4)e^{\lambda_3+\lambda_4}\} \tag{2.40a}$$

$$\frac{\partial g(s)}{\partial \lambda_2} = (s^2+\beta^2)\{(\lambda_1-\lambda_2-1)(\lambda_3^2-\lambda_4^2)e^{\lambda_1+\lambda_2} + (\lambda_2-\lambda_3+1)(\lambda_1^2-\lambda_4^2)e^{\lambda_2+\lambda_3} +$$

$$(\lambda_4-\lambda_2-1)(\lambda_1^2-\lambda_3^2)e^{\lambda_2+\lambda_4} + 2\lambda_2(\lambda_3-\lambda_1)e^{\lambda_1+\lambda_3} + 2\lambda_2(\lambda_1-\lambda_4)e^{\lambda_1+\lambda_4} +$$

$$-2\lambda_2(\lambda_3-\lambda_4)e^{\lambda_3+\lambda_4}\} \tag{2.40b}$$

$$\frac{\partial g(s)}{\partial \lambda_3} = (s^2+\beta^2)\{(\lambda_1-\lambda_3-1)(\lambda_4^2-\lambda_2^2)e^{\lambda_1+\lambda_3} + (\lambda_2-\lambda_3-1)(\lambda_1^2-\lambda_4^2)e^{\lambda_2+\lambda_3} +$$

$$(\lambda_3-\lambda_4+1)(\lambda_1^2-\lambda_4^2)e^{\lambda_3+\lambda_4} + 2\lambda_3(\lambda_1-\lambda_2)e^{\lambda_1+\lambda_2} - 2\lambda_3(\lambda_1-\lambda_4)e^{\lambda_1+\lambda_4} +$$

$$-2\lambda_3(\lambda_2-\lambda_4)e^{\lambda_2+\lambda_4}\} \quad . \tag{2.40c}$$

$$\frac{\partial g(s)}{\partial \lambda_4} = (\lambda_1-\lambda_4-1)(\lambda_2^2-\lambda_3^2)e^{\lambda_1+\lambda_4} + (\lambda_2-\lambda_4-1)(\lambda_3^2-\lambda_1^2)e^{\lambda_2+\lambda_4} +$$

$$(\lambda_3 - \lambda_4 - 1)(\lambda_1^2 - \lambda_2^2)e^{\lambda_3+\lambda_4} + 2\lambda_4(\lambda_2 - \lambda_1)e^{\lambda_1+\lambda_2} - 2\lambda_4(\lambda_1 - \lambda_3)e^{\lambda_1+\lambda_3} +$$

$$-2\lambda_4(\lambda_2 - \lambda_3)e^{\lambda_2+\lambda_3}\} \qquad (2.40d)$$

Equation (2.33) is the solution of the problem in the X-s plane. To find the solution in the X-T plane it is necessary to find the inverse of Y(X,s). This is obtained by summing the residues of the poles of Y(X,s) in the complex s-plane [6]; equivalently we need to find the zeros of g(s) (Equation (2.36d)). For convenience Equation (2.36d) can be written as:

$$g(s) = (s^2 + \beta^2)g_1(s) \qquad (2.41)$$

From Equation 2.41, the apparent poles of the system are $s = \pm i\beta$. The rest of the poles correspond to the zeros of g_1 (s). Efforts (Appendix C) to find these values were unsucessful, because of the range of values of g_1 (s) results in computer overflow. To circumvent this problem Equation (2.31) was treated as an eigenvalue problem, which is equivalent to finding the zeros of g_1 (s). The eigenvalues were found using the Extended Lanczos' Method (Chapter 4). Equation (2.37) can be rewritten in a simplified form for convenience:

$$Y(X,T) = \sum_{i=1}^{4} C_i(s_r)e^{\lambda_i X}e^{s_r T} \qquad (2.42)$$

The poles of the problem appear in complex conjugate pairs. When the conjugate pairs are substituted in Equation 2.42 a real-valued function will result because of the following properties of the coefficients:

$$s_r = s_{r_{Real}} + i s_{r_{Imaginary}}$$

$$\lambda_i(\tilde{s}_r) = \tilde{\lambda}_i(s_r) = \lambda_{i_{Real}}(s_r) - i\lambda_{i_{Imaginary}}(s_r)$$

$$C_i(\tilde{s}_r) = \tilde{C}_i(s_r) = C_{i_{Real}}(s_r) - iC_{i_{Imaginary}}(s_r)$$

$$Y(X,T) = \sum_{r=1}^{\infty}\sum_{i=1}^{4}\{2C_{i_{Real}} e^{(\lambda_{i_{Real}}X + s_{r_{Real}}T)} \cos(\lambda_{i_{imaginary}}X + s_{r_{imaginary}}T) -$$

$$2C_{i_{imaginary}} e^{(\lambda_{i_{Real}}X + s_{r_{Real}}T)} \sin(\lambda_{i_{imaginary}}X + s_{r_{imaginary}}T)\}$$

The resonance characteristics can be analyzed in a manner similar to the simple-simple support case. From Equation (2.41), the simplified form of the denominator will be

$$g'(s) = 2sg_1(s) + (s^2 + \beta^2)g_1(s)$$

The β values that makes g'(s) zero, will correspond to the resonance frequencies. But, the excitation frequencies are real, therefore the resonance frequencies will be

$$\beta^2 = s^2_{r_{Imaginary}} - s^2_{r_{Real}}$$

From the transverse response characteristics instability conditions can also be determined, but the discussion of these is left for Chapter 5.

CHAPTER 3

SECOND-ORDER MODEL

The mathematical model of an axially travelling string under tension will be called the second-order model. This problem has been previously studied in Wickert [1] and Chonan[5], and series solutions have been determined for models with homogenous boundary conditions. In this chapter, the second order model will be studied to see if the results are good approximations to the fourth-order problem. The homogeneous transverse equation of motion is:

$$(\delta V^2 - P)u_{xx} + 2V\delta u_{xt} + \delta u_{tt} = 0 \tag{3.1}$$

with the boundary conditions

$$\begin{aligned} u(x=0,t) &= K\sin(wt) \\ u(x=L,t) &= 0 \end{aligned} \tag{3.2}$$

For convenience, further substitution is made for the coefficients of Equation (3.1):

$$b_2 u_{xx} + b_1 u_{xt} + u_{tt} = 0$$

$$b_2 = V^2 - \tfrac{P}{\delta} \qquad\qquad b_1 = 2V \tag{3.3}$$

u(x,t) : Transverse displacement of the string.
V : Constant axial velocity.
K : Transverse displacement of the excitation.
L : The distance between the supports
δ : Density per unit length
w : Frequency of the excitation

The solution procedure will be similar to that of the fourth order models because of non-homogenous boundary conditions. The Laplace transform of Equation (3.3) is taken with respect to t, with the initial conditions;

$$u(x,t=0)=0 \qquad\qquad u_t(x,t=0)=0 \tag{3.4}$$

Yielding

$$b_2U_{xx}(x,s)+b_1sU_x(x,s)+s^2U(x,s)-b_1u(x,t=0)-su(x,t=0)-u_t(x,t=0)=0 \tag{3.5}$$

Equation (3.5) results in the following ordinary differential equation when the initial conditions are applied.

$$b_2U_{xx}(x,s)+b_1sU_x(x,s)+s^2U(x,s)=0 \tag{3.6}$$

with the boundary conditions

$$U(x=0,s)=K\tfrac{w}{s^2+w^2}\,; \qquad U(x=L,s)=0 \tag{3.7}$$

A solution of the form $U(x,s)=Ae^{\mu x}$ will be assumed for Equation (3.6). Substitution results in the characteristic equation (Equation (3.8b)).

$$(\mu^2b_2+\mu sb_1+s^2)Ae^{\mu x}=0 \tag{3.8a}$$

$$\mu^2b_2+\mu sb_1+s^2=0 \tag{3.8b}$$

The roots of Equation (3.8b) are:

$$\mu_1 = \frac{-b_1+\sqrt{b_1^2-4b_2}}{2b_2}s; \qquad \mu_2 = \frac{-b_1-\sqrt{b_1^2-4b_2}}{2b_2}s \tag{3.9}$$

Assuming distinct roots, U(x,s) will be

$$U(x,s) = A_1 e^{\mu_1 x} + A_2 e^{\mu_2 x} \tag{3.10}$$

The boundary conditions are applied to determine the unknown coefficients, A_1 and A_2.

$$U(x=0,s) = A_1 + A_2 = K\frac{w}{s^2+w^2} \quad \Rightarrow \quad A_2 = -A_1 + \frac{Kw}{(s^2+w^2)} \tag{3.11}$$

Then

$$U(x,s) = A_1\{e^{\mu_1 x} - e^{\mu_2 x}\} + \frac{Kw}{(s^2+w^2)}e^{\mu_2 x} \tag{3.12}$$

Where the following substitutions are made for convenience:

$$\alpha_1 = \frac{-b_1}{2b_2} \quad ; \quad \alpha_2 = \frac{\sqrt{b_1^2-4b_2}}{2b_2}$$

Substituting the above expressions into Equation (3.12) yields

$$U(x,s) = A_1 e^{\alpha_1 sx} 2\sinh(\alpha_2 sx) + \frac{Kw}{(s^2+w^2)}e^{\mu_2 x} \tag{3.13}$$

Applying the other boundary condition

$$U(x=L,s)=0=A_1 e^{\alpha_1 sL} 2\sinh(\alpha_2 sL)+\frac{Kw}{(s^2+w^2)}e^{\mu_2 L}$$

Then A_1 is found to be

$$A_1=\frac{-\frac{Kw}{(s^2+w^2)}e^{\mu_2 L}}{2e^{\alpha_1 sL}\sinh(\alpha_2 sL)}=-\frac{Kwe^{\mu_2 L}e^{-\alpha_1 sL}}{2(s^2+w^2)\sinh(\alpha_2 sL)} \tag{3.14}$$

The final form of the solution to the second order model in the X-s plane is as follows:

$$U(x,s)=\frac{Kwe^{\mu_2 L}e^{\alpha_1 s(x-L)}}{(s^2+w^2)\sinh(\alpha_2 sL)}\sinh(\alpha_2 sx)+\frac{Kw}{(s^2+w^2)}e^{\mu_2 x} \tag{3.15}$$

To find the solution in the x-t plane it is necessary to find the inverse transform of U(x,s). This is obtained by the summation of the residues of the poles of U(x,s) in the complex s-plane [6]. Before the inverse of U(x,s) is determined, Equation (3.15) will be written in a more convenient form as follows:

$$U(x,s)=U_1(x,s)+U_2(x,s)=\frac{f_1(s)}{g_1(s)}+\frac{f_2(s)}{g_2(s)}$$

Then the inverse of U(x,s) becomes

$$U(x,t)=\sum_{i=1}^{2}\sum_{r=1}^{\infty}\frac{f_i(s_{r_i})}{g_i'(s_{r_i})}e^{s_r t} \tag{3.16}$$

where, $s_{r,i}$ corresponds to the poles of $U_i(x,s)$, and $g_i'(s)$ is the derivative of $g_i(s)$ with respect to s. The terms in the inverse of U(x,s) will be treated one at a time. Considering $U_1(x,s)$

$$U_1(x,s) = \frac{Kw\{e^{\mu_2 L}-1\}e^{\alpha_1 s(x-L)}}{(s^2+w^2)\sinh(\alpha_2 sL)}\sinh(\alpha_2 sx) = \frac{f_1(s)}{g_1(s)} \tag{3.17}$$

$$f_1(s) = Kw\{e^{\mu_2 L}-1\}e^{\alpha_1 s(x-L)}\sinh(\alpha_2 sx)$$

$$g_1(s) = (s^2+w^2)\sinh(\alpha_2 sL)$$

and $g_1'(s)$ is found to be

$$g_1'(s) = 2s\sinh(\alpha_2 sL) + (s^2+w^2)\alpha_2 L\cosh(\alpha_2 sL)$$

The poles of $U_1(x,s)$ are the zeros of $g_1(s)$, which are

$$s_{r_1} = \pm iw \quad ; \quad s_{r_1} = \pm i\frac{n\pi}{\alpha_2 L} \quad ; \quad r = 1,2,3,.. \tag{3.18}$$

Summing the residues of the poles for $U_1(x,s)$ results in $u_1(x,t)$.

$$u_1(x,t) = \sum_{r_1}\frac{Kw\{e^{\mu_2(s_{r1})L}-1\}e^{\alpha_1 s_{r1}(x-L)}\sinh(\alpha_2 s_{r1}x)e^{s_{r1}t}}{2s_{r1}\sinh(\alpha_2 s_{r1}L)+(s_{r1}^2+w^2)\alpha_2 L\cosh(\alpha_2 s_{r1}L)} \tag{3.19}$$

Equation (3.19) can be further simplified:

$$u_1(x,t) = \frac{Kw\{e^{\mu_2(iw)L}-1\}e^{\alpha_1 iw(x-L)}\sinh(\alpha_2 iwx)e^{iwt}}{2iw\sinh(\alpha_2 iwL)}+$$

$$\frac{Kw\{e^{\mu_2(-iw)L}-1\}e^{-\alpha_1 iw(x-L)}\sinh(-\alpha_2 iwx).e^{-iwt}}{-2iw\sinh(-\alpha_2 iwL)}+$$

$$\sum_{n=1}^{\infty}\{\frac{Kw\{e^{\mu_2(\frac{in\pi}{\alpha_2 L})L}-1\}e^{\alpha_1\frac{in\pi}{\alpha_2 L}(x-L)}\sinh(\alpha_2\frac{in\pi}{\alpha_2 L}x)e^{\frac{in\pi}{\alpha_2 L}t}}{((\frac{in\pi}{\alpha_2 L})^2+w^2)\alpha_2 L\cosh(in\pi)}+$$

$$\frac{Kw\{e^{\mu_2(-\frac{in\pi}{\alpha_2 L})L}-1\}e^{-\alpha_1\frac{in\pi}{\alpha_2 L}(x-L)}\sinh(-\alpha_2\frac{in\pi}{\alpha_2 L}x)e^{-\frac{in\pi}{\alpha_2 L}t}}{((-\frac{in\pi}{\alpha_2 L})^2+w^2)\alpha_2 L\cosh(-in\pi)}\} \quad (3.20)$$

Further simplification gives the result in real functions as follows:

$$u_1(x,t) = \frac{K\sin(\alpha_2 wx)}{\sin(\alpha_2 wL)}\{\sin(w((\alpha_1-\alpha_2)L+\alpha_1(x-L)+t))-$$

$$\sin(w(\alpha_1(x-L)+t))\}$$

$$-\sum_{n=1}^{\infty}\frac{2Kw\sin(\frac{n\pi}{L}x)}{(w^2+(\frac{n\pi}{\alpha_2 L})^2)\cos(n\pi)}\{\cos(\frac{n\pi}{\alpha_2 L}((\alpha_1-\alpha_2)L+\alpha_1(x-L)+t))-$$

$$-\cos(\frac{n\pi}{\alpha_2 L}(\alpha_1(x-L)+t))\} \quad (3.21)$$

The same procedure is followed for U_2 (x,s). Then

$$U_2(x,s) = \frac{Kw}{(s^2+w^2)} e^{\mu_2 x} \tag{3.22}$$

$$f_1(s) = Kwe^{\mu_2 x}$$

$$g_1(s) = (s^2 + w^2)$$

$$g_1'(s) = 2s$$

The poles for U_2 (x,s) are as follows:

$$s_{r_2} = \pm iw$$

Then U_2 (x,s) becomes

$$u_2(x,t) = \frac{Kw}{2iw} e^{(\alpha_1-\alpha_2)iwx} e^{iwt} + \frac{Kw}{-2iw} e^{-(\alpha_1-\alpha_2)iwx} e^{-iwt}$$

$$u_2(x,t) = -K\sin(w(\alpha_1 - \alpha_2)x + wt) \tag{3.23}$$

The addition of the two parts gives the final solution.

$$u(x,t) = \frac{K\sin(\alpha_2 wx)}{\sin(\alpha_2 .w.L)} \{\sin(w((\alpha_1 - \alpha_2)L + \alpha_1(x-L) + t)) -$$

$$\sin(w(\alpha_1(x-L) + t))\}$$

$$-\sum_{n=1}^{\infty} \frac{2Kw\sin(\frac{n\pi}{L}x)}{(w^2+(\frac{n\pi}{\alpha_2 L})^2)\cos(n\pi)}\{\cos(\frac{n\pi}{\alpha_2 L}(\mu_2 L+\alpha_1(x-L)+t))-$$

$$-\cos(\frac{n\pi}{\alpha_2 L}(\alpha_1(x-L)+t))\}$$

$$-K\sin((\alpha_1-\alpha_2)wx+wt) \qquad (3.24)$$

The instability and resonance characteristics of the second order model can be analyzed from Equation (3.24). For instability to set in the sine or the cosine functions with the time as a variable existing, should become hyperbolic-sine and hyperbolic-cosine. In Equation (3.24) this will be observed when α_2 becomes imaginary ($\cos(\frac{n\pi}{\alpha_2 L}t)$). This parameter was defined in Equation (3.9) as follows:

$$\alpha_2 = \frac{\sqrt{b_1^2-4.b_2}}{2.b_2} \qquad (3.25)$$

Then the instability will start when:

$$b_2 \geq \frac{b_1^2}{4}\ . \qquad (3.26)$$

Resonance, on the other hand, will occur whenever one of the denominator terms of first part of the solution becomes small (as for the second part, there is no denominator). The resonance condition is met when

$$w = \frac{n\pi}{\alpha_2 L} \qquad (3.27)$$

CHAPTER 4

APPLICATION OF THE EXTENDED LANCZOS METHOD

In Chapter 2, Y(x,s) represents the solution of the problem in the complex X-s plane. To find the solution in the X-T plane an inverse transformation is required. As discussed in Chapter 2, the inverse transformation depends on the poles of Y(X,s), which corresponds to the zeros of g(s) (Equation (2.36d)). For convenience g(s) will be written as follows

$$g(s) = (s^2 + \beta^2).g_1(s)$$

From the above form, the apparent poles are $s = \pm i\beta$, but the rest of the poles are the zeros of g_1 (s). The exact solution to the roots of g_1 (s) is not available, therefore numerical methods can be used to find a sufficient number of roots to describe the transverse response. Within the operating range, because of the exponential coefficients, g_1 (s) achieves magnitudes resulting in computer overflows if the values of s are not close enough to the roots (Appendix C). To circumvent this problem Equation (2.8) was treated as an eigenvalue problem in s, this being equivalent to finding the zeros of g_1 (s). The method of obtaining these eigenvalues is the Extended Lanczos' Method presented by Herman [4].

