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ABSTRACT

EXTENDING THE BOUNDARY METHOD FOR SOLVING HUMAN MOTION PROBLEMS USING A 3 DIMENSIONAL COUPLED PENDULUM MODEL

by
Avani Janardan Patel

Mathematical modeling, an evolving area in movement analysis research, couples quantitative measures of human motion with theoretical concepts in physiology and mechanics. Literature reveals two methods to solve for human motion problems, *Forward and Inverse Mechanics.* In this investigation, a new approach called the *Boundary Method* is adopted to solve for human motion. This approach has the advantage of being able to solve for both new motions *and* the net muscular joint forces required to produce those motions but only at those discrete times and body configurations that are believed to be most crucial for accomplishing the task.

The method is applied to solve for the dynamic equations of motion governing a coupled double pendulum in 3 Dimensions.

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by
Avani Janardan Patel

A Thesis
Submitted to the Faculty of
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APPROVAL PAGE

EXTENDING THE BOUNDARY METHOD FOR SOLVING HUMAN MOTION PROBLEMS USING A 3 DIMENSIONAL COUPLED PENDULUM MODEL

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To my dad
for his unconditional love and support

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Objective

The objective of this thesis is to discuss the application of the Boundary Method to solve human motion problems. The specific case studied is the shoulder, modeled as a 3D system of two coupled pendulums with four degrees of freedom.

The equations are derived using the Lagrangian equations of motion. Given the segment masses, segment lengths, distribution of mass in each of the segments, and appropriate side conditions (either initial or boundary) the equations of motion can be solved for the position of the segments at any given time. The Boundary Method enables one to solve for many possible movement paths that can accomplish a given desired motor task. The method breaks each movement path into a discrete and flexible number of contiguous ballistic phases. Each ballistic phase is solved independently as a two point boundary value problem. The two boundary points are the starting body configuration and the target ending configuration. In addition, a time duration for moving from the start to the target must be specified. The equations of motion are used to find a ballistic solution. More precisely a ballistic solution is a motion path that moves the body from start to target in the given duration using only gravity and the initial momentum in the starting configuration. Since gravity does much of the work, in a ballistic solution muscles would only be used to set the correct initial segment velocities for gravity to act upon. Unlike an initial value problem, the initial velocity (momentum) of each segment is not part of the input of a two point boundary value problem; rather it is part of the

solution. So in this sense, the boundary method solves for both the motion path and the necessary net muscle activity required to produce that motion. Therefore the boundary method is neither an inverse nor a forward approach to solving biomechanical questions involving the dynamics of movement but rather is a new approach for solving such problems.

The method searches for a path that exploits gravity and momentum of the system and can be used to optimize mechanical efficiency.

The 3D system of a coupled double pendulum is studied as a test case for the more general form of a system of 'N' coupled pendulum in 3D. This general form has been worked out by the research team in the Human Performance Lab.

1.2 Background Information

Biomechanics is the science which investigates the internal and external forces acting on a human body and the effects produced by these forces. In the last several decades, biomechanics has demonstrated considerable growth evolving from an exercise in the filming of human movement to an applied science with a powerful array of measurement and modeling techniques. The simple descriptive approach which was characteristic of early work has been superceded by attempts to explain the mechanisms underlying movement. Consequently, biomechanics has emerged as an important area of scientific investigation in a variety of disciplines. Included among these are automobile safety, biomedical engineering, ergonomics, exercise science, orthopedic surgery, physical rehabilitation, and sport.

The development and application of mathematical modeling approaches designed to investigate human movement is constantly expanding. These models are useful in

better understanding movement and in estimating clinical parameters that are significant but otherwise difficult to access. For example, joint reaction forces or stability indices are clinically important but difficult to assess directly. Modeling techniques can also be used to gather information regarding the action of muscles such as estimating force, torque, work, power and duration of activity.

1.2.1 Forward and Inverse Dynamics

Modeling of human motion is dominated by two approaches: *forward and inverse dynamics,* primarily because they either solve the dynamical equations of motion in a forward or backward direction. The type of model used in a particular research situation depends on the goal of the study and the type of data that has either been collected or is otherwise made available.

