General parallel-leg mechanism: position analysis

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ABSTRACT

GENERAL PARALLEL-LEG MECHANISM: POSITION ANALYSIS

by

Hans Buus Gangwar

This thesis proposes to analytically express the positions of the legs (each leg has a length and two spin orientations) for a parallel leg mechanism in terms of a single variable. The only limitations imposed are the leg bases are all on the same reference plane and the tool orientation is normal to the mobile upper plate. This is done for a parallel leg mechanism having any number of legs. The variable chosen is the first leg’s vertex position on the upper plate.

The state space is the space the configuration can occupy. By looking at a planar slice of the state space for a specified tool axis position and orientation on the upper plate, the different possible leg positions can be constructed. The three dimensional space can then be generated by overlaying a progressive set of planes.
GENERAL PARALLEL-LEG MECHANISM: POSITION ANALYSIS

by
Hans Buus Gangwar

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GENERAL PARALLEL LEG MECHANISM: POSITION ANALYSIS

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To my dad for his unobtrusive support of all my decisions
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Stewart Platform
Griffs and Duffy's Modified Stewart Platform
Innocenti and Parenti-Castelli's Parallel-Leg Mechanism
Leg Coordinates
Upper Plate Parameters
Base Coordinates for Six Legs
Coordinate Frames
LIST OF LEG SYMBOLS

\( i \)  
leg subscript #  
(when absent, the leg symbol is used as a general representation of all legs)

\( r_1 \)  
constant ground radial distance of leg i from origin

\( x_i \)  
length of leg i  
\((x \in [x_{min}, x_{max}]\))

\( \theta_i \)  
constant ground orientation of leg i about \( z_g \)-axis

\( \eta_i \)  
rotation of leg i about the \( y_B \)-axis at leg base

\( \psi_i \)  
rotation of leg i about the \( x'' \) axis so new coordinate \( x_L \) axis aligned with leg
LIST OF OTHER SYMBOLS

\( \mathbf{B}(r, \theta) \) 3x3 Dual Number matrix expressing first leg base coordinate \( x_B y_B z_B \) placement in terms of ground (\( z_B \) in \( z_g \) direction)

\( \mathbf{D} \) 3x3 matrix of leg top positions along the \( x_G, y_G, z_G \) directions

\( \mathbf{D}^0 \) First column of matrix \( \mathbf{D} \)

\( \mathbf{D}^1 \) Second column of matrix \( \mathbf{D} \)

\( \mathbf{D}^2 \) Third column of matrix \( \mathbf{D} \)

\( \mathbf{d}_l \) 1x3 column matrix representation of \( \mathbf{D} \)

\( \mathbf{d}_{l,x} \) \( x_G \)-component of \( \mathbf{d}_l \)

\( \mathbf{d}_{l,y} \) \( y_G \)-component of \( \mathbf{d}_l \)

\( \mathbf{d}_{l,z} \) \( z_G \)-component of \( \mathbf{d}_l \)

\( \mathbf{L}(\eta, \psi) \) 3x3 Dual Number matrix expressing second leg base coordinate \( x_L y_L z_L \) placement (\( x_L \) in leg direction) in terms of \( x_B y_B z_B \)

\( \mathbf{n} \) Tool axis orientation; assumed to be normal to the upper plate surface at the tool axis position \( P \).

\( \mathbf{n}_x \) \( x_G \)-component of \( \mathbf{n} \)

\( \mathbf{n}_y \) \( y_G \)-component of \( \mathbf{n} \)

\( \mathbf{n}_z \) \( z_G \)-component of \( \mathbf{n} \)

\( \mathbf{P} \) Position of tool axis on the upper plate with respect to the ground

\( \mathbf{P}_x \) \( x_G \)-component of \( \mathbf{P} \)

\( \mathbf{P}_y \) \( y_G \)-component of \( \mathbf{P} \)

\( \mathbf{P}_z \) \( z_G \)-component of \( \mathbf{P} \)
LIST OF OTHER SYMBOLS
(Continued)

\( T \) 3x3 Dual number matrix expressing leg top in terms of ground; equal to \( B(r,\theta)L(\eta,\psi)U(x) \)

\( T^0 \) First column of dual number matrix \( T \)

\( T^1 \) Second column of dual number matrix \( T \)

\( T_p \) 3x3 matrix of primary (rotational) component of dual number matrix \( T \)

\( T_d \) 3x3 matrix of dual (translation multiplied by rotation) component of dual number matrix \( T \)

\( U(x) \) 3x3 Dual Number matrix expressing leg top (or distal) coordinate in terms of \( x_l, y_l, z_l \)

\( V_i \) Fixed distance from \( d_i \) to \( P \)

\( \gamma_i \) Fixed angle between \( |P-d_i| \) and \( |P-d_1| \)
CHAPTER 1
INTRODUCTION

1.1 Advantages and Disadvantages of a Parallel-Leg Mechanism

There are three main advantages of a parallel-leg mechanism over a conventional serial link configuration: identical components, even distribution of weight, and absence of error buildup.

By the inherent nature of a parallel-leg configuration, the links and their ball joint connections do not vary from leg to leg. This component universality thereby reduces the costs of production and part replacement. A serial-link mechanism does not have this intrinsic repetition of parts, thus raising the maintenance and production costs.

Second, in a parallel-leg mechanism, the weight of the tool plate is distributed among the six legs. Thus, the legs do not need to be massive, nor do the servo-motors controlling the movement need to exert a large amount of torque. Whereas in a serial-link mechanism, the weight is *additive*, for the base link has to support all the other links in addition to the tool plate. As a consequence of better weight distribution, parallel-leg mechanisms exhibit better rigidity and higher acceleration.

Lastly, with the parallel-leg mechanism, the error in link orientation is isolated to the specific links (it is not felt between links), so the total error is not additive, and usually ends up being a little greater than the largest error [Behi, 1988]. However, with a serial arrangement, the error is *cumulative*.

Thus a parallel-leg mechanism is superior to a serial-link mechanism in performance, cost of manufacturing, and cost of maintenance.
The disadvantages are subtle. First, analysis of the movement for a serial mechanism is fundamentally simpler. Only forces and moments between links cumulate, and the effects of those are easily analyzed, especially with the effects being felt at the link connections. Thus the motion can be isolated along orthogonal axes. With a parallel mechanism, however, link movement is coupled — when one link moves, the relative orientation of the others also changes. Second, tool path generation in the operation of the mechanism is straightforward with a serial mechanism. Path generation is an everyday requirement for machinists with NC machines. Every different machine job needs a different program. Because the motion of a parallel-leg mechanism is complex, planning a trajectory is time intensive, difficult, and many times not optimal.