The method is based on "quadrature by differentiation" and makes use of the properties of modified Legendre polynomials. The resultant quadrature formula [4] is

$$\int_a^b f(x)dx = \frac{1}{C_0^n} \sum_{k=0}^{n-1} C_{k+1}^n (b-a)^{k+1} [f^{(k)}(a) + (-1)^k f^{(k)}(b)] \tag{4.1}$$

where $(\ )^{(k)}$ denotes the k'th derivative, and

$$C_j^i = \frac{(2j-i)!}{(j-i)!i!} \tag{4.2}$$

In general, the higher the value of n, the better the approximation of the quadrature formula. In this problem, the beam has been divided into N subintervals and the quadrature formula has been applied over each subinterval resulting in the following system of equations (for the k'th subinterval).

$$(Y_{k+1}^{(3)} - Y_k^{(3)}) + b_2(Y_{k+1}^{(1)} - Y_k^{(1)}) + sb_1(Y_{k+1} - Y_k) = -s^2\int_k^{k+1} Y.dx$$

$$= -s^2\{\frac{1}{2N}(Y_k + Y_{k+1}) + \frac{3}{28N^2}(Y_k^{(1)} - Y_{k+1}^{(1)}) + \frac{1}{84N^3}(Y_k^{(2)} + Y_{k+1}^{(2)}) + \frac{1}{1680N^4}(Y_k^{(3)} - Y_{k+1}^{(3)})\}$$

$$\tag{4.3}$$

$$\int_k^{k+1} Y^{(1)}dx = (Y_{k+1} - Y_k) = \frac{1}{2N}(Y_k^{(1)} + Y_{k+1}^{(1)}) + \frac{1}{10N^2}(Y_k^{(2)} - Y_{k+1}^{(2)}) + \frac{1}{120N^3}(Y_k^{(3)} + Y_{k+1}^{(3)})$$

$$\int_k^{k+1} Y^{(2)}dx = (Y_{k+1}^{(1)} - Y_k^{(1)}) = \frac{1}{2N}(Y_k^{(2)} + Y_{k+1}^{(2)}) + \frac{1}{12N^2}(Y_k^{(3)} - Y_{k+1}^{(3)})$$

$$\int_k^{k+1} Y^{(3)}dx = (Y_{k+1}^{(2)} - Y_k^{(2)}) = \frac{1}{2N}(Y_k^{(3)} + Y_{k+1}^{(3)})$$

The subintervals will be called elements, and the ends of the elements will be called nodes, following the standart nomenclature used in the finite element method. There are four unknowns at each node, and from Equations (4.3) there are four equations in 4(N+1) unknowns. All of these equations are combined in a single matrix equation as follows:

$$[M(s)]_{4.(N+1),4.(N+1)}.(Y_k)_{4.(N+1)} = (0)_{4.(N+1)}$$

where

$$(Y_k) = \left(Y_1^{(0)}, Y_1^{(1)}, Y_1^{(2)}, Y_1^{(3)}, \ldots\ldots, Y_{N+1}^{(0)}, Y_{N+1}^{(1)}, Y_{N+1}^{(2)}, Y_{N+1}^{(3)}\right)$$

For a non-trivial solution for (Y_k), it is necessary that the determinant of M(s) be equal to zero. The determinant turns out to be a polynomial in s, where the coefficients are a function of the element number and the system parameters b_1 and b_2. The roots of this polynomial correspond to the approximate eigenvalues of the system. The determinant of the equation has to be solved symbolically as s appears implicitly in M(s). Therefore, Equations (4.3), that form matrix M(s), should be simplified as much as possible before they are substituted into Equation (4.4). Rearranging Equations (4.3):

$$(-b_1 s + \tfrac{s^2}{2N})Y_k + (-b_2 + \tfrac{3s^2}{28N^2})Y_k^{(1)} + \tfrac{s^2}{84N^3}Y_k^{(2)} + (-1 + \tfrac{s^2}{1680N^4})Y_k^{(3)} +$$
$$(b_1 s + \tfrac{s^2}{2N})Y_{k+1} + (b_2 - \tfrac{3s^2}{28N^2})Y_{k+1}^{(1)} + \tfrac{s^2}{84N^3}Y_{k+1}^{(2)} + (1 - \tfrac{s^2}{1680N^4})Y_{k+1}^{(3)} = 0$$

$$Y_k + \tfrac{1}{2N}Y_k^{(1)} + \tfrac{1}{10N^2}Y_k^{(2)} + \tfrac{1}{120N^3}Y_k^{(3)} - Y_{k+1} + \tfrac{1}{2N}Y_{k+1}^{(1)} - \tfrac{1}{10N^2}Y_{k+1}^{(2)} + \tfrac{1}{120N^3}Y_{k+1}^{(3)} = 0$$

$$Y_k^{(1)} + \tfrac{1}{2N}Y_k^{(2)} + \tfrac{1}{12N^2}Y_k^{(3)} - Y_{k+1}^{(1)} + \tfrac{1}{2N}Y_{k+1}^{(2)} - \tfrac{1}{12N^2}Y_{k+1}^{(3)} = 0$$

$$Y_{k+1}^{(2)} - Y_k^{(2)} - \tfrac{1}{2N}Y_k^{(3)} - \tfrac{1}{2N}Y_{k+1}^{(3)} = 0$$

These equations can be written in submatrix form for each subinterval as follows

$$[M_k(s)].(Y)_k = (0) \tag{4.5}$$

with

$$(Y)_k = \left(Y_k^{(0)}, Y_k^{(1)}, Y_k^{(2)}, Y_k^{(3)}, Y_{k+1}^{(0)}, Y_{k+1}^{(1)}, Y_{k+1}^{(2)}, Y_{k+1}^{(3)}\right)$$

Then matrix $[M_k(s)]$ will be

$$\begin{bmatrix} 1 & \frac{1}{2\cdot N} & \frac{1}{10\cdot N^2} & \frac{1}{120\cdot N^3} & -1 & \frac{1}{2\cdot N} & -\frac{1}{10\cdot N^2} & \frac{1}{120\cdot N^3} \\ 0 & 1 & \frac{1}{2\cdot N} & \frac{1}{12\cdot N^2} & 0 & -1 & \frac{1}{2\cdot N} & -\frac{1}{12\cdot N^2} \\ 0 & 0 & 1 & \frac{1}{2\cdot N} & 0 & 0 & -1 & \frac{1}{2\cdot N} \\ -b_1\cdot s+\frac{s^2}{2\cdot N} & -b_2+\frac{3\cdot s^2}{28\cdot N^2} & \frac{s^2}{84\cdot N^3} & -1+\frac{s^2}{1680\cdot N^4} & -b_1\cdot s+\frac{s^2}{2\cdot N} & -b_2+\frac{3\cdot s^2}{28\cdot N^2} & \frac{s^2}{84\cdot N^3} & -1+\frac{s^2}{1680\cdot N^4} \end{bmatrix}$$

Elementary row operations are employed for further simplifications which result in the following matrix.

$$M_1(s) = \begin{bmatrix} 1 & \frac{1}{2\cdot N} & \frac{1}{10\cdot N^2} & \frac{1}{120\cdot N^3} & -1 & \frac{1}{2\cdot N} & -\frac{1}{10\cdot N^2} & \frac{1}{120\cdot N^3} \\ 0 & 1 & \frac{1}{2\cdot N} & \frac{1}{12\cdot N^2} & 0 & -1 & \frac{1}{2\cdot N} & -\frac{1}{12\cdot N^2} \\ 0 & 0 & 1 & \frac{1}{2\cdot N} & 0 & 0 & -1 & \frac{1}{2\cdot N} \\ 0 & 0 & 0 & A_1 & A_2 & A_3 & A_4 & A_5 \end{bmatrix}. \tag{4.6}$$

where

$$A_1 = -\frac{\left(20b_2N^2-\left(5b_1sN-120N^4-s^2\right)\right)}{120N^4};$$

$$A_2 = \frac{s^2}{N};$$

$$A_3 = \frac{s(2b_1N-s)}{2N^2};$$

$$A_4 = \frac{\left(6b_2N^2-s(3b_1N-s)\right)}{6N^3};$$

$$A_5 = -\frac{40b_2N^2-\left(15b_1Ns+4\left(30N^4-s^2\right)\right)}{120N^4};$$

If the beam is divided into N elements, then there will be N such matrices. These matrices are combined in a single matrix form as shown in Figure 6.

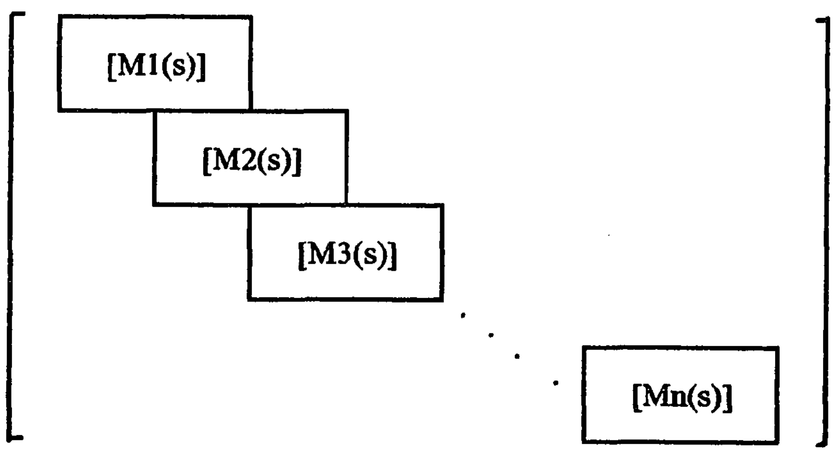

Figure 6 The Arrangement of the Submatrices

Then, the columns that represent the known nodal variables will be taken out of the matrix M(s) to avoid trivial solutions. By way of an example, the simple-simple support case using two elements is shown in Figure 7. In this figure, columns corresponding to the boundary conditions, $Y_1^{(0)} = Y_1^{(2)} = Y_3^{(0)} = Y_3^{(2)} = 0$, are the columns that have the value 1 in the first two and the last two rows.

$$
\begin{bmatrix}
1 & & & & & & & & & & & \\
 & & 1 & & & & & & & & & \\
1 & 1/(2N) & 1/(10N^2) & 1/(120N^3) & -1 & 1/(2N) & -1/(10N^2) & 1/(120N^3) & & & & \\
 & 1 & 1/(2N) & 1/(12N^2) & & -1 & 1/(2N) & -1/(12N^2) & & & & \\
 & & 1 & 1/(2N) & & & -1 & 1/(2N) & & & & \\
 & & & A1 & A2 & A3 & A4 & A5 & & & & \\
 & & & & 1 & 1/(2N) & 1/(10N^2) & 1/(120N^3) & -1 & 1/(2N) & -1/(10N^2) & 1/(120N^3) \\
 & & & & & 1 & 1/(2N) & 1/(12N^2) & & -1 & 1/(2N) & -1/(12N^2) \\
 & & & & & & 1 & 1/(2N) & & & -1 & 1/(2N) \\
 & & & & & & & A1 & A2 & A3 & A4 & A5 \\
 & & & & & & & & 1 & & & \\
 & & & & & & & & & & 1 &
\end{bmatrix}
$$

Figure 7 Matrix [M(s)] for the Simple-simple Support Case, using Two Elements (Blanks stands for zero values)

After the simplification (cancelation of the columns), the determinant of matrix M(s) is found using a symbolic processor(Derive and Mathematica). If the same procedure is carried out for a six- element matrix, then the resultant polynomial obtained by evaluating the determinant of a fixed-simple support using six elements will be:

(24768000000*b2^6*N^12 + 3097843200000*b2^5*N^14 -
85927219200000*b2^4*N^16 + 733813862400000*b2^3*N^18 -
2452936089600000*b2^2*N^20 + 3061081497600000*b2*N^22 -
1057038336000000*N^24 + 70705600000*b2^5*b1*N^11* s -
852576000000*b2^4*b1*N^13* s + 82828800000*b2^3*b1*N^15* s +
15773184000000*b2^2*b1*N^17* s -

6695654400000*b2*b1*N^19* s - 47949926400000*b1*N^21* s +
150812160000*b2^5*N^10* s^2 - 32615200000*b2^4*b1^2*N^10* s^2 -
20657790720000*b2^4*N^12* s^2 + 4200456000000*b2^3*b1^2*N^12* s^2 +
421392706560000*b2^3*N^14* s^2 - 74430662400000*b2^2*b1^2*N^14* s^2 -
3027694878720000*b2^2*N^16* s^2 + 428304326400000*b2*b1^2*N^16* s^2 +
8212426444800000*b2*N^18* s^2 - 739092211200000*b1^2*N^18* s^2 -
6672853094400000*N^20* s^2 - 237316000000*b2^4*b1*N^9* s^3 +
39672200000*b2^3*b1^3*N^9* s^3 + 2798186880000*b2^3*b1*N^11* s^3 -
303753600000*b2^2*b1^3*N^11* s^3 - 3805401600000*b2^2*b1*N^13* s^3 -
534016800000*b2*b1^3*N^13* s^3 - 14100410880000*b2*b1*N^15* s^3 +
892080000000*b1^3*N^15* s^3 - 23252313600000*b1*N^17* s^3 -
507770688000*b2^4*N^8* s^4 + 186892800000*b2^3*b1^2*N^8* s^4 -
17414550000*b2^2*b1^4*N^8* s^4 + 30041345088000*b2^3*N^10* s^4 -
9383595840000*b2^2*b1^2*N^10* s^4 + 659560950000*b2*b1^4*N^10* s^4 -
452069144064000*b2^2*N^12* s^4 + 106758509760000*b2*b1^2*N^12* s^4 -
4395168000000*b1^4*N^12* s^4 + 2390721765120000*b2*N^14* s^4 -
327890643840000*b1^2*N^14* s^4 - 3816398592000000*N^16* s^4 +
186542472000*b2^3*b1*N^7* s^5 - 32835680000*b2^2*b1^3*N^7* s^5 -
488106250*b2*b1^5*N^7* s^5 - 1827800640000*b2^2*b1*N^9* s^5 +
21758520000*b2*b1^3*N^9* s^5 + 36555375000*b1^5*N^9* s^5 +
2377970784000*b2*b1*N^11* s^5 + 919234800000*b1^3*N^11* s^5 -
100517760000*b1*N^13* s^5 + 377202220800*b2^3*N^6* s^6 -
155922336000*b2^2*b1^2*N^6* s^6 + 20139810000*b2*b1^4*N^6* s^6 -
809334375*b1^6*N^6* s^6 - 14432933260800*b2^2*N^8* s^6 +
4422829428000*b2*b1^2*N^8* s^6 - 307655325000*b1^4*N^8* s^6 +
147899487168000*b2*N^10* s^6 - 25230860208000*b1^2*N^10* s^6 -
428339249280000*N^12* s^6 - 55411480000*b2^2*b1*N^5* s^7 +
3050930500*b2*b1^3*N^5* s^7 + 1710115625*b1^5*N^5* s^7 +
388630132800*b2*b1*N^7* s^7 + 53283030000*b1^3*N^7* s^7 -
309455568000*b1*N^9* s^7 - 101556062160*b2^2*N^4* s^8 +
34212362800*b2*b1^2*N^4* s^8 - 2911533750*b1^4*N^4* s^8 +
2508796854000*b2*N^6* s^8 - 469180347600*b1^2*N^6* s^8 -
13383305136000*N^8* s^8 + 7917219910*b2*b1*N^3* s^9 + 22730750*b1^3*N^3*
s^9 - 34252120200*b1*N^5* s^9 + 10740580608*b2*N^2* s^10 -
1817042895*b1^2*N^2* s^10 - 133451752680*N^4* s^10 - 490692327*b1*N* s^11 -
375996372* s^12)/(21499084800000*N^27)

(4.7)

Similarly, the closed form solution for the simple-simple-support case, using six elements will be:

(-2652887036648800294797312000000 + 3745974750823374533017600000*b2 -
1256618730922892800819200000*b2^2 + 155576853941559754752000000*b2^3 -

```
8348228438143795200000000*b2^4 + 19775980451266560000000*b2^5 -
1697890222080000000*b2^6 - 285610673548673780023296000000* s^2 +
135780949683682777497600000000*b2* s^2 - 19493661826831771238400000*b2^2*
s^2 +    11372119725747732480000*b2^3* s^2 - 284788433018880000000*b2^4*
s^2 +    2548528386048000000*b2^5* s^2 - 1151031800100626104320000000*b1^2*
s^2 +    25698223635592642560000000*b2*b1^2* s^2 -
183442671732326400000000*b2^2*b1^2* s^2 + 515538244546560000000*b2^3*b1^2*
s^2 -    4961249740800000000*b2^4*b1^2* s^2 - 19738502570367988531200000* s^4
+    45857790400975994880000*b2* s^4 - 33773998682574028800000*b2^2* s^4 +
97824199103078400000*b2^3* s^4 - 97177542912000000*b2^4* s^4 -
57428346997614182400000*b1^2* s^4 + 699617625112166400000*b2*b1^2* s^4 -
2573773014528000000*b2^2*b1^2* s^4 + 2935632844800000*b2^3*b1^2* s^4 -
2339537630284800000*b1^4* s^4 + 13609141128000000*b2*b1^4* s^4 -
18966538800000*b2^2*b1^4* s^4 - 24382171161914572800000* s^6 +
30990904537190400000*b2* s^6 - 11939324595609600*b2^2* s^6 +
1432714314240000*b2^3* s^6 - 45158410741708800000*b1^2* s^6 +
29406949974720000*b2*b1^2* s^6 - 45359254800000*b2^2*b1^2* s^6 -
131025007080000*b1^4* s^6 + 3211609500000*b2*b1^4* s^6 -
8098672575628800* s^8 + 565130230344000*b2* s^8 - 939404401200*b2^2* s^8 -
83038482936000*b1^2* s^8 + 234838602000*b2*b1^2* s^8 - 7457670000*b1^4*
s^8 -    877912761960* s^10 + 2745450630*b2* s^10 - 362302200*b1^2* s^10 -
2911363*s^12)/2652887036648800294797312000000
```

(4.8)

In Equation (4.7), the number of elements (N) is left as a parameter, whereas in Equation (4.8) its value (N=6) is substituted to show how the magnitudes of the coefficients, even for the most simplified equations, might become large.

For a general case, as the number of elements increases, the coefficients of the determinantal equation become symbolically very complicated. From a practical point of view, the first couple of eigenvalues are the dominant natural frequencies in most systems, as well as for the band-saw. The four, five and six-element solutions have been completed for the band-saw problem with the two possible cases of boundary conditions. The numerical results indicated that a sufficient number of elements for the solution of the band-saw problem, yielding accurate results for the first three eigenvalues in this study, should be around six. The results for the eigenvalues studied showed little variation (less then 10%) with respect to the number of elements. The computed values are the approximations to the exact solution, and to see how close they are to the exact solution, the results were substituted into g(s). As expected, the value of g(s) was close to zero, but the increase in the accuracy with an increase in the number of elements was significant. Therefore the printed solutions in this study are the ones that were obtained using the six-element model, to get the highest accuracy possible with respect to four and five elements. These results are plotted for further discussion in Chapter 5.