Whether an inverse dynamics or a forward dynamics approach is used, the modeling process must always include a reasonable idealization of the moving anatomic and physical structures. Depending on the motion studied and the inherent complexity of the associated model, the body is idealized as a two- or three-dimensional, multi-segment, coupled pendulum system with segments represented as either point masses or rigid bodies referred to as distributed masses.

Underlying every model mathematically are its equations of motion, which relate the actual movement of the body to the applied forces, which cause the movement. How the equations of motion are solved determines whether the modeling approach is forward or inverse. Forward methods require forces as input and solve for the motion as output.

Inverse methods use the equations backwards to solve for the forces necessary to generate a given motion.

Although any given method and choice of dynamic variables may ultimately lead to an equivalent system of differential equations that can be used to solve for either kinetics or kinematics, the choice is not trivial. Different methods and choices for dynamic variables can greatly simplify the explicit form of the equations, reduce the number of redundant equations and/or the number of terms and non-linearities in each equation. For problems involving a large number of degrees of freedom, the choice of formulation method and dynamic variables can greatly affect the ease of formulating the system of differential equations and the efficiency by which solutions can be achieved using modern computational techniques.

For example, the Newtonian free-body diagram approach where the sum of all externally applied force vectors is set equal to the sum of inertial terms often leads to a redundant set of equations that may be difficult to systematically simplify. This is because (reaction) force components that maintain configuration constraints but do no work on the system are included in the free body diagrams that are used to setup the equations.

Finding the equations of motion using the method of Lagrange is straight forward once the appropriate work or energy (Lagrangian) function is obtained, and simply involves taking the appropriate partial and time derivatives. However, when there are a large number of degrees of freedom, this process can become quite tedious and can often lead to equations that are much lengthier than the equivalent reduced equations finally arrived at using the force methods of Newton or D'Alembert (Reference).

The Lagrange method is usually more direct than either the Newton or D'Alembert (Reference) methods for arriving at the correct set of independent motion equations. For this reason it may be preferred in a forward dynamics approach when an analysis of all the interacting forces in the system is of less importance than actually obtaining the motion of the system itself. The Newtonian formulation method is used more frequently in inverse dynamic studies because direct information about all the forces, including the reaction forces that occur between interacting segments, is often desired.

The major advantage of inverse modeling is that its required input (motion kinematics and ground reaction forces) is readily measurable by sophisticated motion analysis systems and force plates available in many research laboratories. The muscle activity predictions obtained from inverse modeling are useful in estimating such important quantities as the mechanical energy expenditure and the power consumption of a performed movement. Such measures can provide a means for quantitatively comparing different movements with respect to the forces, the mechanical energy, and the power required to produce them.

A primary disadvantage of inverse models is that they require kinematics as input. Because of this, inverse models can be used only to analyze movements that have already been performed. They are incapable of generating system kinematics as output and, therefore, cannot be used to predict new movement patterns. That is, they cannot suggest better alternatives to present observable motions but can only compare or contrast existing motions. Therefore, a different approach that can be used to propose other movement options than the research subject is currently affording himself or herself is

highly desirable. When the goal of the research is to predict new physically realizable motions, forward models are ordinarily utilized.

Unlike the inverse method, forward models require forces as input and solve for the motion as output. All forces acting on the system must be input to the model, while accelerations are unknown and constitute model output. These dynamic accelerations are used to solve for position (and velocity) segment trajectories as a function of time. Thus, the causes of the motion are known by forward models, which predict the ensuing motion of the system when such forces are applied.

In this way, the hierarchy of forward modeling more closely mimics the real sequence of physiological events. Movement begins with neural drive to the muscles, resulting in muscle force activity at the joints, and ultimately causing displacements of the joints (i.e., the observed motion). Therefore, forward models will simulate physically realizable movements when provided with input data resembling the net joint force produced from neural input to the muscles. Unfortunately, noninvasive instrumentation to measure such muscle forces does not exist presently. Therefore, forward models must rely on estimates or even educated guesses of forces that must have acted on the system in order to produce an observed movement.

In addition to net muscle force estimates, forward model input in a Newton-Euler formulation of the equations of motion requires inclusion of all other body and contact forces that could significantly contribute to the system's motion such as gravity, ground and joint reaction forces. Forward approaches usually solve the differential equations of motion as initial value problems and therefore both the initial system configuration (segment angles) and the initial system velocity (segment angular velocities) are also

required input. As with inverse solutions all forward solutions require accurate anthropometric segment data.