1.2 Background

Parallel-leg mechanism designs were first introduced back in the 1800’s. It was not until 1947 that one was actually built by Gough. His Universal Tire Testing Rig was the first practical application of a parallel-leg mechanism. Gough’s mechanism allowed the measuring of wear by pressing a rotating tire at different angles against a rough surface [Gough & Whitewall, 1956].

Parallel-leg mechanisms are popularly known as Stewart platforms, after Stewart’s six degree of freedom flight simulator shown in Figure 1 [Stewart, 1965]. However, it was not until 25 years after the Stewart platforms first appearance that theoretical research in robotics began to focus on parallel-leg mechanisms. The first analytical efforts were made by Behi, Griffs and Duffy, and Fichter.
Fichter determined the leg lengths, velocities, and singular positions of the configuration for a Stewart’s platform. His parameters were the position, velocity, and orientation of the upper plate [Fichter, 1986].

Behi [1988] also studied a Stewart’s platform. He analyzed the platform for both reverse and forward displacements. In reverse displacement analysis, the orientation and center position of the upper plate are given, for which the leg lengths are solved. In forward displacement analysis, the leg lengths are given, for which the orientation and center position of the upper plate are solved. Then he looked as well at the workspace of the mechanism – the three dimensional space in which the mechanism operates.
Griffis and Duffy [1989] analyzed a modified Stewart platform with six legs attached at three separate ball joints at both the base and upper plates (shown in Figure 2). Eight reflected solution pairs for the leg lengths were arrived at, given the orientation and center position of the upper plate.

The most intensive study was done by Innocenti and Parenti-Castelli [1990], whom analyzed the mechanism shown in Figure 3. They solved for the orientation and mobile upper plate position in terms of the leg parameters. Their approach was first closure of the loop equation for the legs, then construction of the orientation and center position vectors for the upper plate by means of the three top vertices. Using complex numbers, their result was a 16th order polynomial in terms of one variable.
1.3 Motivation and Objective

In machining, it is important to be able to move a parallel leg mechanism through a prescribed path. Since most tools are symmetric, the mobile upper plate can be rotated about the tool axis without changing the tool orientation. Thus a single degree of freedom exists for the upper plate, a rotation about the tool axis. Having an equation expressing the leg variables in terms of a single point on the upper plate (in addition to the tool axis position and orientation) would enable the generation of the leg positions for different rotations of the upper plate. From inputting the different points on the tracer line of the tool axis position, this process can be repeated to generate a set of such solutions. In addition the tool orientation need not be held constant over tracer path.

The objective is then to determine all possible configurations for a specified tool position and orientation.
1.4 Design Parameters

Each leg is given an axial length $x$, an angular orientation $\eta$ about the $y_B$-axis, and a second angular orientation $\psi$ about the $z_L$ axis as shown in Figure 4. There are six legs, so there are $6 \times 3$ variables. Given the tool axis orientation and position, a point in the upper plate can lie on a circle of solutions. Therefore the objective of this thesis is to express the other variables in terms of the first leg's upper vertex position on the mobile plate, $d_1$. This is done in order to provide the equations to map out the solution space over the range of the single independent variable.

---

**Figure 4 Leg Coordinates**
The tool axis orientation $n$ is assumed to be normal to the upper plate surface at the tool axis position $P$. The distance between $P$ and leg vertex $d_i$ is $V_i$ (see Figure 5).
The last parameters needed are the base positions of each leg. The assumption made is that each leg $i$ is on a circle of radius $r_i$ at an angle $\theta_i$ (Figure 6 illustrates the base coordinates for the case of six legs).

![Figure 6 Base Coordinates for Six Legs](image-url)
CHAPTER 2

POSITION ANALYSIS

2.1 Approach

A parallel leg mechanism's position depends on the tool axis position $\mathbf{P}$ and orientation $\mathbf{n}$ on the upper plate. The main quantity of interest controlling the position of the upper plate is the legs' lengths. Since the legs are attached to the upper plate at the vertices, in order to express the legs' lengths in terms of $\mathbf{P}$ and $\mathbf{n}$, first the equation describing the position of the vertices is needed. To accomplish this, a general equation will be developed in Section 2.2 for any arbitrary leg. This equation will be called the Single Leg Equation. Once the Single Leg Equation is known, the leg variables (length $x$ and angular orientations $\eta$ and $\psi$) can be expressed in terms of the vertex distance $d$. This will be done in Section 2.3. The equations derived in Sections 2.2 and 2.3 are for an arbitrary leg and corresponding vertex. Since the vertices are not independent, but related by the geometry of the upper plate and by specifying the tool axis position and orientation, there is only a single degree of freedom (once one vertex is specified, the plate is fully constrained). As such, the other vertices after the first can be expressed in terms of the first. All that is needed is the distance $V_i$ between vertex $d_i$ and tool axis position $\mathbf{P}$, and the angle $\gamma_i$ between the axis position and the vertices $d_i$ and $d_1$. Section 2.4 derives the equation relating the vertices. From this set of equations, all leg lengths are expressible as a function of the first vertex $d_1$. 
2.2 Single Leg Equation

The purpose of this section is to express the distance from a vertex to the ground in terms of the leg variables. This expression is called the *Single Leg Equation*, and can be used to relate any vertex position in terms of its leg’s variables $x$, $\eta$, and $\psi$.

![Figure 7 Coordinate Frames](image)

*Figure 7 Coordinate Frames*

A leg is broken into the three different coordinate frames shown in Figure 7: its reference base frame $\{B\}$ on the base plate, its leg direction frame $\{L\}$ whose origin coincides with $\{B\}$ but whose x-axis points in the direction of the leg, and its upper connection frame $\{U\}$ which lies on the connection of the leg to the upper plate. By formulating dual number transformations from the ground $\{G\}$ to $\{B\}$, from $\{B\}$ to $\{L\}$,
and from \{L\} to \{U\}, the vertex \(d\) becomes simply the transformation \(T\) from the ground \{G\} to the upper connection (vertex) \{U\}. The appendix describes the dual number transforms in detail.