Getting results from a symbolic processing (Mathematica, Derive and MathCad) is not easy if the problem that is of interest is sufficiently complex. To avoid this problem, the matrix was made upper triangular by elementary row operations. Then the determinant becomes the product of the diagonal elements. The routine for such a procedure has been completed up to six elements. The determinant is resolved into a polynomial with the coeficients given in terms of the problem parameters, and the results are found for a whole range of values in a matter of seconds. These procedures that are the macro programs for Mathematica are presented in Appendix B.

CHAPTER 5

DISCUSSION

Instability and resonance are two conditions that must avoided for an effective design of a dynamic system. Instabilities are associated with the complementary solution, whereas resonance is associated with the forced vibrations, that determine the steady state response. For the band-saw model, the instability is directly related to the applied tension and to the axial velocity of the blade. The velocity at which instability occurs is called the critical velocity and as far as the operating range of a band-saw is concerned, a critical velocity always exists. Therefore, the operating range should be restricted to velocities lower than those causing instability. Operation above some resonance frequencies is possible but the resonance frequencies should be passed through very quickly.

A dynamic system may experience divergence or flutter instability. The divergence instability appears whenever one of the eigenvalues of the system becomes real and positive. This results in displacements that increase exponentially with time. Flutter appears when the real part of one of the eigenvalues become positive, but the imaginary part still exists. This type of instability will result in oscillations with the amplitude increasing exponentially with time.

Numerical results are substituted into the parameters of the problem to gain a better understanding of the system characteristics. This discussion is based on a representative blade thickness and width, which are kept the same throughout the computations for the different models, axial tensions and axial velocities. To understand the instability characteristics and to find the transverse response, the eigenvalues of the system have been computed using the Extended Lanczos' method described in Chapter 4. Then the smallest eigenvalue's real and imaginary parts have been plotted against changing values of tension and velocity.

Table 1 The numerical values for the models

Width of the Blade:	24.5 mm
Thickness of the Blade :	1 mm
Elastic Modulus (E) :	20 E 10 N/m^2
Area Moment of Inertia (I) :	2.12 mm^4
EI :	0.424 Nm^2
Simple-Simple Support ;	
b_2 :	$0.73\, v^2 - 3.685\, P$
b_1 :	$1.708\, v$
Fixed-Simple Support ;	
b_2 :	$0.1825\, v^2 - 0.92125\, P$
b_1 :	$0.854\, v$

In the graphs plotted, the actual tension (P) is varied from 250 N to 1000 N. The blade is under varying tension, hence subject to fatigue failure throughout its operation. Therefore the highest value of the blade tension is kept around 1000 N to assure that the endurance limit is not reached and that the surface, reliability, heat etc. effects are all taken into account.

The graphs presented clearly define the variation of the systems eigenvalues with respect to changing values of tension and velocity. For each of the curves, the tension is kept constant, and the velocity is varied until a critical velocity is encountered.

In the simple-simple support case, the changes in the eigenvalues typically lead to divergence instability. From Figures 8, 10, 12 and 14, it may seem as if the real part of the first eigenvalue leads to flutter instability as they show positive values, but the numbers are so small that they are outside the computational accuracy of the calculations. Therefore the real parts of the first eigenvalue are taken to be zero. Similar argument are valid for the higher modes. It is important to keep in mind that the plots are done with respect to non-dimensional tension ($\frac{PL^2}{EI}$) and non-dimensional velocity ($v\sqrt{\frac{\delta L^2}{EI}}$). Figure 16 clearly indicates that the critical velocities increase with increasing values of tension. Thus, to permit operation at a higher velocity we may increase the tension. The previous studies (Wickert and Mote [1]) have only dealt with the divergence type of instability for the band-saw problem at low tensions, and the for this range of values our results agree with those previously obtained (Figure 17).

For the fixed-simple support case, the distribution of the eigenvalues with respect to the change in the velocity and tension is not typical. At low tensions, flutter instability occurs even for very low velocities (Figure 18). From Figures 18, to 26 the increase in the tension leads to an increase in the critical velocity at which flutter instability starts. From the same graphs, the critical velocity at which flutter occurs gets closer with increasing blade tension to the velocity at which divergence instability would have started if there were no flutter instabilities. Then, the tension in the blade is increased to maximum values permitted by the fatigue strength properties, to provide a large enough operating range for the velocity of the blade, thus increasing the stability region.

For a fixed tension value, the fixed-simple and simple-simple cases have almost the same critical velocities (when the values are converted from non-dimensional to actual velocity values); however the simple-simple support case reaches divergence , and the fixed-simple, flutter instability.

Basically, when analyzing the response characteristics, the blade is studied in three segments as discussed earlier in Chapter 2. When the blade is not cutting, the tension is constant throughout. During cutting, the tension in the blade between the work-piece and the driving pulley will increase as the cutting force increases, but the tension in the other segments will decrease by some fraction of the cutting force. Therefore, the critical speed will first be reached in the segments in which the tension drops. This should be kept in mind in the design of band-saws.

Even though the dynamic system may attain steady state without experiencing instabilities, there is still the problem that the operating frequency may be close to a resonance point. To avoid resonance when it arises, the operating conditions or the forcing frequency could be changed, but from an efficient design point of view, the resonance frequencies of the system should be determined and taken into account in design in order to provide as large a safe operation range as possible. In Chapter 2 the mathematical source of resonance has been studied, and the imaginary part of the eigenvalues (the natural frequencies of the system) are found to be close to the resonance frequencies (Chapter 2). When related to the physical properties of the saw, this provides useful design information.

To reduce the effort and get a quicker solution than the one obtained for the fourth-order problem, the second-order model was considered. The closed form of the second order model is studied in Chapter 3 using the same approach. Confirming our expectations, the solution for the second-order model was close to that obtained for the fourth-order model with simple-simple supports. The second-order model reaches only divergence instability and almost at the same critical velocity as does the fourth-order model when the tension is sufficiently high. This type of a result was expected since the band is long and flexible and the applied tension, being relatively high, will reduce the influence of the fourth-order terms appearing in the equation.

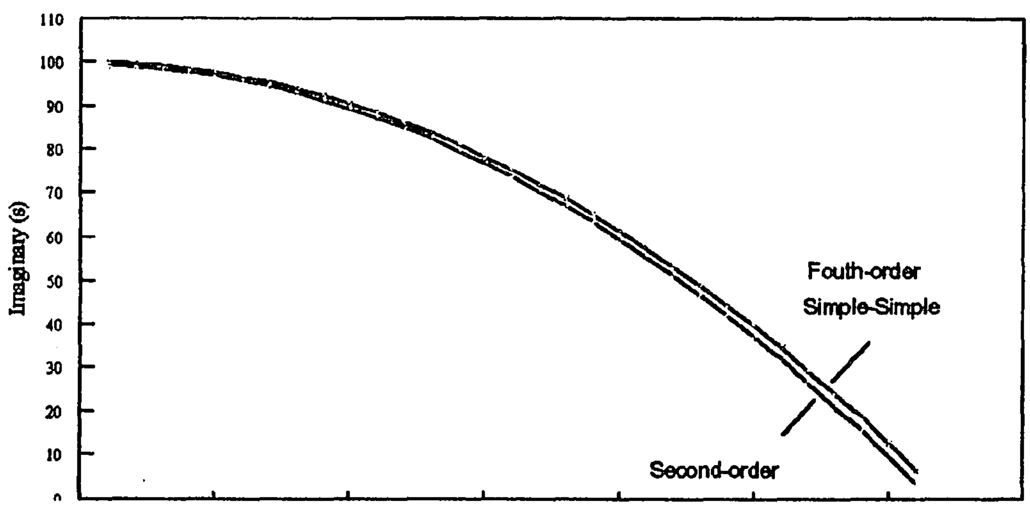

Figure 8 The C omparison between Second-order Model and the Fouth-order Model with Simple-Simple Supports

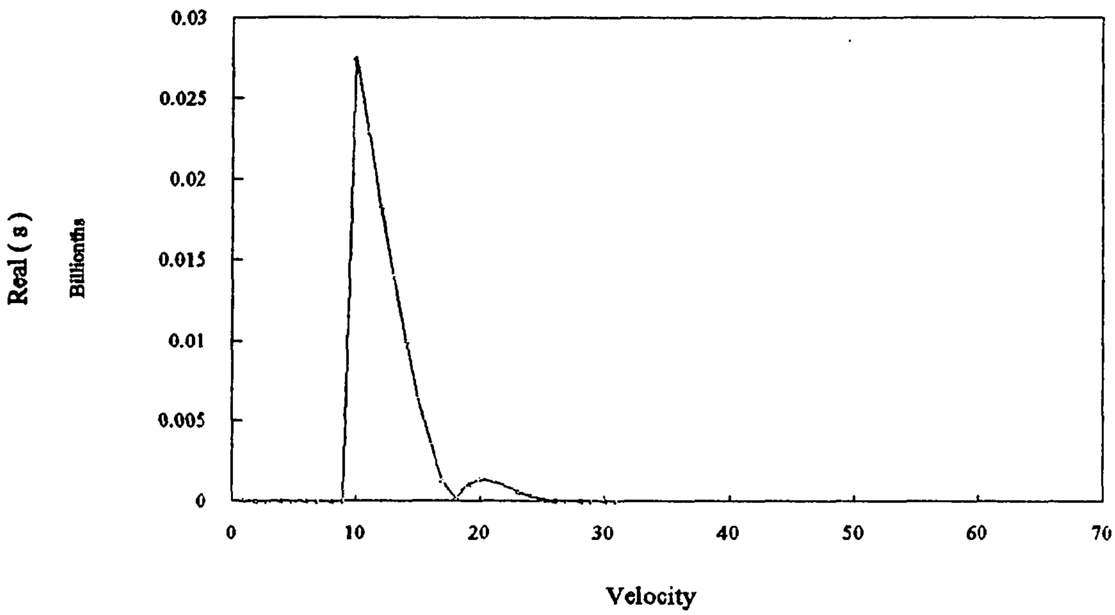

Figure 9 Real part of the First Eigenvalue versus Velocity at Tension=1000 Simple-Simple Support

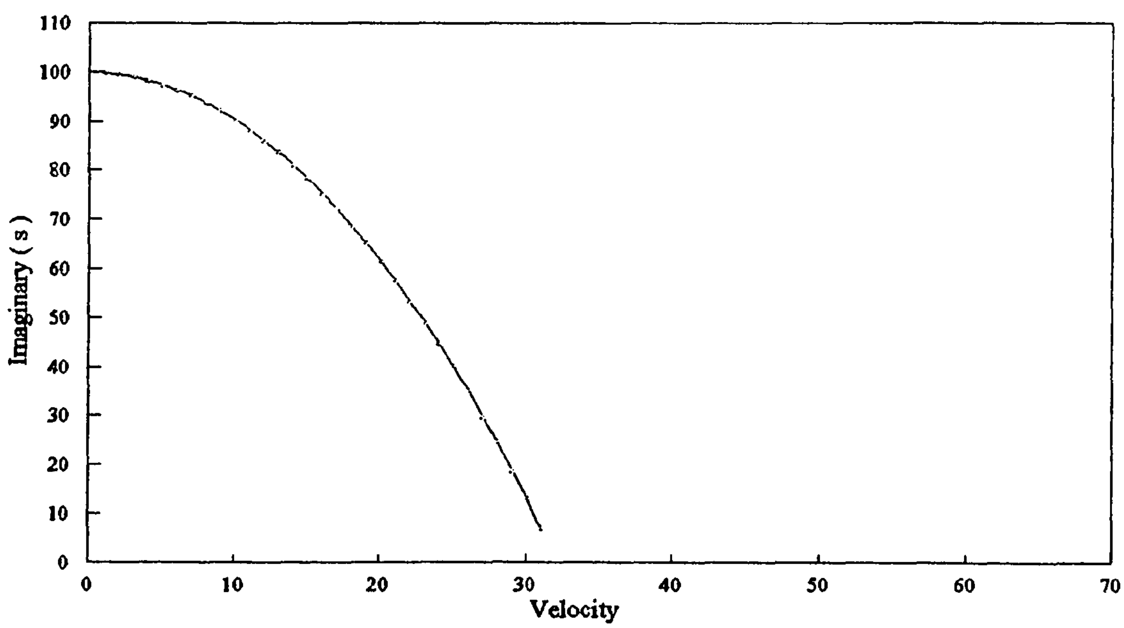

Figure 10 Imaginary part of the Eigenvalue versus Velocity at Tension=1000 Simple-Simple Support

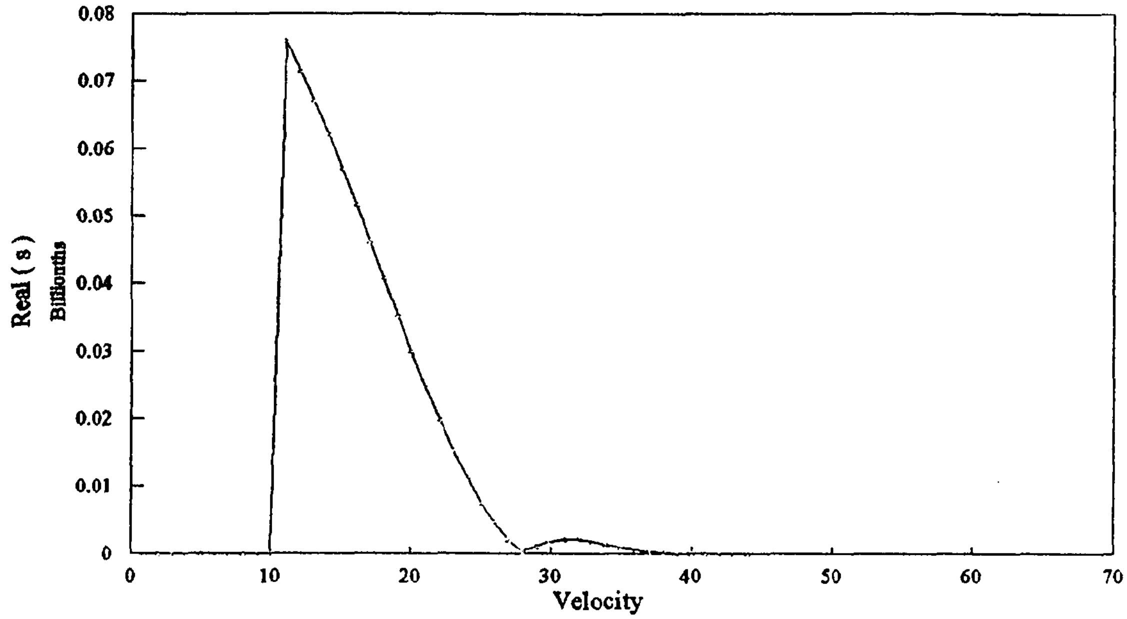

Figure 11 Real part of the First Eigenvalue versus Velocity at Tension=2000 Simple-Simple Support

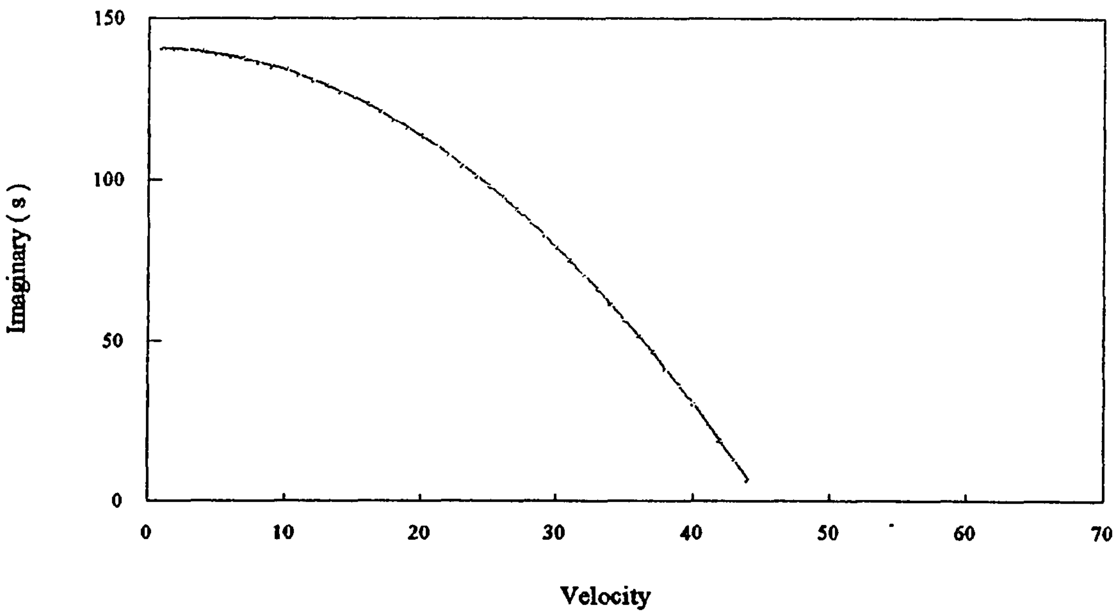

Figure 12 Imaginary part of the Eigenvalue versus Velocity at Tension=2000 Simple-Simple Support

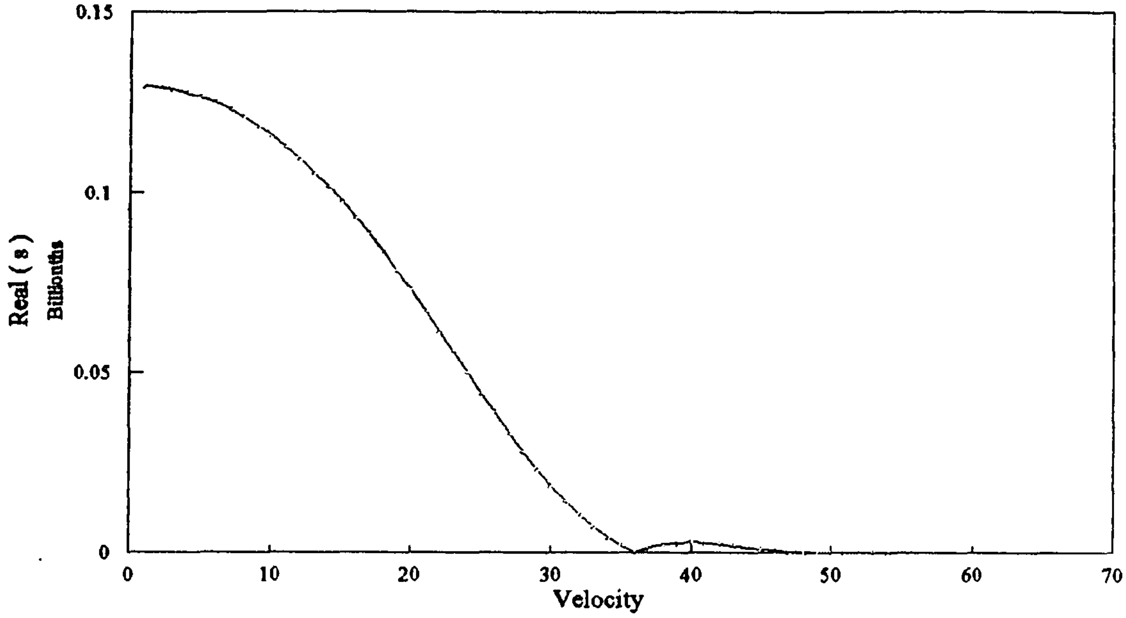

Figure 13 Real part of the First Eigenvalue versus Velocity at Tension=3000 Simple-Simple Support