The potential of the forward modeling approach is that new and better movement solutions can be found that will optimize any sufficiently well defined performance goal. In practice, however, the technique is very difficult to apply towards this goal. Many of the motor tasks for which an optimal solution is desired are highly skilled movements that are likely to be very sensitive to small changes in the dynamic forces applied to the system. This inherent instability of the motion requires a very accurate guess of the net muscle moments that must have acted at each joint of the system in order to produce the improved movement and even educated guesses of what these net muscle moments might be are not available.

After considering the limitations of existing modeling methods, it seems desirable to try to develop a new modeling technique, which would try to address some of the inadequacies of both inverse and forward models. Inverse models can effectively use motion data collected in the lab to critically compare two given performance techniques but cannot output new motions since they are input to the method. Forward models have the potential for finding new movements to skilled motor tasks but to find such new solutions requires very accurate input data (muscle force or EMG) that can not be known or even reasonably guessed a priori.

1.2.2 Lagrangian Equations of Motion: An Energy Based Approach

Specific to any mathematical model are its equations of motion, which describe the interaction between applied forces (kinetics) and the movement of limb segments (kinematics). There are a number of methods that can be used to formulate equations of

motion. Many of greatest names in the history of modern science such as Newton, Euler, D'Alembert, Lagrange, Hamilton, Jacobi, Boltzmann, Gibbs have developed distinct methods for constructing equations of motion.

For purposes of the research undertaken in this thesis, the Lagrangian approach has been adopted. The formulation method of Lagrange uses an energy (rather than force) approach and a more flexible (generalized) coordinate system than the Newtonian approach. Force terms that do not contribute to the motion, such as reaction force components that keep segment lengths constant, do not explicitly appear in the equations of motion. These force components are implicitly accounted for in the Lagrangian approach, either by a clever choice of independent generalized coordinates or by using Lagrange multipliers. For example, choosing segment angles as the generalized coordinate system implicitly allows for body movements to occur without violating the constraints of constant segment lengths.

Lagrange's equations require the concepts of virtual displacement and employ system energy and work as functions of the generalized coordinates to obtain a set of second-order differential equations of motion. To a large extent the method reduces the entire field of dynamics to a single procedure involving the same basic steps, regardless of the number of segments considered, the type of coordinates employed, the number of constraints on the model, and whether or not the constraints are in motion. In summation, the Lagrangian method is characterized by simplicity and is applicable in any suitable coordinates.

1.2.2.1 The Generalized Coordinate. One of the foremost things to be determined in a given system is the number of independent coordinates that represent the model. Any set of time-dependent parameters that give an unambiguous representation of the system configuration can serve as system coordinates. These parameters are known as generalized coordinates, and they are denoted by $[q]^t$,

$$[q]^t = [q_1, q_2, q_3, \ldots\ldots, q_n]$$

The symbol $[\ ]^t$ indicates the transpose of an array or a matrix.

By definition, the first and second derivatives of a generalized coordinate q_i with respect to time are called the *generalized velocity* $\dot{q}_i$ and the *generalized acceleration* $\ddot{q}_i$, respectively.

1.2.2.2 The Lagrangian Function L. The Lagrangian function L is defined as the difference between the total kinetic energy KE and the total potential energy PE in the system

$$L = KE - PE$$

The kinetic energy for a segment is defined as the work done on the segment to increase its velocity from rest to some value **v**. Potential energy exists if the system is under the influence of conservative forces. For the purposes of this discussion, segment potential energy will be defined as the energy possessed by virtue of a segment (or particle) position in a gravity field relative to an arbitrarily selected datum level (usually ground level) in the system.

1.2.2.3 Lagrange's Equations. Lagrangian equations of motion for a conservative system with *n* generalized coordinates are given by:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, \ldots\ldots, n)$$

where L = KE – PE.

1.2.3 The Lagrangian Derived from the Principle of Least Action

1.2.3.1 For a system of one particle. *Principle of Least Action* says that nature selects a motion path which tends to optimize (usually minimize) a real valued functional called the action of the system, or in other words, it selects a path where the *action* is critical.