\(B_i(r,\theta)\) expresses the coordinates of the reference base of leg \(i\) in terms of the ground. Two coordinate transforms are needed. First rotate about \(z_g\) by \(\theta\) to arrive at the new \(x'\) and \(y'\)-axes. Then translate along the new \(x'\)-axis by \(r\) to arrive at the \(x_B\) and \(y_B\)-axes. Since the purpose is to express a coordinate point \((x_B, y_B, z_B)\) as \((x_g, y_g, z_g)\), the order of matrix operation for \(B_i(r,\theta)\) is backwards.

\[
B(r, \theta) := \begin{bmatrix}
\cos(\theta) & -\sin(\theta) & 0 \\
\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -\varepsilon \cdot r \\
0 & \varepsilon \cdot r & 1
\end{bmatrix}
\]

Now each leg \(i\) is presumed to be connected at \(\{B_i\}\) by a ball joint. Looking at the leg \(i\), a second coordinate system can be established at \(\{B_i\}\), where the \(x_L\)-axis points in the leg direction. Since we are only looking at the joint right now to see the directional orientation of the leg, the translation \(r = 0\). To arrive at the new coordinate system, first rotate \(\{B_i\}\) about the \(y_B\)-axis by \(\eta_i\) until the leg is in the \(x'y''\) (or \(x''y_B\)) plane, then rotate about the \(z''\) axis by \(\psi_i\) to align \(x_L\) with the leg. So undoing this operation to arrive at the coordinate transformation,

\[
L = Y(\eta, 0)Z(\psi, 0)
\]

\[
L(\eta, \psi) := \begin{bmatrix}
\cos(\eta) & 0 & \sin(\eta) \\
0 & 1 & 0 \\
-\sin(\eta) & 0 & \cos(\eta)
\end{bmatrix} \begin{bmatrix}
\cos(\psi) & -\sin(\psi) & 0 \\
\sin(\psi) & \cos(\psi) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Next, each leg has a length \( x_i \), so to move from the coordinate system at \( \{L\} \) to the coordinate system at \( \{U\} \) (upper part or distal end of the leg), a simple translation occurs along the \( x_L \)-axis.

\[
U(x) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon \cdot x \\ 0 & \varepsilon \cdot x & 1 \end{bmatrix}
\]

Combining the equations, the position of the top of each leg can be expressed in terms of the ground coordinate. \( T = B \times L \times U \)

\[
T = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon \cdot r \\ 0 & \varepsilon \cdot r & 1 \end{bmatrix} \times \begin{bmatrix} \cos(\eta) & 0 & \sin(\eta) \\ \sin(\eta) & 0 & \cos(\eta) \\ -\sin(\eta) & 0 & \cos(\eta) \end{bmatrix} \times \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Evaluating the matrix symbolically

\[
T(r, \theta, \eta, \psi, x)^{<0>} \rightarrow \begin{bmatrix} \cos(\theta) \cdot \cos(\eta) \cdot \cos(\psi) - \sin(\theta) \cdot \sin(\psi) - \sin(\theta) \cdot \varepsilon \cdot r \cdot \sin(\eta) \cdot \cos(\psi) \\ \sin(\theta) \cdot \cos(\eta) \cdot \cos(\psi) + \cos(\theta) \cdot \sin(\psi) + \cos(\theta) \cdot \varepsilon \cdot r \cdot \sin(\eta) \cdot \cos(\psi) \\ \varepsilon \cdot r \cdot \sin(\psi) - \sin(\eta) \cdot \cos(\psi) \end{bmatrix}
\]

\[
T(r, \theta, \eta, \psi, x)^{<1>} \rightarrow \begin{bmatrix} -\cos(\theta) \cdot \cos(\eta) \cdot \sin(\psi) - \sin(\theta) \cdot \sin(\psi) + \sin(\theta) \cdot \varepsilon \cdot r \cdot \sin(\eta) \cdot \sin(\psi) + \varepsilon \cdot r \cdot \cos(\theta) \cdot \sin(\eta) + \varepsilon^2 \cdot r \cdot \cos(\theta) \cdot \cos(\eta) \\ -\sin(\theta) \cdot \cos(\eta) \cdot \sin(\psi) + \cos(\theta) \cdot \cos(\psi) - \cos(\theta) \cdot \varepsilon \cdot r \cdot \sin(\eta) \cdot \sin(\psi) + \varepsilon \cdot r \cdot \sin(\theta) \cdot \sin(\eta) - \varepsilon^2 \cdot r \cdot \cos(\theta) \cdot \cos(\eta) \\ \varepsilon \cdot r \cdot \cos(\psi) + \sin(\eta) \cdot \sin(\psi) + \cos(\eta) \cdot \varepsilon \cdot x \end{bmatrix}
\]

\[
T(r, \theta, \eta, \psi, x)^{<2>} \rightarrow \begin{bmatrix} \varepsilon \cdot r \cdot \cos(\theta) \cdot \cos(\eta) \cdot \sin(\psi) + \varepsilon \cdot r \cdot \sin(\theta) \cdot \cos(\psi) - \varepsilon^2 \cdot r \cdot \sin(\theta) \cdot \sin(\psi) + \cos(\theta) \cdot \sin(\eta) + \sin(\theta) \cdot \varepsilon \cdot r \cdot \cos(\eta) \\ \varepsilon \cdot r \cdot \sin(\theta) \cdot \cos(\eta) \cdot \sin(\psi) - \varepsilon \cdot r \cdot \cos(\theta) \cdot \cos(\psi) + \varepsilon^2 \cdot r \cdot \cos(\theta) \cdot \sin(\eta) \cdot \sin(\psi) + \sin(\theta) \cdot \varepsilon \cdot r \cdot \cos(\eta) - \cos(\theta) \cdot \varepsilon \cdot r \cdot \cos(\eta) \\ -\varepsilon \cdot r \cdot \cos(\psi) - \varepsilon \cdot r \cdot \sin(\eta) \cdot \sin(\psi) + \cos(\eta) \end{bmatrix}
\]

Using the property \( \varepsilon^2 = 0 \), the result is

\[
T(r, \theta, \eta, \psi, x)^{<0>} \rightarrow \begin{bmatrix} \cos(\theta) \cdot \cos(\eta) \cdot \cos(\psi) - \sin(\theta) \cdot \sin(\psi) - \sin(\theta) \cdot \varepsilon \cdot r \cdot \sin(\eta) \cdot \cos(\psi) \\ \sin(\theta) \cdot \cos(\eta) \cdot \cos(\psi) + \cos(\theta) \cdot \sin(\psi) + \cos(\theta) \cdot \varepsilon \cdot r \cdot \sin(\eta) \cdot \cos(\psi) \\ \varepsilon \cdot r \cdot \sin(\psi) - \sin(\eta) \cdot \cos(\psi) \end{bmatrix}
\]
Separating the primary component $T_p$ and dual component $T_d$