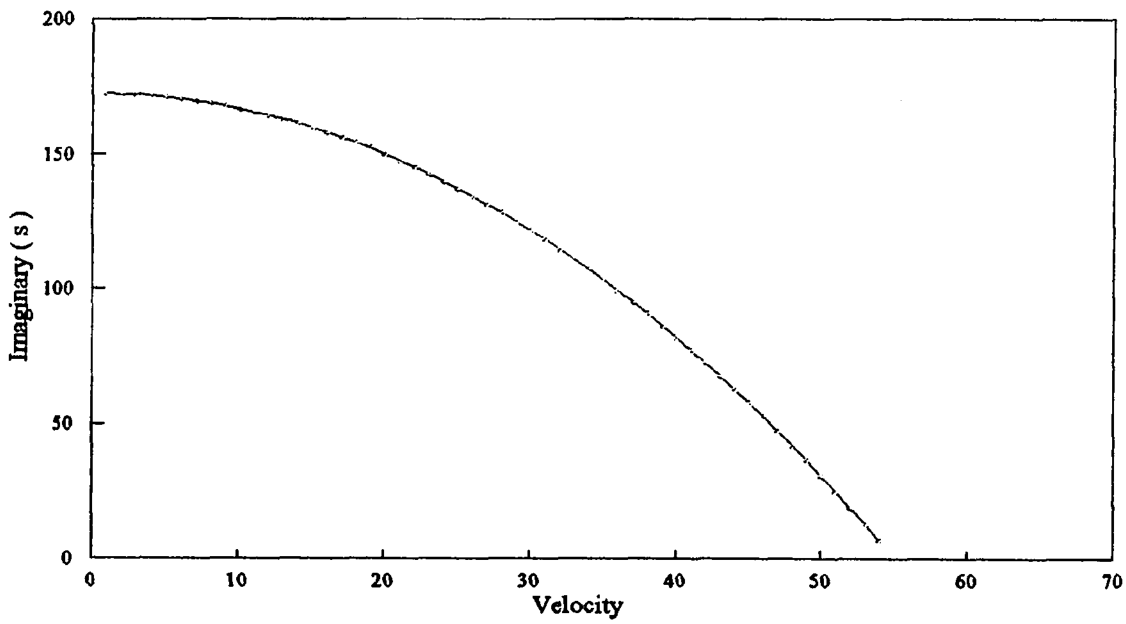

Figure 14 Imaginary part of the Eigenvalue versus Velocity at Tension=3000 Simple-Simple Support

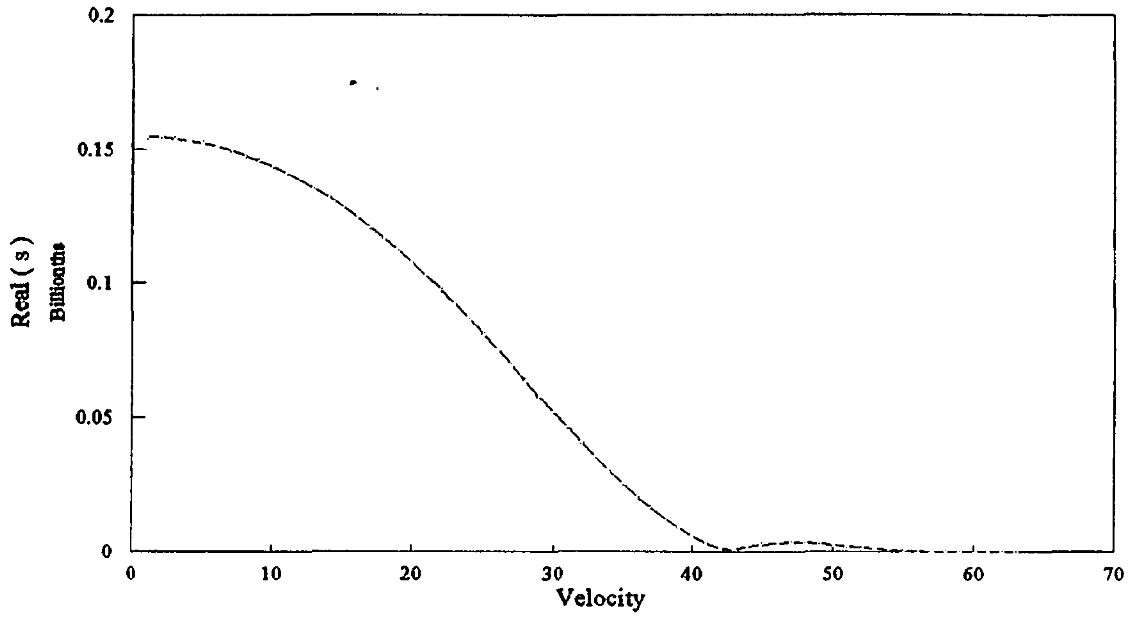

Figure 15 Real part of the First Eigenvalue versus Velocity at Tension=4000 Simple-Simple Support

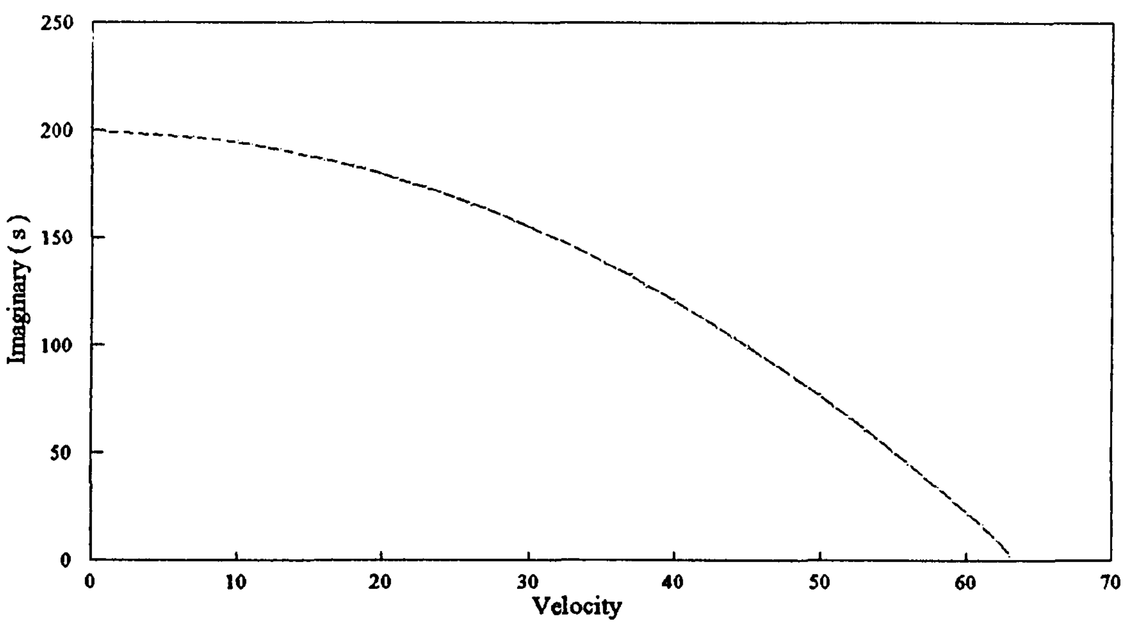

Figure 16 Imaginary part of the Eigenvalue versus Velocity at Tension=4000 Simple-Simple Support

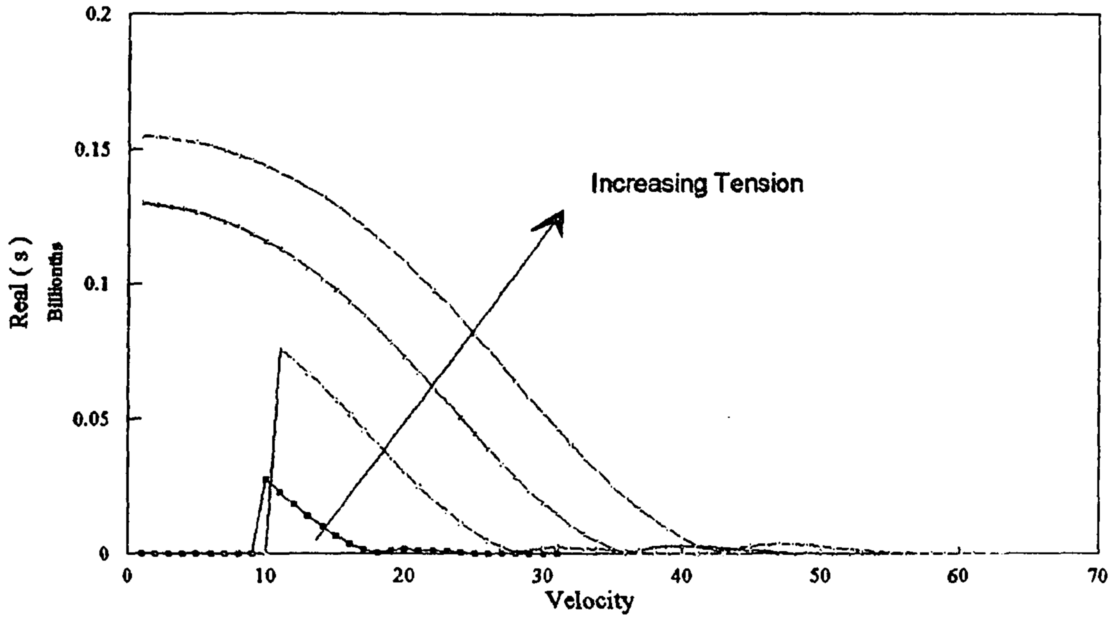

Figure 17 Real part of the First Eigenvalue versus Velocity, Simple-Simple Support

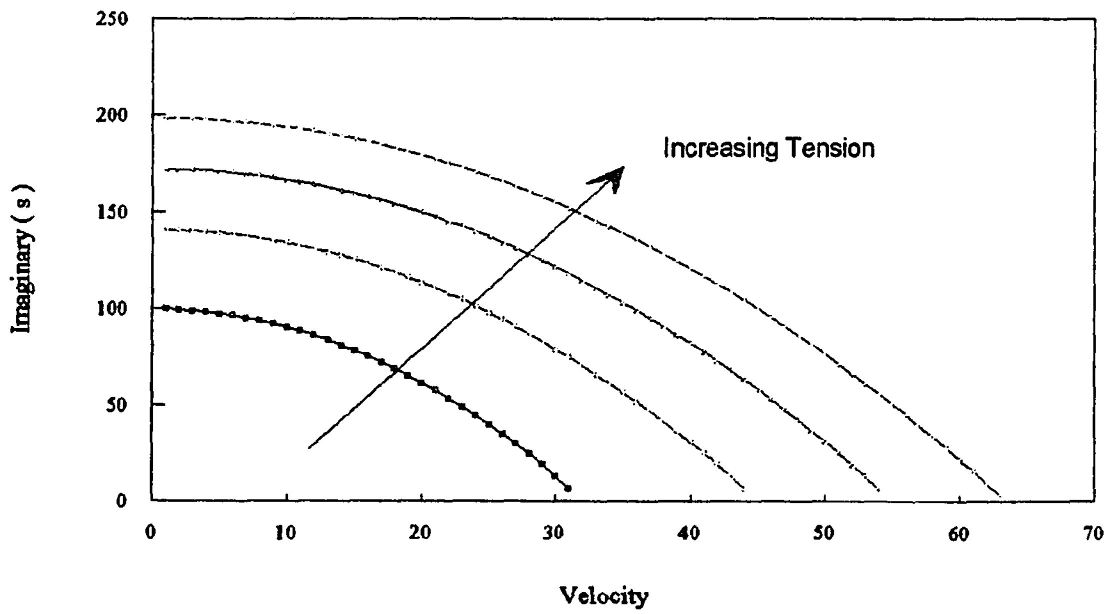

Figure 18 Imaginary part of the Eigenvalue versus Velocity, Simple-Simple Support

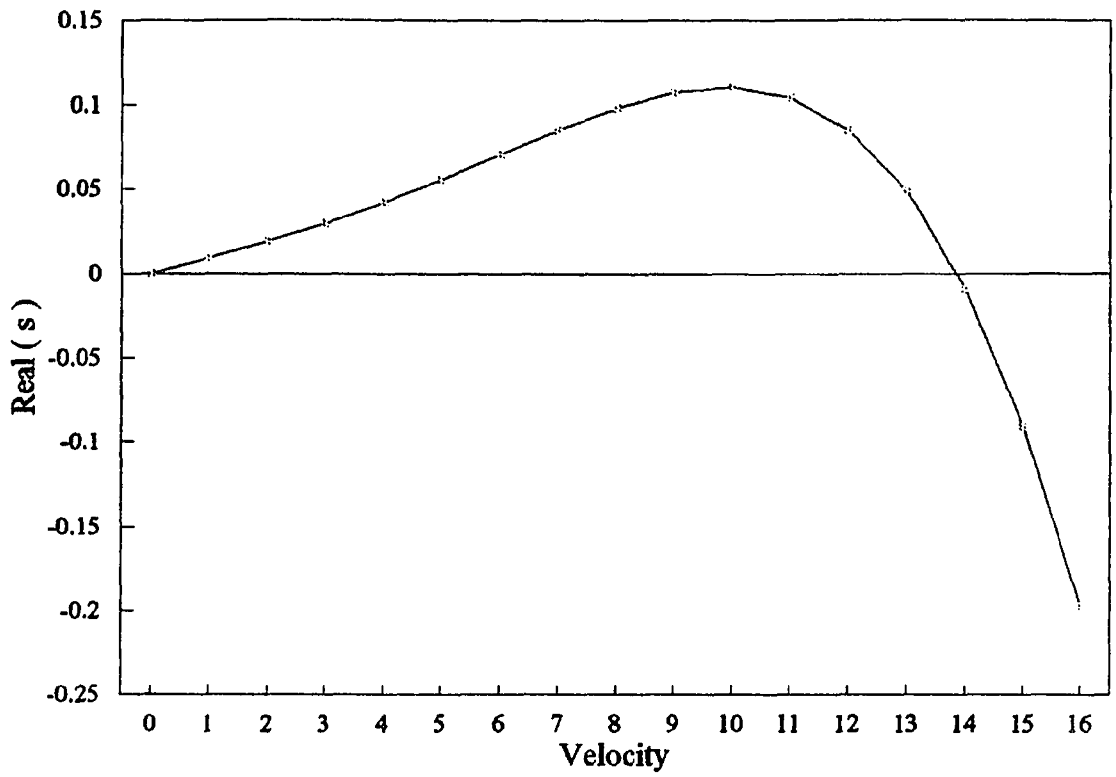

Figure 19 Real part of the Eigenvalue versus Velocity at Tension=250
Fixed-Simple Support

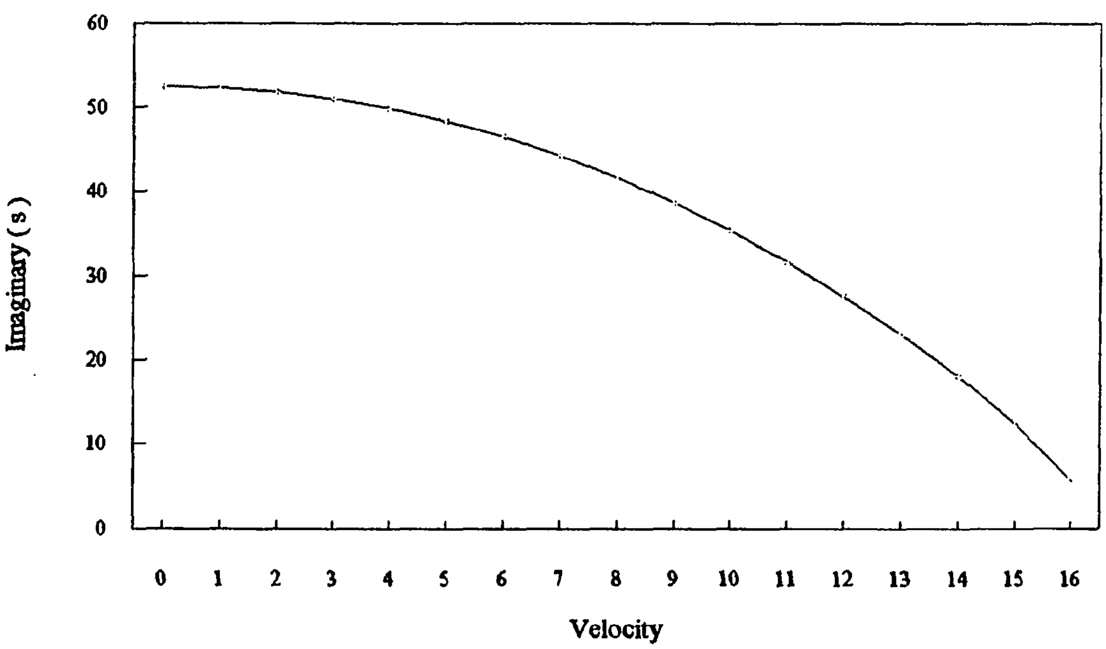

Figure 20 Imaginary part of the Eigenvalue versus Velocity at Tension=250
Fixed-Simple Support