$T(r, \theta, \eta, \psi, x) = \begin{bmatrix}
-\cos(\theta) \cdot \cos(\eta) \cdot \sin(\psi) - \sin(\theta) \cdot \cos(\psi) + \varepsilon \cdot \sin(\theta) \cdot \sin(\eta) \cdot \sin(\psi) + \varepsilon \cdot \varepsilon \cdot \cos(\theta) \cdot \sin(\eta) \\
-\sin(\theta) \cdot \cos(\eta) \cdot \sin(\psi) + \cos(\theta) \cdot \cos(\psi) - \varepsilon \cdot \cos(\theta) \cdot \sin(\eta) \cdot \sin(\psi) + \varepsilon \cdot \sin(\theta) \cdot \sin(\eta) \\
\varepsilon \cdot \varepsilon \cdot \cos(\theta) \cdot \cos(\eta) + \sin(\eta) \cdot \sin(\psi) + \varepsilon \cdot \varepsilon \cdot \cos(\theta) \cdot \cos(\eta)
\end{bmatrix}$

So $[D] = [DR] [R]^T$.

(Note: $[R]^{-1} = [R]^T$ because $[R]$ is obtained from the rotation matrices $X, Y, Z$ whose inverses are their transposes; since $[R]$ is composed directly from these matrices, its inverse is its transpose – decompose then take its inverse step by step to see). (Fischer 1999, p. 83)
where \([D]\) is the matrix result of the cross of \(d\) by the unit vectors of frame \(\{U\}\)

\[
[D] = \begin{bmatrix}
  dx \\ dy \\ dz
\end{bmatrix} \times \begin{bmatrix}
  e_{u,x} \\ e_{u,y} \\ e_{u,z}
\end{bmatrix} = \begin{bmatrix}
  0 & -dz & dy \\ dz & 0 & -dx \\ -dy & dx & 0
\end{bmatrix} \begin{bmatrix}
  e_{u,x} \\ e_{u,y} \\ e_{u,z}
\end{bmatrix}
\]

As a check, the 1-1 element should be zero

\[-r \cdot \sin(\theta) \cdot \sin(\eta) \cdot \cos(\theta) \cdot \cos(\eta) \cdot (\cos(\psi)^2 + \sin(\psi)^2) + \cos(\theta) \cdot \sin(\eta) \cdot r \cdot \sin(\theta) \cdot \cos(\eta) = 0\]

As a check, the 2-2 element should be zero

\[r \cdot \sin(\theta) \cdot \sin(\eta) \cdot \cos(\psi)^2 + \cos(\theta) \cdot \cos(\eta) + r \cdot \sin(\theta) \cdot \sin(\eta) \cdot \sin(\psi)^2 \cdot \cos(\theta) \cdot \cos(\eta) - \cos(\theta) \cdot \sin(\eta) \cdot r \cdot \sin(\theta) \cdot \cos(\eta)\]

\[r \cdot \sin(\theta) \cdot \cos(\eta) - \cos(\theta) \cdot \sin(\eta) \cdot r \cdot \sin(\theta) \cdot \cos(\eta) = 0\]

Now, the 2-1 element = \(d_z\)

\[r \cdot \cos(\theta)^2 \cdot \sin(\eta) \cdot \cos(\eta) \cdot (\cos(\psi)^2 + \sin(\psi)^2) - x \cdot (\cos(\theta)^2 + \sin(\psi)^2) \cdot \sin(\eta) \cdot \cos(\psi) - \cos(\theta)^2 \cdot \sin(\eta) \cdot r \cdot \cos(\eta)\]

\[-x \cdot \sin(\eta) \cdot \cos(\psi) = d_z\]

And, the 3-1 element = \(d_x\)

\[\begin{align*}
\left[ r \cdot (\cos(\psi)^2 + \sin(\psi)^2) \cdot \cos(\theta) - x \cdot \sin(\eta) \cdot (\cos(\eta)^2 + \sin(\eta)^2) \cdot \sin(\psi) \right] + \cos(\eta) \cdot x \cdot \cos(\theta) \cdot \cos(\psi) \\
r \cdot \cos(\theta) - x \cdot \sin(\eta) \cdot \sin(\psi) + x \cdot \cos(\theta) \cdot \cos(\eta) \cdot \cos(\psi) = d_x
\end{align*}\]
Lastly, the 1-3 element = \( d_y \)

\[
\begin{align*}
    r \sin(\theta) \cdot \sin(\eta)^2 \cdot (\cos(\psi)^2 + \sin(\psi)^2) + (\sin(\eta)^2 + \cos(\eta)^2) \cdot \sin(\psi) \cdot \cos(\theta) + \cos(\eta) \cdot x \cdot \sin(\theta) \cdot \cos(\psi) + r \sin(\theta) \cdot \cos(\eta)^2 \\
    r \sin(\theta) \cdot (\sin(\eta)^2 + \cos(\eta)^2) + x \cdot \cos(\theta) \cdot \sin(\psi) + x \cdot \sin(\theta) \cdot \cos(\eta) \cdot \cos(\psi) \\
    r \cdot \sin(\theta) + x \cdot \cos(\theta) \cdot \sin(\psi) + x \cdot \sin(\theta) \cdot \cos(\eta) \cdot \cos(\psi) = d_y \\
\end{align*}
\]

or combining the results for \( d_x, d_y, \) and \( d_z \).

\[
\begin{align*}
    d(r, \theta, \eta, \psi, x) &= \\
    &= \begin{bmatrix} 
    r \cdot \cos(\theta) - x \cdot \sin(\theta) \cdot \sin(\psi) + x \cdot \cos(\theta) \cdot \cos(\eta) \cdot \cos(\psi) \\
    r \cdot \sin(\theta) + x \cdot \cos(\theta) \cdot \sin(\psi) + x \cdot \sin(\theta) \cdot \cos(\eta) \cdot \cos(\psi) \\
    -x \cdot \sin(\eta) \cdot \cos(\psi) 
    \end{bmatrix} \\
    &= \begin{bmatrix} 
    r \cdot \cos(\theta) - x \cdot \sin(\theta) \cdot \sin(\psi) + x \cdot \cos(\theta) \cdot \cos(\eta) \cdot \cos(\psi) \\
    r \cdot \sin(\theta) + x \cdot \cos(\theta) \cdot \sin(\psi) + x \cdot \sin(\theta) \cdot \cos(\eta) \cdot \cos(\psi) \\
    -x \cdot \sin(\eta) \cdot \cos(\psi) 
    \end{bmatrix}
\end{align*}
\]

This equation is applicable to any leg, whatever the top constraint is.

### 2.3 Leg Variable Dependence

The *Single Leg Equation* can solved for \( x, \eta, \psi \) in terms of \( d_x, d_y, \) and \( d_z \). Now, the subscript \( i \) is introduced, for clarity of differentiation between legs.

\[
\begin{align*}
    d_{i,x} &= \begin{bmatrix} 
    r \cdot \cos(\theta_i) - x \cdot \sin(\theta_i) \cdot \sin(\psi_i) + x \cdot \cos(\theta_i) \cdot \cos(\eta_i) \cdot \cos(\psi_i) \\
    r \cdot \sin(\theta_i) + x \cdot \cos(\theta_i) \cdot \sin(\psi_i) + x \cdot \sin(\theta_i) \cdot \cos(\eta_i) \cdot \cos(\psi_i) \\
    -x \cdot \sin(\eta_i) \cdot \cos(\psi_i) 
    \end{bmatrix} \\
    d_{i,y} &= \begin{bmatrix} 
    r \cdot \cos(\theta_i) - x \cdot \sin(\theta_i) \cdot \sin(\psi_i) + x \cdot \cos(\theta_i) \cdot \cos(\eta_i) \cdot \cos(\psi_i) \\
    r \cdot \sin(\theta_i) + x \cdot \cos(\theta_i) \cdot \sin(\psi_i) + x \cdot \sin(\theta_i) \cdot \cos(\eta_i) \cdot \cos(\psi_i) \\
    -x \cdot \sin(\eta_i) \cdot \cos(\psi_i) 
    \end{bmatrix} \\
    d_{i,z} &= \begin{bmatrix} 
    r \cdot \cos(\theta_i) - x \cdot \sin(\theta_i) \cdot \sin(\psi_i) + x \cdot \cos(\theta_i) \cdot \cos(\eta_i) \cdot \cos(\psi_i) \\
    r \cdot \sin(\theta_i) + x \cdot \cos(\theta_i) \cdot \sin(\psi_i) + x \cdot \sin(\theta_i) \cdot \cos(\eta_i) \cdot \cos(\psi_i) \\
    -x \cdot \sin(\eta_i) \cdot \cos(\psi_i) 
    \end{bmatrix}
\end{align*}
\]

Multiplying the first equation by \( \cos(\theta_i) \) and the second equation by \( \sin(\theta_i) \).

\[
\begin{align*}
    \begin{bmatrix} 
    -x \cdot \sin(\theta_i) \cdot \cos(\theta_i) \cdot \sin(\psi_i) + x \cdot \cos(\theta_i)^2 \cdot \cos(\eta_i) \cdot \cos(\psi_i) \\
    x \cdot \sin(\theta_i) \cdot \cos(\theta_i) \cdot \sin(\psi_i) + x \cdot \sin(\theta_i)^2 \cdot \cos(\eta_i) \cdot \cos(\psi_i) \\
    -x \cdot \sin(\eta_i) \cdot \cos(\psi_i) 
    \end{bmatrix} &= \begin{bmatrix} 
    \cos(\theta_i) \cdot d_{i,x} - r \cdot \cos(\theta_i) \\
    \sin(\theta_i) \cdot d_{i,y} - r \cdot \sin(\theta_i) \\
    d_{i,z} 
    \end{bmatrix}
\end{align*}
\]
Then adding the first and second equation to form another
\[
\begin{bmatrix}
-x_i \cdot \sin(\eta_i) \cdot \cos(\psi_i) \\
-x_i \cdot \sin(\psi_i) \\
\end{bmatrix} = \begin{bmatrix}
\cos(\theta_i) \cdot (d_{i,x} - r_i \cdot \cos(\theta_i)) \\
\sin(\theta_i) \cdot (d_{i,y} - r_i \cdot \sin(\theta_i)) \\
\end{bmatrix}
\]
Eq. 1

Another similar transform can be performed by multiplying the first equation by \(\sin(\theta_i)\) and the second by \(\cos(\theta_i)\).
\[
\begin{bmatrix}
-x_i \cdot \sin(\theta_i) \cdot \sin(\psi_i) + x_i \cdot \sin(\theta_i) \cdot \cos(\theta_i) \cdot \cos(\eta_i) \cdot \cos(\psi_i) \\
-x_i \cdot \sin(\eta_i) \cdot \cos(\psi_i) \\
\end{bmatrix} = \begin{bmatrix}
\sin(\theta_i) \cdot (d_{i,x} - r_i \cdot \cos(\theta_i)) \\
\cos(\theta_i) \cdot (d_{i,y} - r_i \cdot \sin(\theta_i)) \\
\end{bmatrix}
\]
Eq. 2

Then subtracting the first from the second
\[
\begin{bmatrix}
x_i \cdot \sin(\psi_i) \\
-x_i \cdot \sin(\eta_i) \cdot \cos(\psi_i) \\
\end{bmatrix} = \begin{bmatrix}
\cos(\theta_i) \cdot (d_{i,y} - r_i \cdot \sin(\theta_i)) - \sin(\theta_i) \cdot (d_{i,x} - r_i \cdot \cos(\theta_i)) \\
\cos(\theta_i) \cdot (d_{i,x} - r_i \cdot \cos(\theta_i)) + \sin(\theta_i) \cdot (d_{i,y} - r_i \cdot \sin(\theta_i)) \\
\end{bmatrix}
\]
Eq. 3

Combining Equations 1 and 2
\[
\begin{bmatrix}
x_i \cdot \sin(\psi_i) \\
x_i \cdot \cos(\eta_i) \cdot \cos(\psi_i) \\
-x_i \cdot \sin(\eta_i) \cdot \cos(\psi_i) \\
\end{bmatrix} = \begin{bmatrix}
\cos(\theta_i) \cdot (d_{i,y} - r_i \cdot \sin(\theta_i)) - \sin(\theta_i) \cdot (d_{i,x} - r_i \cdot \cos(\theta_i)) \\
\cos(\theta_i) \cdot (d_{i,x} - r_i \cdot \cos(\theta_i)) + \sin(\theta_i) \cdot (d_{i,y} - r_i \cdot \sin(\theta_i)) \\
\end{bmatrix}
\]
Eq. 3