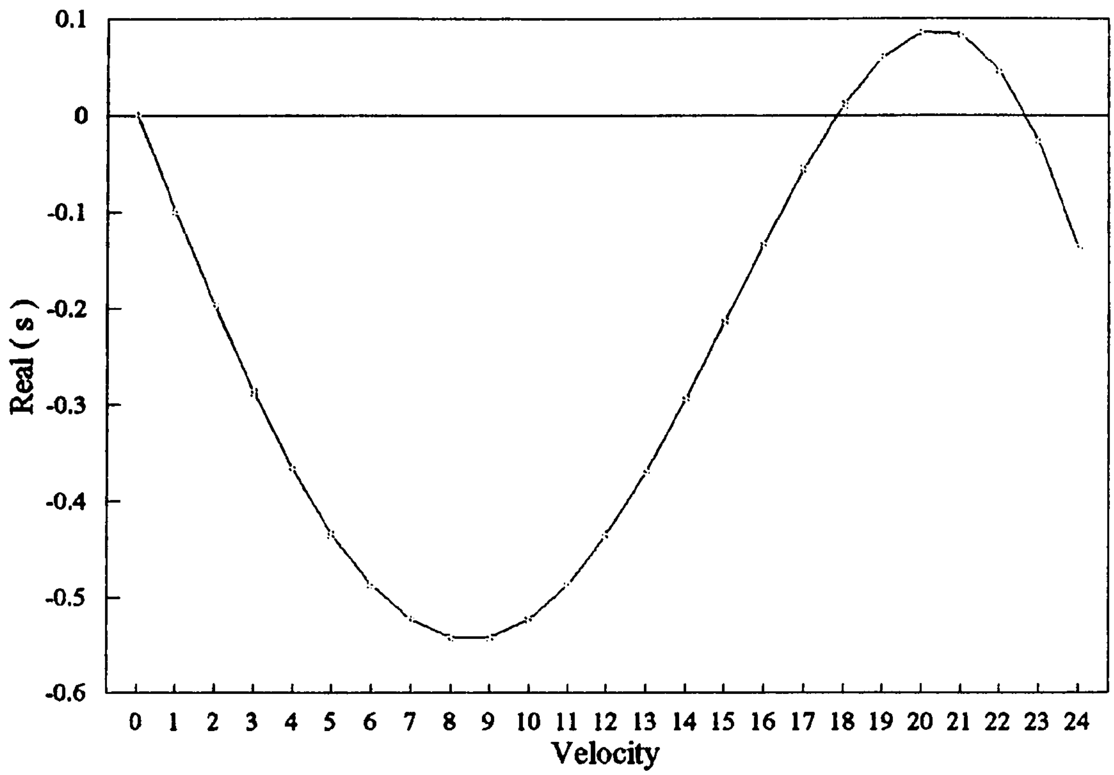

Figure 21 Real part of the Eigenvalue versus Velocity at Tension=575
Fixed-Simple Support

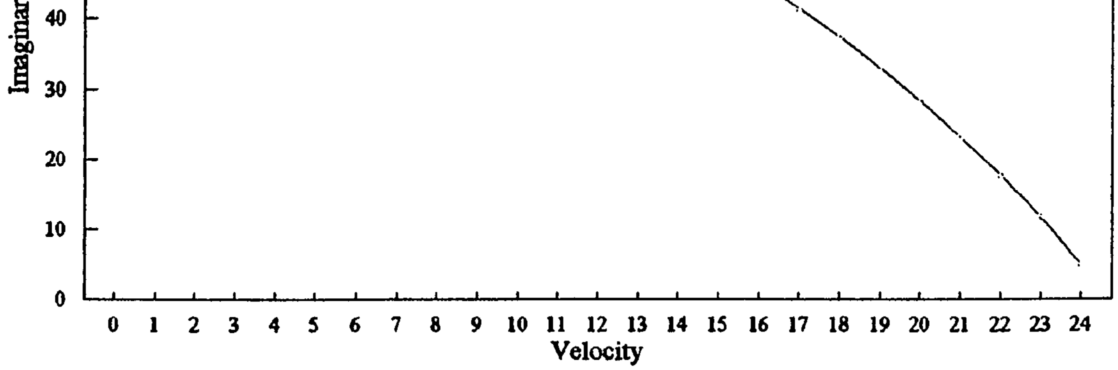

Figure 22 Imaginary part of the Eigenvalue versus Velocity at Tension=575
Fixed-Simple Support

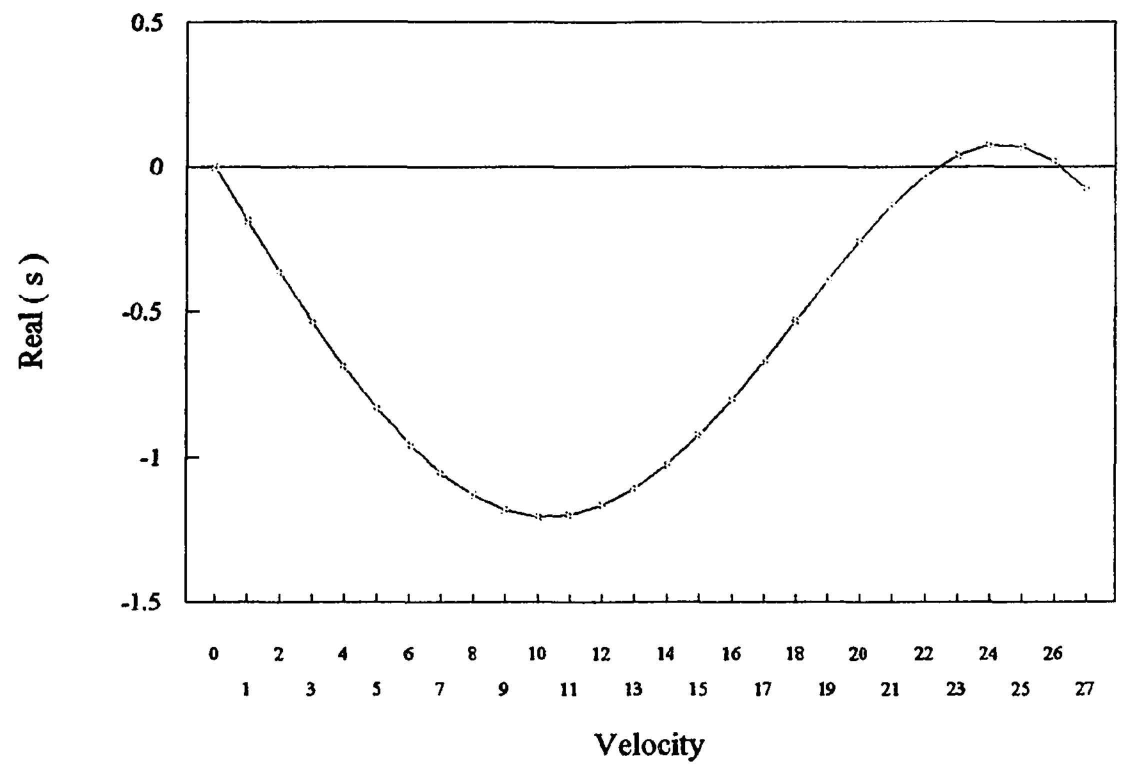

Figure 23 Real part of the Eigenvalue versus Velocity at Tension=750
Fixed-Simple Support

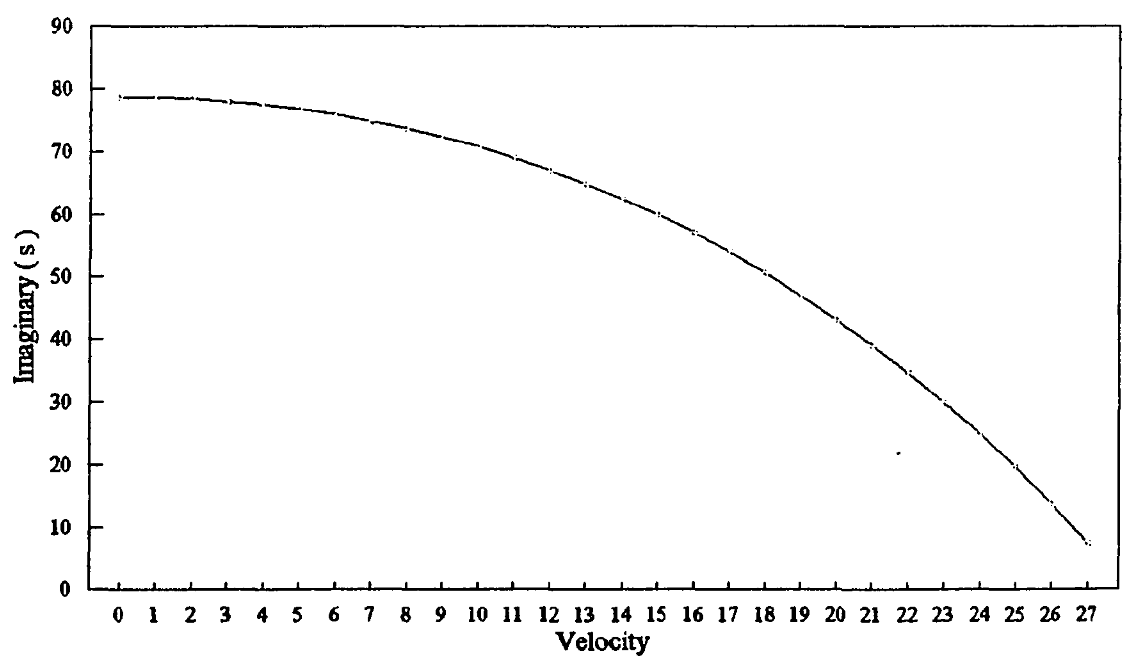

Figure 24 Imaginary part of the Eigenvalue versus Velocity at Tension=750
Fixed-Simple Support

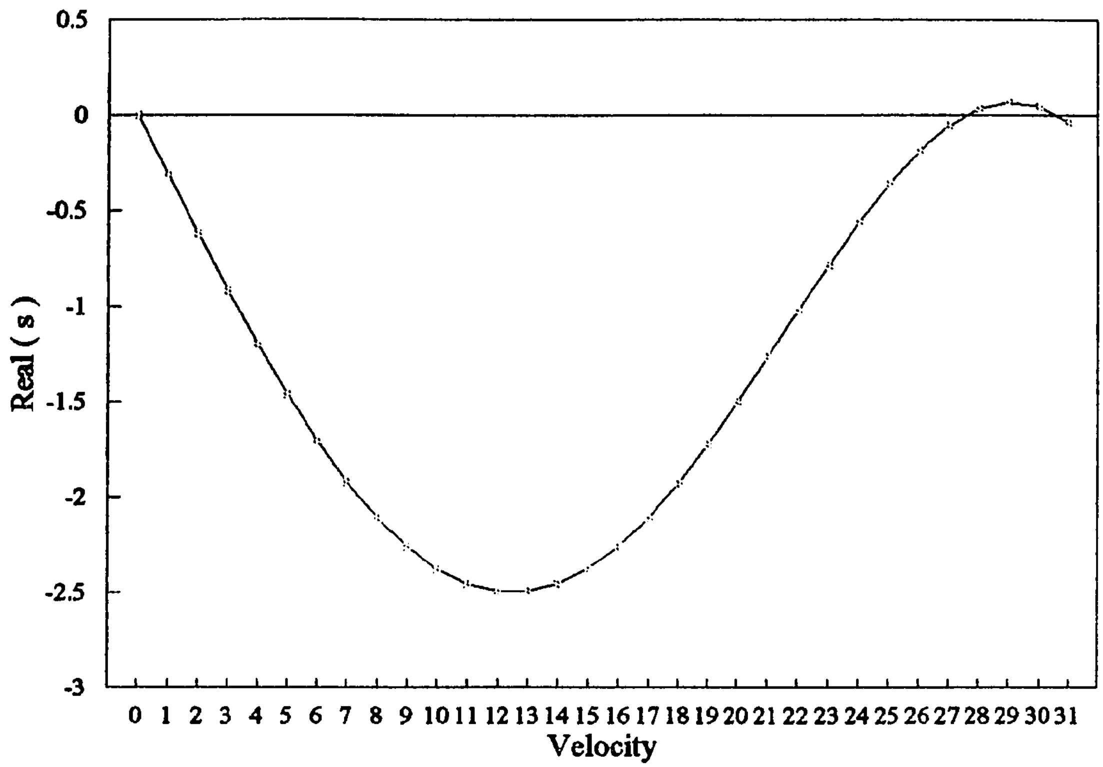

Figure 25 Real part of the Eigenvalue versus Velocity at Tension=1000
Fixed-Simple Support

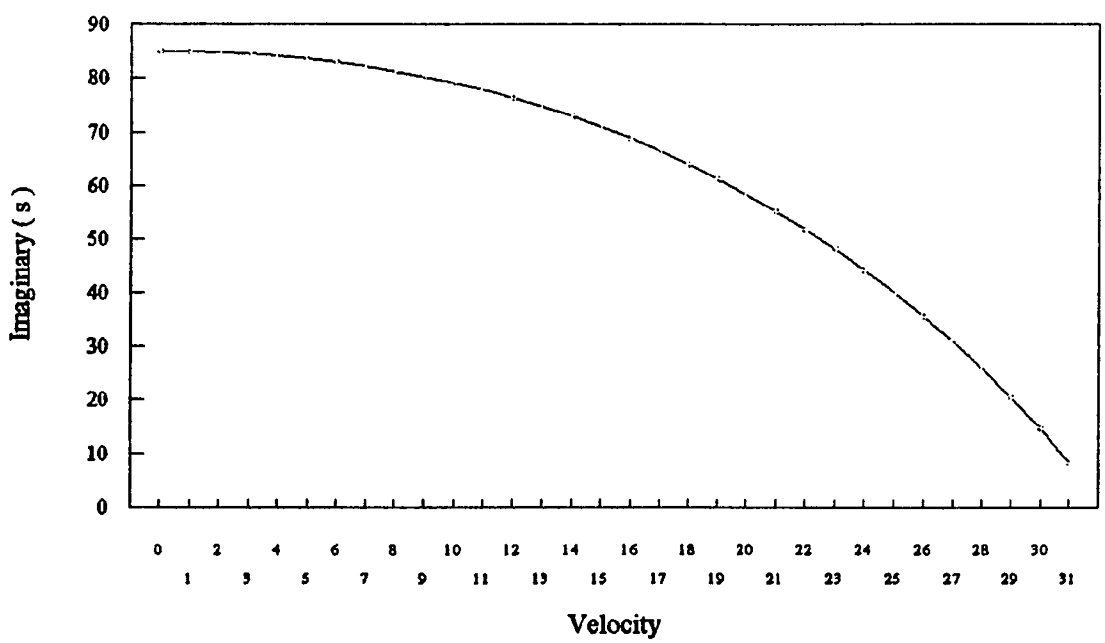

Figure 26 Imaginary part of the Eigenvalue versus Velocity at Tension=1000
Fixed-Simple Support

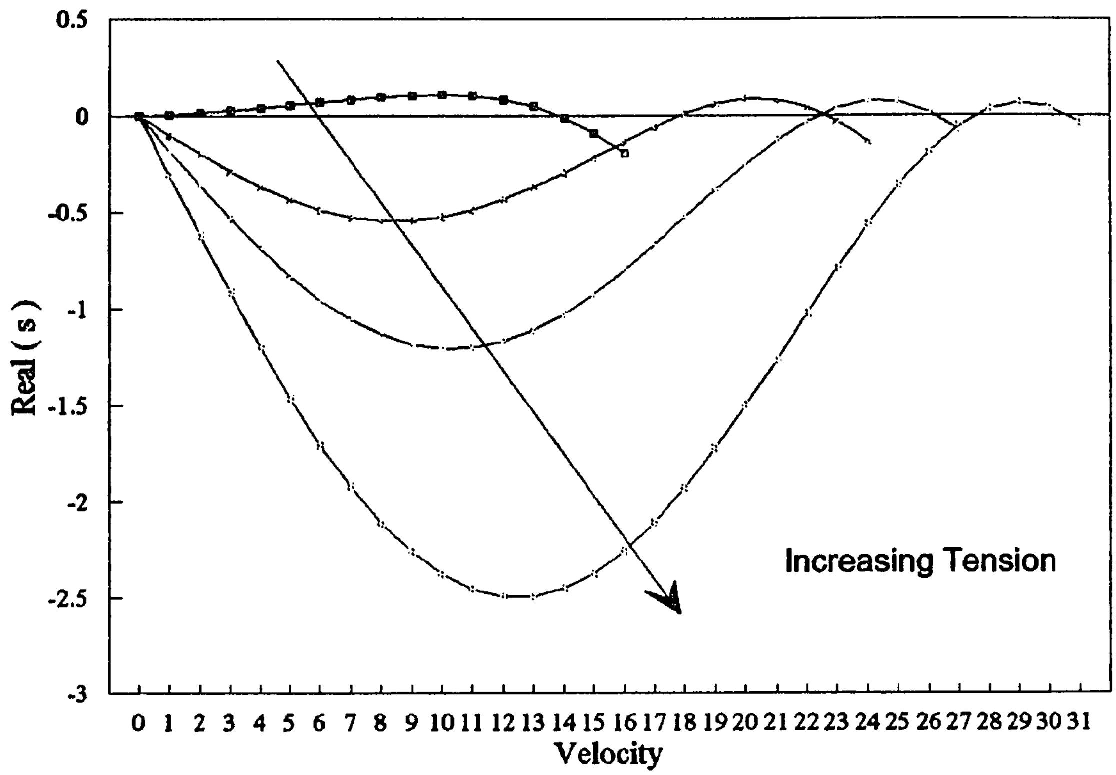

Figure 27 Real part of the Eigenvalue versus Velocity , Fixed-Simple Support

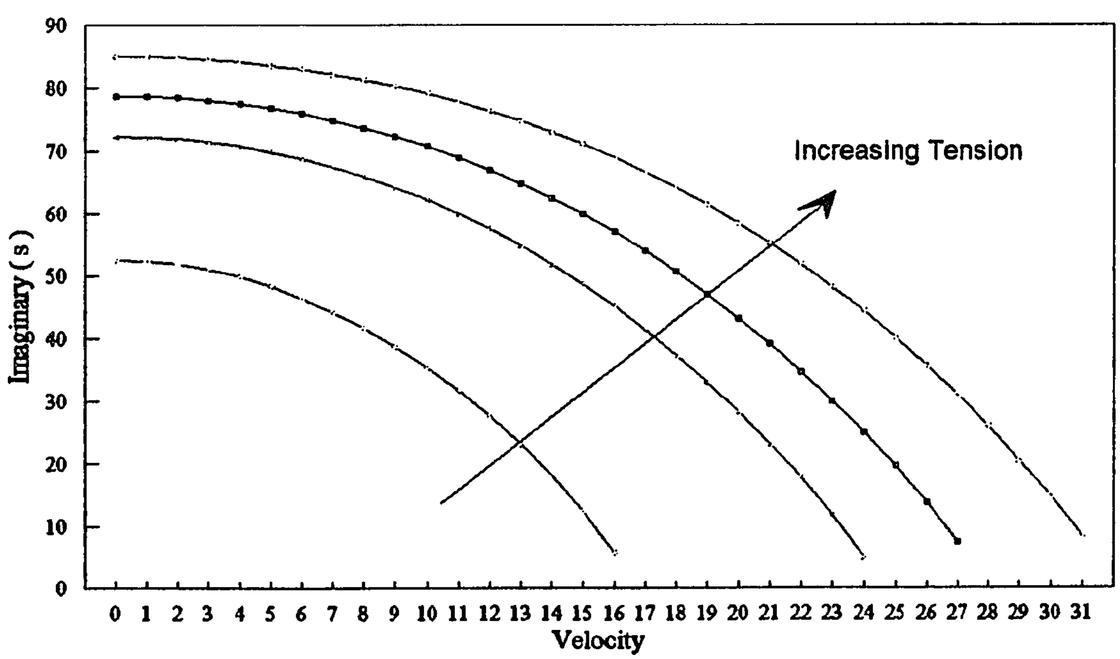

Figure 28 Imaginary part of the Eigenvalue versus Velocity , Fixed-Simple Support

APPENDIX A

To show that the governing differential operator in Equation (2.6) is not self-adjoint, the adjoint operator will be found. Comparing the adjoint and the original operators, non-self-adjointness will be observed. Equation (2.6)is the equation of motion for the axially moving Bernoulli-Euler beam and is as follows:

$$Y_{XXXX} + b_2 . Y_{XX} + b_1 . Y_{XT} + Y_{TT} = 0 \qquad \text{(AA.1)}$$

For simplicity , the boundary conditions will be taken homogenous and simple-simple supports. Then

$$u(X = 0) = u(X = 1) = u_{XX}(X = 0) = u_{XX}(X = 1) = 0 \qquad \text{(AA.2)}$$

A solution of the form $Y(X, T) = U(X)e^{\mu T}$ will be assumed and substituted into Equation AA.1. The resultant ordinary differential equation will be as follows:

$$U_{XXXX} + b_2 U_{XX} + \mu b_1 U_X + \mu^2 U = 0 \qquad \text{(AA.3)}$$

Then eigenvalue problem will be defined such that

$$u_{XXXX} + b_2 u_{XX} + \mu b_1 u_X + \mu^2 u = 0 \qquad \text{(AA.4)}$$

The governing differential operators will be

$$L = \frac{d^4}{dX^4} + b_2\frac{d^2}{dX^2} + b_1\mu\frac{d}{dX} \tag{AA.5}$$

and

$$M = \mu^2$$

To find the adjoint functions the following integral should be completed.

$$\int_0^1 \Phi L(u)dX = \int_0^1 \{\Phi\frac{d^4u}{dX^4} + \Phi b_2\frac{d^2u}{dX^2} + \Phi b_1\mu\frac{du}{dX}\}dx \tag{AA.6}$$

Applying integration by parts, the above equation will yield

$$\{\Phi.u_{XXX} - \Phi_X.u_{XX} + \Phi_{XX}.u_X - \Phi_{XXX}.u\}|_0^1 + \int_0^1 u\Phi_{XXXX}dx +$$
$$b_2\{\{\Phi.u_X - \Phi_X.u\}|_0^1 + \int_0^1 u\Phi_{XX}dx\} + b_1\mu\{\Phi u|_0^1 - \int_0^1 u\Phi_X dX\}$$

Because the boundary conditions are defined to be homogeneous all the terms evaluated at the boundaries will drop out. Then

$$\int_0^1 \{\Phi u_{XXXX} + \Phi b_2 u_{XX} + \Phi b_1 \mu u_X\}dx = \int_0^1 \{\Phi_{XXXX}u + \Phi_{XX}b_2u - \Phi_X b_1\mu u\}dx$$

The adjoint operator is not equal to the operator of the governing differential equation (Equation AA.5). Therefore self-adjointness is not possible.