Taking the square of each row and adding the lines.
\[
(x_i)^2 = (d_{i,y} - r_i \cdot \sin(\theta_i))^2 + (d_{i,x} - r_i \cdot \cos(\theta_i))^2 + (d_{i,z})^2
\]
\[
x_i = \sqrt{(d_{i,x})^2 + (d_{i,y})^2 + (d_{i,z})^2 + (r_i)^2 - 2 \cdot r_i \cdot (d_{i,y} \cdot \sin(\theta_i) - d_{i,x} \cdot \cos(\theta_i))}
\]

(The leg length can not be negative, so only the positive square root is taken.)
From the first row of Eq. 3

\[
\sin(\psi_i) = \frac{\cos(\theta_i) \cdot (d_i \cdot y - r_i \cdot \sin(\theta_i)) - \sin(\theta_i) \cdot (d_i \cdot x - r_i \cdot \cos(\theta_i))}{x_i}
\]

The square root can be either +/- since \(d_{i,z}\) in the third row of Eq. 3 cannot be negative, either \(\sin(\eta_i)\) is negative or \(\cos(\psi_i)\) is negative.

\[
(-x \cdot \sin(\eta_i)) \cos(\psi_i) = d_{i,z} \\
x \cdot \cos(\eta_i) \cos(\psi_i) = \cos(\theta_i) \cdot (d_i \cdot x - r_i \cdot \cos(\theta_i)) + \sin(\theta_i) \cdot d_i \cdot y - r_i \cdot \sin(\theta_i)
\]

So there are two solutions based off the sign of \(\cos(\psi_i)\).

### 2.4 Vertex Dependence

The position of a vertex depends on both the tool orientation \(n\) and position \(P\) on the upper plate and internal geometry. The tool orientation of the upper plate describes the plane in which the plate lies. It is also the normal vector to that plane. The vector curl or cross product of two lines in the upper plate's plane will determine the normal vector. Since the normal vector is normalized \((n_x^2 + n_y^2 + n_z^2 = 1)\), the lines used in the cross product need to be normalized as well (divided by the line length). The lines from \(P\) to
\( d_1 \) and \( P \) to \( d_i \) are used, whose line lengths are \( V_1 \) and \( V_i \) respectively. (Note: this is done in order to develop equations relating the vertex \( d_i \) in terms of \( d_1 \).)

\[
\begin{pmatrix} \frac{P - d_1}{V_1} \\ \frac{P - d_i}{V_i} \end{pmatrix} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \text{ or } \begin{bmatrix} e_{u,x} & e_{u,y} & e_{u,z} \\ p_{x - d_1,x} & p_{y - d_1,y} & p_{z - d_1,z} \\ p_{x - d_i,x} & p_{y - d_i,y} & p_{z - d_i,z} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = V_1, V_i
\]

Which can be rewritten as

\[ V_1 \cdot V_i \cdot n_x = \langle P_y - d_{1,y} \rangle \cdot \langle P_z - d_{1,z} \rangle - \langle P_x - d_{1,x} \rangle \cdot \langle P_y - d_{1,y} \rangle \]

\[ V_1 \cdot V_i \cdot n_y = \langle P_z - d_{1,z} \rangle \cdot \langle P_x - d_{1,x} \rangle - \langle P_y - d_{1,y} \rangle \cdot \langle P_z - d_{1,z} \rangle \]

\[ V_1 \cdot V_i \cdot n_z = \langle P_x - d_{1,x} \rangle \cdot \langle P_y - d_{1,y} \rangle - \langle P_y - d_{1,y} \rangle \cdot \langle P_x - d_{1,x} \rangle \]

or expressing in terms of \( d_i \).

\[ \langle P_z - d_{1,z} \rangle \cdot d_{1,y} - \langle P_y - d_{1,y} \rangle \cdot d_{1,z} = \langle P_z - d_{1,z} \rangle \cdot P_y - \langle P_y - d_{1,y} \rangle \cdot P_z + V_1 \cdot V_i \cdot n_x \]

\[ \langle P_z - d_{1,z} \rangle \cdot d_{1,x} - \langle P_x - d_{1,x} \rangle \cdot d_{1,z} = \langle P_z - d_{1,z} \rangle \cdot P_x - \langle P_x - d_{1,x} \rangle \cdot P_z - V_1 \cdot V_i \cdot n_y \] \hspace{1cm} \text{Eq. 4}

\[ -\langle P_y - d_{1,y} \rangle \cdot d_{1,x} + \langle P_x - d_{1,x} \rangle \cdot d_{1,y} = \langle P_x - d_{1,x} \rangle \cdot P_y - \langle P_y - d_{1,y} \rangle \cdot P_x - V_1 \cdot V_i \cdot n_z \] \hspace{1cm} \text{Eq. 5}

Since the upper plate's geometry is fixed, the angle \( \gamma_i \) between the lines from \( P \) to \( d_1 \) and \( P \) to \( d_i \) does not change.

\[ \frac{P - d_1}{V_1} \cdot \frac{P - d_i}{V_i} = \cos(\gamma_i) \]

or

\[ \langle P_x - d_{1,x} \rangle \cdot \langle P_x - d_{1,x} \rangle + \langle P_y - d_{1,y} \rangle \cdot \langle P_y - d_{1,y} \rangle + \langle P_z - d_{1,z} \rangle \cdot \langle P_z - d_{1,z} \rangle = V_1 \cdot V_i \cdot \cos(\gamma_i) \]

which can be rewritten as
\[
(P_x - d_{1,x})d_{1,x} + (P_y - d_{1,y})d_{1,y} + (P_z - d_{1,z})d_{1,z} = P^2 - (P_x d_{1,x} + P_y d_{1,y} + P_z d_{1,z}) - V_1 V_i \cos(\gamma_i)
\]

This equation is used with Eq. 4 and Eq. 5 to find an expression for \( d_i \):

\[
(P_x - d_{1,x})d_{1,x} + (P_y - d_{1,y})d_{1,y} = (P_x - d_{1,x})P_y - (P_y - d_{1,y})P_x - V_1 V_i n_z
\]

\[
(P_z - d_{1,z})d_{1,y} - (P_y - d_{1,y})d_{1,z} = (P_z - d_{1,z})P_y - (P_y - d_{1,y})P_z + V_1 V_i n_x
\]

\[
(P_x - d_{1,x})d_{1,y} + (P_y - d_{1,y})d_{1,z} = P^2 - (P_x d_{1,x} + P_y d_{1,y} + P_z d_{1,z}) - V_1 V_i \cos(\gamma_i)
\]

Expressing in matrix notation

\[
\begin{bmatrix}
-P_y - d_{1,y} & P_x - d_{1,x} & 0 \\
0 & P_z - d_{1,z} & -P_y - d_{1,y} \\
P_x - d_{1,x} & P_y - d_{1,y} & P_z - d_{1,z}
\end{bmatrix}
\begin{bmatrix}
d_{1,x} \\
d_{1,y} \\
d_{1,z}
\end{bmatrix} =
\begin{bmatrix}
P_x - d_{1,x}P_y - (P_y - d_{1,y})P_x - V_1 V_i n_z \\
(P_z - d_{1,z})P_y - (P_y - d_{1,y})P_z + V_1 V_i n_x \\
P^2 - (P_x d_{1,x} + P_y d_{1,y} + P_z d_{1,z}) - V_1 V_i \cos(\gamma_i)
\end{bmatrix}
\]

or moving the coefficients of \( d_i \) to the right side

\[
\begin{bmatrix}
d_{1,x} \\
d_{1,y} \\
d_{1,z}
\end{bmatrix} =
\begin{bmatrix}
-P_y - d_{1,y} & P_x - d_{1,x} & 0 \\
0 & P_z - d_{1,z} & -P_y - d_{1,y} \\
P_x - d_{1,x} & P_y - d_{1,y} & P_z - d_{1,z}
\end{bmatrix}^{-1}
\begin{bmatrix}
P_x - d_{1,x}P_y - (P_y - d_{1,y})P_x - V_1 V_i n_z \\
(P_z - d_{1,z})P_y - (P_y - d_{1,y})P_z + V_1 V_i n_x \\
P^2 - (P_x d_{1,x} + P_y d_{1,y} + P_z d_{1,z}) - V_1 V_i \cos(\gamma_i)
\end{bmatrix}
\]

Which can be expressed as

\[
\begin{bmatrix}
d_{1,x} \\
d_{1,y} \\
d_{1,z}
\end{bmatrix} =
\begin{bmatrix}
\frac{(d_{1,y} - P_y)^2 + (d_{1,z} - P_z)^2}{C_1} & \frac{(d_{1,z} - P_z)(d_{1,z} - P_z)}{C_0} & \frac{(d_{1,z} - P_z)}{C_1} \\
\frac{(d_{1,x} - P_x)}{C_1} & \frac{(d_{1,z} - P_z)}{C_1} & \frac{(d_{1,y} - P_y)}{C_1} \\
\frac{(d_{1,x} - P_x)^2 + (d_{1,y} - P_y)^2 + (d_{1,z} - P_z)^2}{C_0}
\end{bmatrix}
\begin{bmatrix}
d_{1,y} - P_y \\
d_{1,z} - P_z \\
(d_{1,y} - P_y)^2 + (d_{1,z} - P_z)^2
\end{bmatrix}
\]

\[
\begin{bmatrix}
d_{1,y} - P_y \\
d_{1,z} - P_z \\
(d_{1,y} - P_y)^2 + (d_{1,z} - P_z)^2
\end{bmatrix} =
\begin{bmatrix}
(d_{1,x} - P_x)^2 + (d_{1,y} - P_y)^2 + (d_{1,z} - P_z)^2
\end{bmatrix}
\]

where

\[
C_1 = \frac{(d_{1,x} - P_x)^2 + (d_{1,y} - P_y)^2 + (d_{1,z} - P_z)^2}{(d_{1,x} - P_x)^2 + (d_{1,y} - P_y)^2 + (d_{1,z} - P_z)^2}
\]

\[
C_0 = \frac{(d_{1,y} - P_y)^2 + (d_{1,z} - P_z)^2}{(d_{1,x} - P_x)^2 + (d_{1,y} - P_y)^2 + (d_{1,z} - P_z)^2}
\]
CHAPTER 3
CONCLUSION

It is important to be able to move a parallel-leg mechanism through a prescribed path. Since most tools are symmetric, the mobile upper plate of a parallel leg mechanism can be rotated about its tool axis without changing the tool orientation. Thus a single degree of freedom exists for the upper plate, a rotation about the tool axis. Having an equation expressing the leg variables in terms of a single point on the upper plate (in addition to the tool axis position and orientation) would enable the generation of the leg positions for different rotations of the upper plate. The point chosen is the vertex $d_i$, where the first leg connects onto the upper plate.

Thus, all leg variables $x$, $\eta$, & $\psi$ have been expressed first as functions of $d_i$. Then the vertices have been expressed as functions of the single variable, $d_i$, and the two parameters, tool axis orientation $\mathbf{n}$ and tool axis position $P$.

For the leg $i$, the solution is expressible in terms of $d_i$.

\[
x_i = \sqrt{\left(\frac{d_{i,x}}{x_i}\right)^2 + \left(\frac{d_{i,y}}{x_i}\right)^2 + \left(\frac{d_{i,z}}{x_i}\right)^2 + \left(\frac{\eta_i}{x_i}\right)^2 - 2 \frac{d_{i,y}}{x_i} \cdot \frac{\sin(\eta_i)}{x_i} - \frac{d_{i,z}}{x_i} \cdot \cos(\eta_i)}
\]

\[
\sin(\psi_i) = -\frac{d_{i,z}}{x_i \cdot \cos(\psi_i)}
\]

\[
\cos(\psi_i) = \frac{d_{i,x} \cdot \cos(\eta_i) + d_{i,y} \cdot \sin(\eta_i) - r_i}{x_i \cdot \cos(\psi_i)}
\]

The square root can be either +/-.

However taking $\eta_i$ and $\psi_i$ into account together, the Single Leg Equation still needs to be satisfied. As such, $d_z$ can not be negative, so either $\sin(\eta_i)$ or $\cos(\psi_i)$ is negative.
These equations combined with

\[
\begin{bmatrix}
  d_{1,x} \\
  d_{1,y} \\
  d_{1,z}
\end{bmatrix} =
\begin{bmatrix}
  \frac{(d_{1,x} - P_x)^2 + (d_{1,y} - P_y)^2}{C_1} & \frac{(d_{1,y} - P_y)(d_{1,z} - P_z)}{C_0} & \frac{(d_{1,z} - P_z)}{C_1} \\
  \frac{(d_{1,x} - P_x)}{C_1} & \frac{(d_{1,y} - P_y)}{C_0} & \frac{(d_{1,z} - P_z)}{C_1} \\
  \frac{(d_{1,x} - P_x)(d_{1,y} - P_y)^2 + (d_{1,y} - P_y)^2}{C_0} & \frac{(d_{1,y} - P_y)^2}{C_1} & \frac{(d_{1,z} - P_z)}{C_1}
\end{bmatrix}
\begin{bmatrix}
  d_{1,y} - P_y - d_{1,x} - P_x - V_1 \cdot V_1 \\
  d_{1,y} - P_y + d_{1,x} + P_x + V_1 \cdot V_1 \\
  d_{1,z}^2 - (P_x d_{1,x} + P_y d_{1,y} + P_z d_{1,z}) - V_1 \cdot V_1 \cos(y_i)
\end{bmatrix}
\]

where

\[
C_1 = \left[ \left( \frac{(d_{1,x} - P_x)^2 + (d_{1,y} - P_y)^2 + (d_{1,z} - P_z)^2}{C_1} \right) \right]
\]

\[
C_0 = \left[ \left( \frac{(d_{1,x} - P_x)^2 + (d_{1,y} - P_y)^2 + (d_{1,z} - P_z)^2}{C_1} \right) \right]
\]

expresses all the legs in terms of \( d_1 \).