APPENDIX B

The macro programs to find the polynomials for the four, five and the six elements with simple-simple supports are as follows:

Six elements:

```
n=6; x = -(20*b2*n^2-(5*b1*n*s-120*n^4-s^2))/(120*n^4); x = Expand[x]; y = s^2/n;
y = Expand[y]; z = s*(2*b1*n-s)/(2*n^2); z = Expand[z]; w =
(6*b2*n^2-s*(3*b1*n-s))/(6*n^3); w = Expand[w]; u =
-(40*b2*n^2-(15*b1*n*s+4*(30*n^4-s^2)))/(120*n^4); u = Expand[u]; A1=y/n+z;
A1=Expand[A1]; B1=w+2*n*x-5*y/(12*n^2); C1=u-x+y/(12*n^3); B1=Expand[B1];
C1=Expand[C1]; C10=x; D1=x; E1=y; F1=z; G1=w; H1=u;
A4 =
(41472*n^11*x^4-13824*n^8*x^3*(7*y-3*n*z)-288*n^5*x^2*(35*y^2-n*y*(253*z+12
*n*(14*n*u-11*w))+4*n^2*z*(61*z+21*n*(2*n*u-3*w)))-12*n^2*x*(2*y^3-n*y^2*(7
7*z+48*n*(2*n*u-w))+8*n^2*y*(49*z^2+3*n*z*(82*n*u-45*w)+36*n^2*(2*n*u-w)^2
)-16*n^3*z*(29*z^2+6*n*z*(34*n*u-21*w)+9*n^2*(20*n^2*u^2-36*n*u*w+13*w^2))
)+z*(y-4*n*(z+6*n^2*u-3*n*w))^3)/(41472*n^11*x^4); A4 = Expand[A4]; B4 =
(13824*n^9*x^4-288*n^6*x^3*(143*y-4*n*(35*z-6*n^2*u-21*n*w))-12*n^3*x^2*(12
7*y^2-4*n*y*(151*z+66*n*(17*n*u-4*w))+16*n^2*(40*z^2+3*n*z*(120*n*u-41*w)+
9*n^2*(8*n^2*u^2-34*n*u*w+9*w^2)))-x*(3*y^3-4*n*y^2*(8*z+3*n*(71*n*u-7*w))
+16*n^2*y*(7*z^2+6*n*z*(53*n*u-6*w)+9*n^2*(136*n^2*u^2-86*n*u*w+5*w^2))-6
4*n^3*(2*z^3+15*n*z^2*(7*n*u-w)+18*n^2*z*(34*n^2*u^2-25*n*u*w+2*w^2)+27*
n^3*(12*n^3*u^3-32*n^2*u^2*w+15*n*u*w^2-w^3)))+u*(y-4*n*(z+6*n^2*u-3*n*w))
^3)/(41472*n^11*x^4); B4 = Expand[B4]; A5 =
-(82944*n^11*x^4+13824*n^8*x^3*(17*y-23*n*z)+288*n^5*x^2*(39*y^2-n*y*(341*z
+12*n*(22*n*u-15*w))+4*n^2*z*(101*z+3*n*(46*n*u-45*w)))+12*n^2*x*(2*y^3-3*
n*y^2*(27*z+16*n*(2*n*u-w))+8*n^2*y*(53*z^2+3*n*z*(90*n*u-49*w)+36*n^2*(2*
n*u-w)^2)-48*n^3*z*(11*z^2+2*n*z*(42*n*u-25*w)+3*n^2*(36*n^2*u^2-52*n*u*w+
17*w^2)))-z*(y-4*n*(z+6*n^2*u-3*n*w))^3)/(13824*n^10*x^4); A5 = Expand[A5];
B5 =
-(103680*n^9*x^4+288*n^6*x^3*(247*y-4*n*(79*z+138*n^2*u-69*n*w))+12*n^3*x^
2*(139*y^2-12*n*y*(57*z+2*n*(267*n*u-52*w))+48*n^2*(16*z^2+n*z*(208*n*u-53
*w)+3*n^2*(56*n^2*u^2-82*n*u*w+13*w^2)))+x*(3*y^3-4*n*y^2*(8*z+3*n*(75*n*
u-7*w))+16*n^2*y*(7*z^2+18*n*z*(19*n*u-2*w)+9*n^2*(152*n^2*u^2-94*n*u*w+5
*w^2))-64*n^3*(2*z^3+3*n*z^2*(39*n*u-5*w)+18*n^2*z*(42*n^2*u^2-29*n*u*w+2
*w^2)+27*n^3*(28*n^3*u^3-48*n^2*u^2*w+19*n*u*w^2-w^3)))-u*(y-4*n*(z+6*n^2
*u-3*n*w))^3)/(13824*n^10*x^4); B5 = Expand[B5]; A6 =
(41472*n^11*x^4+6912*n^8*x^3*(23*y-27*n*z)+144*n^5*x^2*(73*y^2-n*y*(575*z+
12*n*(34*n*u-25*w))+4*n^2*z*(155*z+3*n*(58*n*u-63*w)))+6*n^2*x*(4*y^3-n*y^2
*(157*z+96*n*(2*n*u-w))+8*n^2*y*(101*z^2+3*n*z*(170*n*u-93*w)+72*n^2*(2*n*
u-w)^2)-16*n^3*z*(61*z^2+6*n*z*(74*n*u-45*w)+9*n^2*(52*n^2*u^2-84*n*u*w+29
```

```
*w^2)))-z*(y-4*n*(z+3*n*(2*n*u-w)))^3)/(6912*n^12*x^4); A6 = Expand[A6]; B6 =
(55296*n^9*x^4+144*n^6*x^3*(373*y-4*n*(109*z+3*n*(50*n*u-29*w)))+6*n^3*x^2
*(263*y^2-4*n*y*(317*z+6*n*(437*n*u-94*w))+16*n^2*(86*z^2+39*n*z*(24*n*u-7
*w)+9*n^2*(64*n^2*u^2-110*n*u*w+21*w^2)))+x*(3*y^3-2*n*y^2*(16*z+3*n*(145
*n*u-14*w))+16*n^2*y*(7*z^2+3*n*z*(109*n*u-12*w)+9*n^2*(142*n^2*u^2-89*n*
u*w+5*w^2))-32*n^3*(4*z^3+3*n*z^2*(73*n*u-10*w)+18*n^2*z*(74*n^2*u^2-53*n
*u*w+4*w^2)+27*n^3*(36*n^3*u^3-76*n^2*u^2*w+33*n*u*w^2-2*w^3)))-u*(y-4*n*
(z+3*n*(2*n*u-w)))^3)/(6912*n^12*x^4); B6 = Expand[B6]; A7 =
(1327104*n^11*x^4+13824*n^8*x^3*(81*y-175*n*z)+576*n^5*x^2*(30*y^2-n*y*(41
3*z+24*n*(16*n*u-9*w))+4*n^2*z*(167*z+3*n*(130*n*u-97*w)))+24*n^2*x*(y^3-3
*n*y^2*(17*z+8*n*(2*n*u-w))+8*n^2*y*(37*z^2+3*n*z*(66*n*u-35*w)+18*n^2*(2*
n*u-w)^2)-48*n^3*z*(9*z^2+2*n*z*(42*n*u-23*w)+3*n^2*(60*n^2*u^2-68*n*u*w+1
9*w^2)))-z*(y-4*n*(z+3*n*(2*n*u-w)))^3)/(331776*n^12*x^4); A7 = Expand[A7]; B7
=
(1119744*n^9*x^4+576*n^6*x^3*(431*y-4*n*(178*z+15*n*(44*n*u-13*w)))+24*n^3
*x^2*(101*y^2-24*n*y*(23*z+n*(355*n*u-47*w))+48*n^2*(15*z^2+2*n*z*(179*n*u
-29*w)+3*n^2*(200*n^2*u^2-190*n*u*w+17*w^2)))+x*(3*y^3-4*n*y^2*(8*z+3*n*(9
6*n*u-7*w))+16*n^2*y*(7*z^2+36*n*z*(13*n*u-w)+9*n^2*(236*n^2*u^2-136*n*u*
w+5*w^2))-64*n^3*(2*z^3+15*n*z^2*(12*n*u-w)+36*n^2*z*(42*n^2*u^2-25*n*u*w
+w^2)+27*n^3*(112*n^3*u^3-132*n^2*u^2*w+40*n*u*w^2-w^3)))-u*(y-4*n*(z+3*n
*(2*n*u-w)))^3)/(331776*n^12*x^4); B7 = Expand[B7]; A8 =
-(82944*n^11*x^4+13824*n^8*x^3*(27*y-37*n*z)+576*n^5*x^2*(22*y^2-n*y*(223*z
+24*n*(8*n*u-5*w))+4*n^2*z*(73*z+3*n*(38*n*u-35*w)))+24*n^2*x*(y^3-n*y^2*(4
3*z+24*n*(2*n*u-w))+8*n^2*y*(29*z^2+3*n*z*(50*n*u-27*w)+18*n^2*(2*n*u-w)^2
)-16*n^3*z*(19*z^2+6*n*z*(26*n*u-15*w)+9*n^2*(28*n^2*u^2-36*n*u*w+11*w^2))
)-z*(y-4*n*(z+6*n^2*u-3*n*w))^3)/(82944*n^11*x^4); A8 = Expand[A8]; B8 =
-(152064*n^9*x^4+576*n^6*x^3*(181*y-4*n*(62*z+108*n^2*u-57*n*w))+24*n^3*x^
2*(77*y^2-8*n*y*(49*z+3*n*(181*n*u-31*w))+16*n^2*(29*z^2+6*n*z*(77*n*u-17*
w)+27*n^2*(16*n^2*u^2-22*n*u*w+3*w^2)))+x*(3*y^3-4*n*y^2*(8*z+3*n*(80*n*u-
7*w))+16*n^2*y*(7*z^2+12*n*z*(31*n*u-3*w)+9*n^2*(172*n^2*u^2-104*n*u*w+5*
w^2))-64*n^3*(2*z^3+3*n*z^2*(44*n*u-5*w)+36*n^2*z*(26*n^2*u^2-17*n*u*w+w^
2)+27*n^3*(48*n^3*u^3-68*n^2*u^2*w+24*n*u*w^2-w^3)))-u*(y-4*n*(z+6*n^2*u-3
*n*w))^3)/(82944*n^11*x^4); B8 = Expand[B8]; A9 =
(13824*n^8*x^3*(9*y-7*n*z)+576*n^5*x^2*(18*y^2-n*y*(137*z+24*n*(4*n*u-3*w))
+4*n^2*z*(35*z+3*n*(10*n*u-13*w)))+24*n^2*x*(y^3-3*n*y^2*(13*z+8*n*(2*n*u-
w))+8*n^2*y*(25*z^2+3*n*z*(42*n*u-23*w)+18*n^2*(2*n*u-w)^2)-48*n^3*z*(5*z^
2+2*n*z*(18*n*u-11*w)+3*n^2*(12*n^2*u^2-20*n*u*w+7*w^2)))-z*(y-4*n*(z+3*n*
(2*n*u-w)))^3)/(27648*n^10*x^4); A9 = Expand[A9]; B9 =
(13824*n^9*x^4+576*n^6*x^3*(83*y-4*n*(22*z+3*n*(4*n*u-5*w)))+24*n^3*x^2*(6
5*y^2-24*n*y*(13*z+n*(103*n*u-23*w))+48*n^2*(7*z^2+2*n*z*(35*n*u-11*w)+3*n
^2*(8*n^2*u^2-22*n*u*w+5*w^2)))+x*(3*y^3-4*n*y^2*(8*z+3*n*(72*n*u-7*w))+16
*n^2*y*(7*z^2+36*n*z*(9*n*u-w)+9*n^2*(140*n^2*u^2-88*n*u*w+5*w^2))-64*n^3
*(2*z^3+3*n*z^2*(36*n*u-5*w)+36*n^2*z*(18*n^2*u^2-13*n*u*w+w^2)+27*n^3*(1
6*n^3*u^3-36*n^2*u^2*w+16*n*u*w^2-w^3)))-u*(y-4*n*(z+3*n*(2*n*u-w)))^3)/(276
48*n^10*x^4); B9 = Expand[B9]; A10 =
-(13824*n^8*x^3*(3*y-n*z)+576*n^5*x^2*(16*y^2-n*y*(97*z+48*n*(n*u-w))+4*n^2
```

```
*z*(19*z+3*n*(2*n*u-5*w)))+24*n^2*x*(y^3-n*y^2*(37*z+24*n*(2*n*u-w))+8*n^2*
y*(23*z^2+3*n*z*(38*n*u-21*w)+18*n^2*(2*n*u-w)^2)-16*n^3*z*(13*z^2+6*n*z*(1
4*n*u-9*w)+9*n^2*(4*n^2*u^2-12*n*u*w+5*w^2)))-z*(y-4*n*(z+6*n^2*u-3*n*w))^
3)/(13824*n^9*x^3); A10 = Expand[A10];  B10 =
-(576*n^6*x^3*(43*y-4*n*(8*z-3*n*w))+24*n^3*x^2*(59*y^2-8*n*y*(34*z+3*n*(67
*n*u-19*w))+16*n^2*(17*z^2+6*n*z*(17*n*u-8*w)-27*n^2*w*(2*n*u-w)))+x*(3*y^
3-4*n*y^2*(8*z+3*n*(68*n*u-7*w))+16*n^2*y*(7*z^2+12*n*z*(25*n*u-3*w)+9*n^2
*(124*n^2*u^2-80*n*u*w+5*w^2))-64*n^3*(2*z^3+3*n*z^2*(32*n*u-5*w)+36*n^2*
z*(14*n^2*u^2-11*n*u*w+w^2)-27*n^3*w*(20*n^2*u^2-12*n*u*w+w^2)))-u*(y-4*n
*(z+6*n^2*u-3*n*w))^3)/(13824*n^9*x^3); B10 = Expand[B10];  st1 = -A1*A4; st1 =
Expand[st1]; st2 = -B1*A5; st2 = Expand[st2]; st3 = -C1*A6*2*n^3; st3 = Expand[st3];
A = st1+st2+st3; A= Expand[A]; st1=-A1*B4; st1= Expand[st1]; st2=-B1*B5;
st2=Expand[st2]; st3 = -C1*B6*2*n^3; st3=Expand[st3]; B2= st1+st2+st3;
B2=Expand[B2]; st1 = -2*n^3*D1*A6; st1=Expand[st1]; st2 = -E1*A7;
st2=Expand[st2]; st3 = -F1*A8; st3=Expand[st3]; st4 = -G1*A9; st4=Expand[st4]; st5 =
-H1*A10; st5=Expand[st5]; st5 = st5/C10; st5=Expand[st5]; Ap=st1+st2+st3+st4+st5;
Ap=Expand[Ap]; st1 = -2*n^3*D1*B6;   st1=Expand[st1];  st2 = -E1*B7;
st2=Expand[st2]; st3 = -F1*B8; st3=Expand[st3]; st4 = -G1*B9; st4=Expand[st4]; st5 =
-H1*B10; st5=Expand[st5]; st5 = st5/C10; st5=Expand[st5]; Bp=st1+st2+st3+st4+st5;
Bp=Expand[Bp];  dt = A*Bp; dt=Expand[dt]; dt1 = B2*Ap; dt1=Expand[dt1]; dt=dt-dt1;
dt=Expand[dt]; dt=dt*C10^(n-2)/(4*n^4); dt=Expand[dt]; dt=Together[dt]
```

Five elements, simple-simple:

```
n=5; x=-(20*b2*n^2-(5*b1*n*s-120*n^4-s^2))/(120*n^4); x=Expand[x]; y= s^2/n;
y=Expand[y]; z=s*(2*b1*n-s)/(2*n^2); z=Expand[z];
w=(6*b2*n^2-s*(3*b1*n-s))/(6*n^3); w=Expand[w];
u=-(40*b2*n^2-(15*b1*n*s+4*(30*n^4-s^2)))/(120*n^4); u=Expand[u];
A1=y/n+z; B1=w+2*n*x-5*y/(12*n^2); C1=u-x+y/(12*n^3); C10=x; D1=x; E1=y; F1=z;
G1=w; H1=u; A1=Expand[A1]; B1=Expand[B1]; C1=Expand[C1];
A4=(1152*n^8*x^3-2016*n^5*x^2*(y-n*z)-12*n^2*x*(2*y^2-n*y*(41*z+24*n*(2*n*u
-w))+4*n^2*z*(17*z+3*n*(10*n*u-9*w)))+z*(y-4*n*(z+6*n^2*u-3*n*w))^2)/(1728*n
^8*x^3)
B4=-(48*n^6*x^3+12*n^3*x^2*(43*y-4*n*(16*z+12*n^2*u-15*n*w))+x*(3*y^2-4*n*
y*(5*z+3*n*(37*n*u-4*w))+16*n^2*(2*z^2+3*n*z*(19*n*u-3*w)+9*n^2*(6*n^2*u^2
-9*n*u*w+w^2)))-u*(y-4*n*(z+6*n^2*u-3*n*w))^2)/(1728*n^8*x^3)
A5=-(2880*n^8*x^3+288*n^5*x^2*(11*y-23*n*z)+12*n^2*x*(2*y^2-3*n*y*(15*z+8*
n*(2*n*u-w))+12*n^2*z*(7*z+n*(18*n*u-13*w)))-z*(y-4*n*(z+6*n^2*u-3*n*w))^2)/(
576*n^7*x^3)
B5=-(2832*n^6*x^3+12*n^3*x^2*(55*y-12*n*(8*z+28*n^2*u-9*n*w))+x*(3*y^2-4*n
*y*(5*z+3*n*(41*n*u-4*w))+16*n^2*(2*z^2+3*n*z*(23*n*u-3*w)+9*n^2*(14*n^2*u
^2-13*n*u*w+w^2)))-u*(y-4*n*(z+6*n^2*u-3*n*w))^2)/(576*n^7*x^3)
```

```
A6=(1440*n^8*x^3+144*n^5*x^2*(17*y-29*n*z)+6*n^2*x*(4*y^2-n*y*(85*z+48*n*(
2*n*u-w))+4*n^2*z*(37*z+3*n*(26*n*u-21*w)))-z*(y-4*n*(z+3*n*(2*n*u-w)))^2)/(2
88*n^9*x^3)
B6=(1560*n^6*x^3+6*n^3*x^2*(95*y-4*n*(38*z+3*n*(32*n*u-13*w)))+x*(3*y^2-2*
n*y*(10*z+3*n*(77*n*u-8*w))+8*n^2*(4*z^2+3*n*z*(41*n*u-6*w)+9*n^2*(18*n^2*
u^2-21*n*u*w+2*w^2)))-u*(y-4*n*(z+3*n*(2*n*u-w)))^2)/(288*n^9*x^3)
A7=(41472*n^8*x^3+576*n^5*x^2*(16*y-65*n*z)+24*n^2*x*(y^2-3*n*y*(11*z+4*n
*(2*n*u-w))+12*n^2*z*(7*z+n*(30*n*u-17*w)))-z*(y-4*n*(z+3*n*(2*n*u-w)))^2)/(13
824*n^9*x^3)
B7=(24768*n^6*x^3+24*n^3*x^2*(59*y-12*n*(11*z+5*n*(20*n*u-3*w)))+x*(3*y^2-
4*n*y*(5*z+6*n*(31*n*u-2*w))+16*n^2*(2*z^2+3*n*z*(44*n*u-3*w)+9*n^2*(56*n^
2*u^2-34*n*u*w+w^2)))-u*(y-4*n*(z+3*n*(2*n*u-w)))^2)/(13824*n^9*x^3)
A8=-(3456*n^8*x^3+576*n^5*x^2*(8*y-19*n*z)+24*n^2*x*(y^2-n*y*(25*z+12*n*(2
*n*u-w))+4*n^2*z*(13*z+3*n*(14*n*u-9*w)))-z*(y-4*n*(z+6*n^2*u-3*n*w))^2)/(345
6*n^8*x^3)
B8=-(4608*n^6*x^3+24*n^3*x^2*(35*y-4*n*(17*z+72*n^2*u-21*n*w))+x*(3*y^2-4*
n*y*(5*z+6*n*(23*n*u-2*w))+16*n^2*(2*z^2+3*n*z*(28*n*u-3*w)+9*n^2*(24*n^2*
u^2-18*n*u*w+w^2)))-u*(y-4*n*(z+6*n^2*u-3*n*w))^2)/(3456*n^8*x^3)
A9=(576*n^5*x^2*(4*y-5*n*z)+24*n^2*x*(y^2-3*n*y*(7*z+4*n*(2*n*u-w))+12*n^2
*z*(3*z+n*(6*n*u-5*w)))-z*(y-4*n*(z+3*n*(2*n*u-w)))^2)/(1152*n^7*x^3)
B9=(576*n^6*x^3+24*n^3*x^2*(23*y-12*n*(3*z+n*(4*n*u-3*w)))+x*(3*y^2-4*n*y*(
5*z+6*n*(19*n*u-2*w))+16*n^2*(2*z^2+3*n*z*(20*n*u-3*w)+9*n^2*(8*n^2*u^2-10
*n*u*w+w^2)))-u*(y-4*n*(z+3*n*(2*n*u-w)))^2)/(1152*n^7*x^3)
A10=-(576*n^5*x^2*(2*y-n*z)+24*n^2*x*(y^2-n*y*(19*z+12*n*(2*n*u-w))+4*n^2*z
*(7*z+3*n*(2*n*u-3*w)))-z*(y-4*n*(z+6*n^2*u-3*n*w))^2)/(576*n^6*x^2)
B10=-(24*n^3*x^2*(17*y-4*n*(5*z-3*n*w))+x*(3*y^2-4*n*y*(5*z+6*n*(17*n*u-2*w
))+16*n^2*(2*z^2+3*n*z*(16*n*u-3*w)-9*n^2*w*(6*n*u-w)))-u*(y-4*n*(z+6*n^2*u-
3*n*w))^2)/(576*n^6*x^2) B10=Expand[B10]; st1 = -A1*A4; st1 = Expand[st1]; st2 =
-B1*A5;   st2 = Expand[st2];   st3 = -C1*A6*2*n^3;   st3 = Expand[st3];   A =
st1+st2+st3;   A= Expand[A];   st1=-A1*B4;   st1= Expand[st1];   st2=-B1*B5;
 st2=Expand[st2];   st3 = -C1*B6*2*n^3;   st3=Expand[st3];   B2= st1+st2+st3;
 B2=Expand[B2];   st1 = -2*n^3*D1*A6;   st1=Expand[st1];   st2 = -E1*A7;
st2=Expand[st2];   st3 = -F1*A8;   st3=Expand[st3];   st4 = -G1*A9;
st4=Expand[st4];   st5 = -H1*A10;   st5=Expand[st5];   st5 = st5/C10;
st5=Expand[st5];   Ap=st1+st2+st3+st4+st5;   Ap=Expand[Ap];   st1 =
-2*n^3*D1*B6; st1=Expand[st1];   st2 = -E1*B7;   st2=Expand[st2];   st3 =
-F1*B8;   st3=Expand[st3];   st4 = -G1*B9;   st4=Expand[st4];   st5 = -H1*B10;
  st5=Expand[st5];   st5 = st5/C10;   st5=Expand[st5];   Bp=st1+st2+st3+st4+st5;
Bp=Expand[Bp];   dt = A*Bp;   dt=Expand[dt];   dt1 = B2*Ap;   dt1=Expand[dt1];
  dt=dt-dt1;   dt=Expand[dt];   dt=dt*C10^(n-2)/(4*n^4);   dt=Expand[dt];
dt=Together[dt]
```

Four elements, simple-simple

```
x=-(20*b2*n^2-(5*b*n*s-120*n^4-s^2))/(120*n^4); x=Expand[x]; y= s^2/n;
y=Expand[y]; z=s*(2*b*n-s)/(2*n^2); z=Expand[z]; w=(6*b2*n^2-s*(3*b*n-s))/(6*n^3);
w=Expand[w]; u=-(40*b2*n^2-(15*b*n*s+4*(30*n^4-s^2)))/(120*n^4); u=Expand[u];
m4={{1,1/(12*n^2),0,-1,1/(2*n),-1/(12*n^2),0,0,0,0,0,0,0,0,0,0},
{0,1/(2*n),0,0,-1,1/(2*n),0,0,0,0,0,0,0,0,0,0},
{0,0,-1,1/n,-5/(12*n^2),1/(12*n^3),0,0,0,0,0,0,0,0,0,0},
{0,0,0,1,1/(2*n),1/(12*n^2),0,-1,1/(2*n),-1/(12*n^2),0,0,0,0,0,0},
{0,0,0,0,1,1/(2*n),0,0,-1,1/(2*n),0,0,0,0,0,0},
{0,0,0,0,0,1/(2*n^3),-1,2/n,-23/(12*n^2),2/(3*n^3),0,0,0,0,0,0},
{0,0,0,0,0,0,1,1/(2*n),1/(10*n^2),1/(120*n^3),-1,1/(2*n),-1/(10*n^2),1/(120*n^3),0,0},
{0,0,0,0,0,0,0,1,1/(2*n),1/(12*n^2),0,-1,1/(2*n),-1/(12*n^2),0,0},
{0,0,0,0,0,0,0,0,1,1/(2*n),0,0,-1,1/(2*n),0,0}, {0,0,0,0,0,0,0,0,0,x,y,z,w,u,0,0},
{0,0,0,0,0,0,0,0,0,0,1,1/(2*n),1/(10*n^2),1/(120*n^3),1/(2*n),1/(120*n^3)},
{0,0,0,0,0,0,0,0,0,0,0,1,1/(2*n),1/(12*n^2),-1,-1/(12*n^2)},
{0,0,0,0,0,0,0,0,0,0,0,0,1,1/(2*n),0,1/(2*n)}, {0,0,0,0,0,0,0,0,0,0,0,0,0,x,z,u},
{0,0,0,y/n+z,2*n*x-5*y/(12*n^2)+w,-x+y/(12*n^3)+u,0,0,0,0,0,0,0,0,0,0},
{0,0,0,0,0,x,y,z,w,u,0,0,0,0,0,0}}; dt=Det[m4]; dt=Together[dt]
```

Six elements, fixed-simple support:

```
n=6; x = -(20*b*n^2-(5*b1*n*s-120*n^4-s^2))/(120*n^4); x = Expand[x]; y =  s^2/n; y
= Expand[y]; z = s*(2*b1*n-s)/(2*n^2); z = Expand[z]; w =
(6*b*n^2-s*(3*b1*n-s))/(6*n^3); w = Expand[w]; u =
-(40*b*n^2-(15*b1*n*s+4*(30*n^4-s^2)))/(120*n^4); u = Expand[u];
A1=-6*n^2*x+(3*y)/(4*n)+z; B1=w+6*n*x-y/(4*n^2); C1=u-2*x+y/(24*n^3);
A1=Expand[A1]; B1=Expand[B1]; C1=Expand[C1]; C10=x; D1=x; E1=y; F1=z; G1=w;
H1=u;
A4=(622080*n^11*x^4-27648*n^8*x^3*(3*y+23*n*z)-1152*n^5*x^2*(41*y^2-3*n*y*
(85*z+4*n*(12*n*u-11*w))+12*n^2*z*(16*z-3*n*(2*n*u+3*w)))-24*n^2*x*(5*y^3-n
*y^2*(187*z+120*n*(2*n*u-w))+24*n^2*y*(39*z^2+n*z*(194*n*u-107*w)+30*n^2*(
2*n*u-w)^2)-16*n^3*z*(67*z^2+6*n*z*(74*n*u-47*w)+9*n^2*(28*n^2*u^2-68*n*u*
w+27*w^2)))+5*z*(y-4*n*(z+6*n^2*u-3*n*w))^3)/(331776*n^11*x^4);
B4=(442368*n^9*x^4-3456*n^6*x^3*(35*y-4*n*(4*z-40*n^2*u+3*n*w))-24*n^3*x^2
*(301*y^2-8*n*y*(175*z+3*n*(356*n*u-99*w))+16*n^2*(89*z^2+6*n*z*(88*n*u-43
*w)-9*n^2*(52*n^2*u^2+16*n*u*w-17*w^2)))-x*(15*y^3-4*n*y^2*(40*z+3*n*(344*
n*u-35*w))+16*n^2*y*(35*z^2+12*n*z*(127*n*u-15*w)+9*n^2*(636*n^2*u^2-408*
n*u*w+25*w^2))-64*n^3*(10*z^3+3*n*z^2*(164*n*u-25*w)+36*n^2*z*(74*n^2*u^2
-57*n*u*w+5*w^2)+27*n^3*(16*n^3*u^3-116*n^2*u^2*w+64*n*u*w^2-5*w^3)))+5*
u*(y-4*n*(z+6*n^2*u-3*n*w))^3)/(331776*n^11*x^4);
```

```
A5=-(953856*n^11*x^4+27648*n^8*x^3*(93*y-127*n*z)+1152*n^5*x^2*(107*y^2-3
*n*y*(311*z+4*n*(60*n*u-41*w))+12*n^2*z*(92*z+3*n*(42*n*u-41*w)))+24*n^2*x
*(11*y^3-n*y^2*(445*z+264*n*(2*n*u-w))+24*n^2*y*(97*z^2+n*z*(494*n*u-269*w
)+66*n^2*(2*n*u-w)^2)-16*n^3*z*(181*z^2+6*n*z*(230*n*u-137*w)+9*n^2*(196*n
^2*u^2-284*n*u*w+93*w^2)))-11*z*(y-4*n*(z+6*n^2*u-3*n*w))^3)/(110592*n^10*x
^4);
B5=-(1161216*n^9*x^4+3456*n^6*x^3*(225*y-4*n*(72*z+128*n^2*u-63*n*w))+24*
n^3*x^2*(763*y^2-8*n*y*(469*z+15*n*(292*n*u-57*w))+16*n^2*(263*z^2+6*n*z*(
568*n*u-145*w)+9*n^2*(308*n^2*u^2-448*n*u*w+71*w^2)))+x*(33*y^3-4*n*y^2*(
88*z+3*n*(824*n*u-77*w))+16*n^2*y*(77*z^2+12*n*z*(313*n*u-33*w)+9*n^2*(16
68*n^2*u^2-1032*n*u*w+55*w^2))-64*n^3*(22*z^3+3*n*z^2*(428*n*u-55*w)+36*n
^2*z*(230*n^2*u^2-159*n*u*w+11*w^2)+27*n^3*(304*n^3*u^3-524*n^2*u^2*w+20
8*n*u*w^2-11*w^3)))-11*u*(y-4*n*(z+6*n^2*u-3*n*w))^3)/(110592*n^10*x^4);
A6=(953856*n^11*x^4+359424*n^8*x^3*(9*y-11*n*z)+1152*n^5*x^2*(175*y^2-3*n
*y*(467*z+4*n*(84*n*u-61*w))+12*n^2*z*(128*z+3*n*(50*n*u-53*w)))+24*n^2*x*
(19*y^3-n*y^2*(749*z+456*n*(2*n*u-w))+24*n^2*y*(161*z^2+n*z*(814*n*u-445*w
)+114*n^2*(2*n*u-w)^2)-16*n^3*z*(293*z^2+6*n*z*(358*n*u-217*w)+9*n^2*(260*
n^2*u^2-412*n*u*w+141*w^2)))-19*z*(y-4*n*(z+3*n*(2*n*u-w)))^3)/(165888*n^12*
x^4);
B6=(1216512*n^9*x^4+3456*n^6*x^3*(309*y-4*n*(92*z+n*(136*n*u-75*w)))+24*n
^3*x^2*(1259*y^2-8*n*y*(761*z+3*n*(2140*n*u-453*w))+16*n^2*(415*z^2+6*n*z*
(776*n*u-221*w)+9*n^2*(340*n^2*u^2-560*n*u*w+103*w^2)))+x*(57*y^3-4*n*y^2
*(152*z+3*n*(1384*n*u-133*w))+16*n^2*y*(133*z^2+12*n*z*(521*n*u-57*w)+9*n
^2*(2724*n^2*u^2-1704*n*u*w+95*w^2))-64*n^3*(38*z^3+15*n*z^2*(140*n*u-19*
w)+36*n^2*z*(358*n^2*u^2-255*n*u*w+19*w^2)+27*n^3*(368*n^3*u^3-748*n^2*u
^2*w+320*n*u*w^2-19*w^3)))-19*u*(y-4*n*(z+3*n*(2*n*u-w)))^3)/(165888*n^12*x
^4);
A7 =
(1327104*n^11*x^4+13824*n^8*x^3*(81*y-175*n*z)+576*n^5*x^2*(30*y^2-n*y*(41
3*z+24*n*(16*n*u-9*w))+4*n^2*z*(167*z+3*n*(130*n*u-97*w)))+24*n^2*x*(y^3-3
*n*y^2*(17*z+8*n*(2*n*u-w))+8*n^2*y*(37*z^2+3*n*z*(66*n*u-35*w)+18*n^2*(2*
n*u-w)^2)-48*n^3*z*(9*z^2+2*n*z*(42*n*u-23*w)+3*n^2*(60*n^2*u^2-68*n*u*w+1
9*w^2)))-z*(y-4*n*(z+3*n*(2*n*u-w)))^3)/(331776*n^12*x^4); A7 = Expand[A7]; B7
=
(1119744*n^9*x^4+576*n^6*x^3*(431*y-4*n*(178*z+15*n*(44*n*u-13*w)))+24*n^3
*x^2*(101*y^2-24*n*y*(23*z+n*(355*n*u-47*w))+48*n^2*(15*z^2+2*n*z*(179*n*u
-29*w)+3*n^2*(200*n^2*u^2-190*n*u*w+17*w^2)))+x*(3*y^3-4*n*y^2*(8*z+3*n*(9
6*n*u-7*w))+16*n^2*y*(7*z^2+36*n*z*(13*n*u-w)+9*n^2*(236*n^2*u^2-136*n*u*
w+5*w^2))-64*n^3*(2*z^3+15*n*z^2*(12*n*u-w)+36*n^2*z*(42*n^2*u^2-25*n*u*w
+w^2)+27*n^3*(112*n^3*u^3-132*n^2*u^2*w+40*n*u*w^2-w^3)))-u*(y-4*n*(z+3*n
*(2*n*u-w)))^3)/(331776*n^12*x^4);B7 = Expand[B7]; A8 =
-(82944*n^11*x^4+13824*n^8*x^3*(27*y-37*n*z)+576*n^5*x^2*(22*y^2-n*y*(223*z
+24*n*(8*n*u-5*w))+4*n^2*z*(73*z+3*n*(38*n*u-35*w)))+24*n^2*x*(y^3-n*y^2*(4
3*z+24*n*(2*n*u-w))+8*n^2*y*(29*z^2+3*n*z*(50*n*u-27*w)+18*n^2*(2*n*u-w)^2
)-16*n^3*z*(19*z^2+6*n*z*(26*n*u-15*w)+9*n^2*(28*n^2*u^2-36*n*u*w+11*w^2))
)-z*(y-4*n*(z+6*n^2*u-3*n*w))^3)/(82944*n^11*x^4); A8 = Expand[A8]; B8 =
```

```
-(152064*n^9*x^4+576*n^6*x^3*(181*y-4*n*(62*z+108*n^2*u-57*n*w))+24*n^3*x^
2*(77*y^2-8*n*y*(49*z+3*n*(181*n*u-31*w))+16*n^2*(29*z^2+6*n*z*(77*n*u-17*
w)+27*n^2*(16*n^2*u^2-22*n*u*w+3*w^2)))+x*(3*y^3-4*n*y^2*(8*z+3*n*(80*n*u-
7*w))+16*n^2*y*(7*z^2+12*n*z*(31*n*u-3*w)+9*n^2*(172*n^2*u^2-104*n*u*w+5*
w^2))-64*n^3*(2*z^3+3*n*z^2*(44*n*u-5*w)+36*n^2*z*(26*n^2*u^2-17*n*u*w+w^
2)+27*n^3*(48*n^3*u^3-68*n^2*u^2*w+24*n*u*w^2-w^3)))-u*(y-4*n*(z+6*n^2*u-3
*n*w))^3)/(82944*n^11*x^4); B8 = Expand[B8]; A9 =
(13824*n^8*x^3*(9*y-7*n*z)+576*n^5*x^2*(18*y^2-n*y*(137*z+24*n*(4*n*u-3*w))
+4*n^2*z*(35*z+3*n*(10*n*u-13*w)))+24*n^2*x*(y^3-3*n*y^2*(13*z+8*n*(2*n*u-
w))+8*n^2*y*(25*z^2+3*n*z*(42*n*u-23*w)+18*n^2*(2*n*u-w)^2)-48*n^3*z*(5*z^
2+2*n*z*(18*n*u-11*w)+3*n^2*(12*n^2*u^2-20*n*u*w+7*w^2)))-z*(y-4*n*(z+3*n*
(2*n*u-w)))^3)/(27648*n^10*x^4); A9 = Expand[A9]; B9 =
(13824*n^9*x^4+576*n^6*x^3*(83*y-4*n*(22*z+3*n*(4*n*u-5*w)))+24*n^3*x^2*(6
5*y^2-24*n*y*(13*z+n*(103*n*u-23*w))+48*n^2*(7*z^2+2*n*z*(35*n*u-11*w)+3*n
^2*(8*n^2*u^2-22*n*u*w+5*w^2)))+x*(3*y^3-4*n*y^2*(8*z+3*n*(72*n*u-7*w))+16
*n^2*y*(7*z^2+36*n*z*(9*n*u-w)+9*n^2*(140*n^2*u^2-88*n*u*w+5*w^2))-64*n^3
*(2*z^3+3*n*z^2*(36*n*u-5*w)+36*n^2*z*(18*n^2*u^2-13*n*u*w+w^2)+27*n^3*(1
6*n^3*u^3-36*n^2*u^2*w+16*n*u*w^2-w^3)))-u*(y-4*n*(z+3*n*(2*n*u-w)))^3)/(276
48*n^10*x^4); B9 = Expand[B9]; A10 =
-(13824*n^8*x^3*(3*y-n*z)+576*n^5*x^2*(16*y^2-n*y*(97*z+48*n*(n*u-w))+4*n^2
*z*(19*z+3*n*(2*n*u-5*w)))+24*n^2*x*(y^3-n*y^2*(37*z+24*n*(2*n*u-w))+8*n^2*
y*(23*z^2+3*n*z*(38*n*u-21*w)+18*n^2*(2*n*u-w)^2)-16*n^3*z*(13*z^2+6*n*z*(1
4*n*u-9*w)+9*n^2*(4*n^2*u^2-12*n*u*w+5*w^2)))-z*(y-4*n*(z+6*n^2*u-3*n*w))^
3)/(13824*n^9*x^3); A10 = Expand[A10]; B10 =
-(576*n^6*x^3*(43*y-4*n*(8*z-3*n*w))+24*n^3*x^2*(59*y^2-8*n*y*(34*z+3*n*(67
*n*u-19*w))+16*n^2*(17*z^2+6*n*z*(17*n*u-8*w)-27*n^2*w*(2*n*u-w)))+x*(3*y^
3-4*n*y^2*(8*z+3*n*(68*n*u-7*w))+16*n^2*y*(7*z^2+12*n*z*(25*n*u-3*w)+9*n^2
*(124*n^2*u^2-80*n*u*w+5*w^2))-64*n^3*(2*z^3+3*n*z^2*(32*n*u-5*w)+36*n^2*
z*(14*n^2*u^2-11*n*u*w+w^2)-27*n^3*w*(20*n^2*u^2-12*n*u*w+w^2)))-u*(y-4*n
*(z+6*n^2*u-3*n*w))^3)/(13824*n^9*x^3); B10 = Expand[B10]; st1 = -A1*A4; st1 =
Expand[st1]; st2 = -B1*A5; st2 = Expand[st2]; st3 = -C1*A6*2*n^3; st3 = Expand[st3];
A = st1+st2+st3; A= Expand[A]; st1=-A1*B4; st1= Expand[st1]; st2=-B1*B5;
st2=Expand[st2]; st3 = -C1*B6*2*n^3; st3=Expand[st3]; B= st1+st2+st3; B=Expand[B];
st1 = -2*n^3*D1*A6; st1=Expand[st1]; st2 = -E1*A7; st2=Expand[st2]; st3 = -F1*A8;
st3=Expand[st3]; st4 = -G1*A9; st4=Expand[st4]; st5 = -H1*A10; st5=Expand[st5]; st5 =
st5/C10; st5=Expand[st5]; Ap=st1+st2+st3+st4+st5; Ap=Expand[Ap]; st1 =
-2*n^3*D1*B6; st1=Expand[st1]; st2 = -E1*B7; st2=Expand[st2]; st3 = -F1*B8;
st3=Expand[st3]; st4 = -G1*B9; st4=Expand[st4]; st5 = -H1*B10; st5=Expand[st5]; st5 =
st5/C10; st5=Expand[st5]; Bp=st1+st2+st3+st4+st5; Bp=Expand[Bp]; dt = A*Bp;
dt=Expand[dt]; dt1 = B*Ap; dt1=Expand[dt1]; dt=dt-dt1; dt=Expand[dt];
dt=dt*C10^(n-2)/(4*n^4); dt=Expand[dt]; dt=Together[dt]; NSolve[dt==0,s]
```