From inputting the different points on the tracer line of the tool axis position \( P \), this process can be repeated to generate a set of such solutions. In addition, when constructing the tracer path, the tool orientation need not be held constant.

Another approach would be to map the state space numerically by constructing a series of different tool axis positions’ \( P \) and orientations’ \( n \). Each different value of \( P \) and \( n \) corresponds to a different plane the mobile upper plate can lie in. The three dimensional space can then be generated by overlaying a progressive set of planes and removing the unattainable configurations.

This analysis can be extended to consider velocity by simply taking the derivatives with respect to time of \( x_i, y_i, \psi_i \), where only the variables \( d_i, n, \) and \( P \) have time derivatives. The second derivative could also be taken to find the accelerations of the leg variables. Acceleration and velocity are important in design of mechanisms because the control systems need to produce them.
Also, instead of constraining the leg bases to lie on the same ground plane, the Single Leg Equation can be modified to incorporate an elevation \( z \) in the \( z_g \)-direction at base point \( B \).

Using

\[
B(r, \theta, z) := \begin{bmatrix}
\cos(\theta) & -\sin(\theta) & 0 \\
\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 1 & -ez & 0 \\
0 & 1 & -er & -ez & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

It can be shown that the Modified Single Leg Equation would be:

\[
d(r, \theta, z, \eta, \psi, x) :=
\begin{bmatrix}
r\cos(\theta) - x(\sin(\theta)\sin(\psi) - \cos(\theta)\cos(\eta)\cos(\psi)) \\
r\sin(\theta) + z(\cos(\theta) - \sin(\theta)) + x(\sin(\eta)\cos(\psi)\sin(\psi) + x(\sin(\psi)\cos(\theta) + \cos(\eta)\cos(\psi)\sin(\theta)) \\
z - x\sin(\eta)\cos(\psi)
\end{bmatrix}
\]

From this equation the leg parameters could be expressed as functions of the variables \( d_1, n, \) and \( P \).

A last modification that could also be done is not to constrain the tool axis orientation to be normal to the mobile upper plate. Then \( n \) would not be the tool axis orientation, and the upper plate would be able to eccentrically rotate about the tool axis instead.

The reason why an elevation and eccentricity were not addressed was because in parallel leg mechanisms the leg bases do primarily lie on the same base plane and the tool is also primarily normal to the mobile upper plate.
APPENDIX

DUAL NUMBER REPRESENTATION

Dual numbers are used to determine the leg vertex (top) position in terms of the ground. The following is an overview of dual numbers. Refer to Fischer’s Dual-Number Methods in Kinematics, Statics and Dynamics for an in-depth derivation.

A dual number is comprised of a primary (real) component and a dual ($\varepsilon$) component, where $\varepsilon^2 = 0$. This is a useful representation of simultaneous rotation and translation. The rotation is done in the primary plane and the translation is done in the $\varepsilon$ plane. (The reason why translation is done in the $\varepsilon$ plane is because from one coordinate transform to another, you never multiply separate translations, but multiply instead the translation by the next rotation; thus in dual numbers, a rotation times a translation ends up in the $\varepsilon$ plane, pure rotations stay in the primary plane, and any multiplied translations disappear.) Now a dual angle is the rotation (vector curl) about a coordinate axis and a translation along that same coordinate axis (the translation does not effect the plane of rotation, but simply elevates the rotated plane). The breakdown of coordinate transformation is the same as if done with a pure rotation, i.e. a 3X3 matrix $M$, whose $\text{det}(M) = 1$ in all cases. However, to represent the $\cos(\theta + \varepsilon \alpha)$, a Taylor series expansion about the primary component is done. (Note, the expansion of $f(a + \varepsilon a_0) = f(a) + \varepsilon a_0$ [$df(a) / da$]; all other terms disappear because $\varepsilon^2 = 0$.) On the next page are the dual angle rotations and translations about the X, Y, & Z axes respectively.
So \( M \) transforms coordinates in \( b \) into \( a \). However to do this, it is important to remember \( (\theta + \epsilon r) \) is measured in \( a \), not \( b \).

To decompose a dual vector, it is important to realize that \( N = N_0 + (r \times N_0)\epsilon \), where \( N_0 \) is the angular orientation: projections along \( i, j, \) & \( k \) axes.

Now the dual vectors for transformations about the \( X, Y, \) & \( Z \) axes are

\[
M = \begin{bmatrix} i_b & j_b & k_b \end{bmatrix}
\]

\[
M = \begin{bmatrix} \cos(\theta + \epsilon r) & -\sin(\theta + \epsilon r) & 0 \\
\sin(\theta + \epsilon r) & \cos(\theta + \epsilon r) & 0 \\
0 & 0 & 1 \end{bmatrix}
\]

\( c(\theta, r) = \cos(\theta) - \epsilon \cdot r \cdot \sin(\theta) \quad s(\theta, r) = \sin(\theta) + \epsilon \cdot r \cdot \cos(\theta) \)

\[
X(\sigma, r) = \begin{bmatrix} 1 & 0 & 0 \\
0 & c(\sigma, r) & -s(\sigma, r) \\
0 & s(\sigma, r) & c(\sigma, r) \end{bmatrix} \quad Y(\eta, r) = \begin{bmatrix} c(\eta, r) & 0 & s(\eta, r) \\
0 & 1 & 0 \\
-s(\eta, r) & 0 & c(\eta, r) \end{bmatrix} \quad Z(\theta, r) = \begin{bmatrix} c(\theta, r) & -s(\theta, r) & 0 \\
s(\theta, r) & c(\theta, r) & 0 \\
0 & 0 & 1 \end{bmatrix}
\]
REFERENCES