Five elements, fixed-simple support:

```
b=-100; c=20; n=5; x=-(20*b*n^2-(5*c*n*s-120*n^4-s^2))/(120*n^4); x=Expand[x]; y=
s^2/n; y=Expand[y]; z=s*(2*c*n-s)/(2*n^2); z=Expand[z];
w=(6*b*n^2-s*(3*c*n-s))/(6*n^3); w=Expand[w];
u=-(40*b*n^2-(15*c*n*s+4*(30*n^4-s^2)))/(120*n^4); u=Expand[u];
A1=-6*n^2*x+(3*y)/(4*n)+z; B1=w+6*n*x-y/(4*n^2); C1=u-2*x+y/(24*n^3);
A1=Expand[A1]; B1=Expand[B1]; C1=Expand[C1]; C10=x; D1=x; E1=y; F1=z; G1=w;
H1=u;
A4=(19008*n^8*x^3-3456*n^5*x^2*(2*y+n*z)-24*n^2*x*(5*y^2-n*y*(97*z+60*n*(2
*n*u-w))+4*n^2*z*(37*z+3*n*(14*n*u-17*w)))+5*z*(y-4*n*(z+6*n^2*u-3*n*w))^2)/
(13824*n^8*x^3);
B4=(8352*n^6*x^3-24*n^3*x^2*(91*y-4*n*(29*z-78*n^2*u-21*n*w))-x*(15*y^2-4*n
*y*(25*z+6*n*(87*n*u-10*w))+16*n^2*(10*z^2+9*n*z*(28*n*u-5*w)+9*n^2*(8*n^2
*u^2-34*n*u*w+5*w^2)))+5*u*(y-4*n*(z+6*n^2*u-3*n*w))^2)/(13824*n^8*x^3);
A5=-(32832*n^8*x^3+3456*n^5*x^2*(10*y-21*n*z)+24*n^2*x*(11*y^2-n*y*(247*z+
132*n*(2*n*u-w))+4*n^2*z*(115*z+3*n*(98*n*u-71*w)))-11*z*(y-4*n*(z+6*n^2*u-3
*n*w))^2)/(4608*n^7*x^3);
B5=-(31392*n^6*x^3+24*n^3*x^2*(301*y-4*n*(131*z+462*n^2*u-147*n*w))+x*(33
*y^2-4*n*y*(55*z+6*n*(225*n*u-22*w))+16*n^2*(22*z^2+9*n*z*(84*n*u-11*w)+9*
n^2*(152*n^2*u^2-142*n*u*w+11*w^2)))-11*u*(y-4*n*(z+6*n^2*u-3*n*w))^2)/(460
8*n^7*x^3);
A6=(32832*n^8*x^3+3456*n^5*x^2*(14*y-25*n*z)+24*n^2*x*(19*y^2-n*y*(407*z+
228*n*(2*n*u-w))+4*n^2*z*(179*z+3*n*(130*n*u-103*w)))-19*z*(y-4*n*(z+3*n*(2
*n*u-w)))^2)/(6912*n^9*x^3);
B6=(33696*n^6*x^3+24*n^3*x^2*(461*y-4*n*(187*z+15*n*(34*n*u-13*w)))+x*(57*
y^2-4*n*y*(95*z+6*n*(369*n*u-38*w))+16*n^2*(38*z^2+9*n*z*(132*n*u-19*w)+9*
n^2*(184*n^2*u^2-206*n*u*w+19*w^2)))-19*u*(y-4*n*(z+3*n*(2*n*u-w)))^2)/(6912
*n^9*x^3);
A4=Expand[A4]; B4=Expand[B4]; A5=Expand[A5]; B5=Expand[B5]; A6=Expand[A6];
B6=Expand[B6];

A7=(41472*n^8*x^3+576*n^5*x^2*(16*y-65*n*z)+24*n^2*x*(y^2-3*n*y*(11*z+4*n
*(2*n*u-w))+12*n^2*z*(7*z+n*(30*n*u-17*w)))-z*(y-4*n*(z+3*n*(2*n*u-w)))^2)/(13
824*n^9*x^3); A7=Expand[A7];
B7=(24768*n^6*x^3+24*n^3*x^2*(59*y-12*n*(11*z+5*n*(20*n*u-3*w)))+x*(3*y^2-
4*n*y*(5*z+6*n*(31*n*u-2*w))+16*n^2*(2*z^2+3*n*z*(44*n*u-3*w)+9*n^2*(56*n^
2*u^2-34*n*u*w+w^2)))-u*(y-4*n*(z+3*n*(2*n*u-w)))^2)/(13824*n^9*x^3);
B7=Expand[B7];
A8=-(3456*n^8*x^3+576*n^5*x^2*(8*y-19*n*z)+24*n^2*x*(y^2-n*y*(25*z+12*n*(2
*n*u-w))+4*n^2*z*(13*z+3*n*(14*n*u-9*w)))-z*(y-4*n*(z+6*n^2*u-3*n*w))^2)/(345
6*n^8*x^3); A8=Expand[A8];
B8=-(4608*n^6*x^3+24*n^3*x^2*(35*y-4*n*(17*z+72*n^2*u-21*n*w))+x*(3*y^2-4*
n*y*(5*z+6*n*(23*n*u-2*w))+16*n^2*(2*z^2+3*n*z*(28*n*u-3*w)+9*n^2*(24*n^2*
u^2-18*n*u*w+w^2)))-u*(y-4*n*(z+6*n^2*u-3*n*w))^2)/(3456*n^8*x^3);
```

```
B8=Expand[B8];
A9=(576*n^5*x^2*(4*y-5*n*z)+24*n^2*x*(y^2-3*n*y*(7*z+4*n*(2*n*u-w))+12*n^2
*z*(3*z+n*(6*n*u-5*w)))-z*(y-4*n*(z+3*n*(2*n*u-w)))^2)/(1152*n^7*x^3);
A9=Expand[A9];
B9=(576*n^6*x^3+24*n^3*x^2*(23*y-12*n*(3*z+n*(4*n*u-3*w)))+x*(3*y^2-4*n*y*(
5*z+6*n*(19*n*u-2*w))+16*n^2*(2*z^2+3*n*z*(20*n*u-3*w)+9*n^2*(8*n^2*u^2-10
*n*u*w+w^2)))-u*(y-4*n*(z+3*n*(2*n*u-w)))^2)/(1152*n^7*x^3); B9=Expand[B9];
A10=-(576*n^5*x^2*(2*y-n*z)+24*n^2*x*(y^2-n*y*(19*z+12*n*(2*n*u-w))+4*n^2*z
*(7*z+3*n*(2*n*u-3*w)))-z*(y-4*n*(z+6*n^2*u-3*n*w))^2)/(576*n^6*x^2);
A10=Expand[A10];
B10=-(24*n^3*x^2*(17*y-4*n*(5*z-3*n*w))+x*(3*y^2-4*n*y*(5*z+6*n*(17*n*u-2*w
))+16*n^2*(2*z^2+3*n*z*(16*n*u-3*w)-9*n^2*w*(6*n*u-w)))-u*(y-4*n*(z+6*n^2*u-
3*n*w))^2)/(576*n^6*x^2); B10=Expand[B10];
 st1 = -A1*A4; st1 = Expand[st1]; st2 = -B1*A5;     st2 = Expand[st2];     st3 =
-C1*A6*3*n^3;     st3 = Expand[st3];     A = st1+st2+st3;     A= Expand[A];
st1=-A1*B4;     st1= Expand[st1];     st2=-B1*B5;     st2=Expand[st2];     st3 =
-C1*B6*3*n^3;     st3=Expand[st3];     B= st1+st2+st3;     B=Expand[B];     st1 =
-3*n^3*D1*A6;     st1=Expand[st1];     st2 = -E1*A7;     st2=Expand[st2];     st3 =
-F1*A8;     st3=Expand[st3];     st4 = -G1*A9;     st4=Expand[st4];     st5 = -H1*A10;
   st5=Expand[st5];     st5=st5/C10; st5=Expand[st5]; Ap=st1+st2+st3+st4+st5;
Ap=Expand[Ap];     st1 = -3*n^3*D1*B6; st1=Expand[st1];     st2 = -E1*B7;
st2=Expand[st2];     st3 = -F1*B8;     st3=Expand[st3];     st4 = -G1*B9;
st4=Expand[st4];     st5 = -H1*B10;     st5=Expand[st5];     st5 = st5/C10;
st5=Expand[st5];     Bp=st1+st2+st3+st4+st5;     Bp=Expand[Bp];     dt = A*Bp;
dt=Expand[dt];     dt1 = B*Ap;     dt1=Expand[dt1];     dt=dt-dt1;     dt=Expand[dt];
dt=dt*C10^(n-2)/(18*n^5);     dt=Expand[dt];     dt=Together[dt];  NSolve[dt==0,s]
```

Four elements, fixed-simple support:

```
n=4; x = -(20*b*n^2-(5*c*n*s-120*n^4-s^2))/(120*n^4); x = Expand[x]; y =  s^2/n; y =
Expand[y]; z = s*(2*c*n-s)/(2*n^2); z = Expand[z]; w = (6*b*n^2-s*(3*c*n-s))/(6*n^3);
w = Expand[w]; u = -(40*b*n^2-(15*c*n*s+4*(30*n^4-s^2)))/(120*n^4); u =
Expand[u]; m4={{1,1/(2*n),0,0,-1,1/(2*n),0,0,0,0,0,0,0,0,0,0},
{0,-1/(6*n^2),0,-1,n^(-1),-1/(3*n^2),0,0,0,0,0,0,0,0,0,0},
{0,0,-1,3/(4*n),-1/(4*n^2),1/(24*n^3),0,0,0,0,0,0,0,0,0,0},
{0,0,0,1,0,0,0,0,0,0,0,0,0,-(100*n^2*w-66*n^3*x+15*y-50*n*z)/(960*n^5*x)+(300*n^
3*u-219*n^3*x+10*y-25*n*z)/(1440*n^5*x)-(120*n^3*u-60*n^2*w-48*n^3*x-5*y+20
*n*z)/(576*n^5*x),(21*n^3*x-5*y+5*n*z)/(24*n^3*x)-(z*(120*n^3*u-60*n^2*w-48*n
^3*x-5*y+20*n*z))/(576*n^5*x^2),-(100*n^2*w-66*n^3*x+15*y-50*n*z)/(960*n^5*x)
-(u*(120*n^3*u-60*n^2*w-48*n^3*x-5*y+20*n*z))/(576*n^5*x^2)+(81*n^3*x-15*y+2
5*n*z)/(1440*n^5*x)}, {0,0,0,0,1,1/(2*n),0,0,-1,1/(2*n),0,0,0,0,0,0},
{0,0,0,0,0,1/(3*n^3),-1,7/(4*n),-3/(2*n^2),1/(2*n^3),0,0,0,0,0,0},
{0,0,0,0,0,0,1,1/(2*n),1/(10*n^2),1/(120*n^3),-1,1/(2*n),-1/(10*n^2),1/(120*n^3),0,0},
{0,0,0,0,0,0,0,1,1/(2*n),1/(12*n^2),0,-1,1/(2*n),-1/(12*n^2),0,0},
```

```
{0,0,0,0,0,0,0,0,1,1/(2*n),0,0,-1,1/(2*n),0,0}, {0,0,0,0,0,0,0,0,0,x,y,z,w,u,0,0},
{0,0,0,0,0,0,0,0,0,0,1,1/(2*n),1/(10*n^2),1/(120*n^3),1/(2*n),1/(120*n^3)},
{0,0,0,0,0,0,0,0,0,0,0,1,1/(2*n),1/(12*n^2),-1,-1/(12*n^2)},
{0,0,0,0,0,0,0,0,0,0,0,0,1,1/(2*n),0,1/(2*n)}, {0,0,0,0,0,0,0,0,0,0,0,0,0,x,z,u},
{0,x,y,z,w,u,0,0,0,0,0,0,0,0,0,0}, {0,0,0,0,0,x,y,z,w,u,0,0,0,0,0,0}};
dt=Det[m4]; dt=Together[dt]; NSolve[dt==0,s]
```

APPENDIX C

To solve the zeros of the function g(s), several routines have been used before the Lanczos' Method. In all of these methods the solution depended on the initial conditions and step lengths which resulted in overflows as well as divergence in the solution.

The first one was the nonlinear root solver in Mathematica. The solution depended on the initial values assumed, and in most cases convergence was not possible. The second method was the perturbation analysis. The zero velocity conditions were used for the initial values of the routine. Then, the velocity was increased in steps, where the solution found was substituted into the next as the initial guess. In this approach the initial guess sometimes has to be found numerically leading to accumulation of errors from the start. Also the step length dependence was highly critical, because if a large step length was chosen then the solution may not converge. The third used the Taylor expansion of the expressions in the equation, but to get enough accuracy more then hundred terms for each of the expression in the equation. This from the practical point of view was not possible.

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