The Lagrangian is defined as follows:

$$\mathbf{L(X, \dot{X}) = K(X, \dot{X}) - P(X)}$$

where K = Kinetic Energy (a function of X and $\dot{X}$)

P = Potential Energy (a function of X)

L is a function of *position (X)* and *velocity($\dot{X}$)*.

Action is defined as:

$$\mathbf{A[x(t)] = \int_{t_0}^{t_f} L(x(t), \dot{x}(t))dt} \qquad (1.1)$$

Consider a single particle with an initial configuration x_0 and a target configuration x_f.

To travel from x_0 to x_f, a multitude of paths can be chosen, for example, $x_1(t)$, $x_2(t)$,....., $x_N(t)$. This is indicated in Figure 1.1

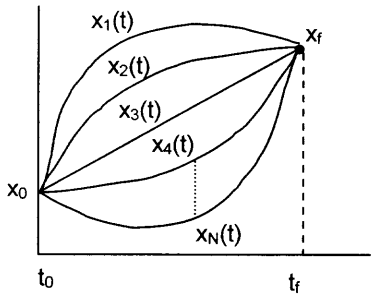


Figure 1.1 Particle can follow 'N' paths.

Suppose we select any arbitrary path x(t). Path x(t) can be approximated by dividing it into 'N' equally spaced time-points separated by time Δt. As Δt gets smaller and smaller, we get a better approximation for the path x(t).

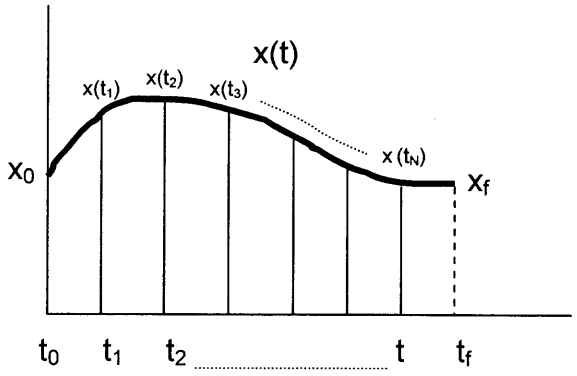


Figure 1.2 An arbitrary path x(t) divided in 'N' equally spaced time points.

As indicated in Figure 1.2, each time-point has a corresponding value of the path associated to it. The initial and final configurations, x_0 and x_f ,respectively, are fixed. Hence, path x(t) is a vector approximated as:

$$\vec{x}(t) = [x(t_1), x(t_2), x(t_3), ..., x(t_N)]$$

In this approximation, the path $\vec{x}(t)$ can be considered as a point in 'N' space. Thus the action functional, defined by Equation (1.1), assigns a number to each path (point) in its domain.

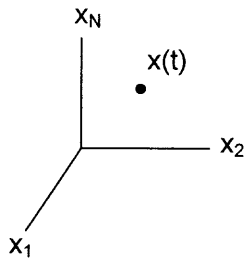


Figure 1.3 Path X(t) is a point in 'N' space.

Once the path has been selected, the derived velocities from it are given by the vector:

$$\vec{\dot{x}}(t) = [\dot{x}(t_1), \dot{x}(t_2), \dot{x}(t_3), ..., \dot{x}(t_N)]$$

The Lagragian, which is a function of position and velocity, assigns a value of L to each 'x_i', where i goes from 1 to N.

Hence the 'Action' for the path x(t) is approximated by:

$$\sum_{i=1}^{N} L_i \cdot \Delta t = A[X(t)]$$

or in the limit as $\Delta t \to 0$, $\quad A[X(t)] = \int_{t_0}^{t_f} L dt$

i.e, $\quad A[X(t)] = \int_{t_0}^{t_f} L(X(t), \dot{X}(t)) dt \qquad (1.2)$

Now we will consider all nearby paths to a given path X(t) by constructing a test function (point) $\eta(t)$ and scale it by a factor ε. $\eta(t)$ is any arbitrary test path that satisfies

$\eta(t) = 0$ when $t = t_0$ and $t = t_T$. The new path $x_N(t)$ is given by: $\boldsymbol{x_N(t) = x(t) + \varepsilon\eta(t)}$. The action for the new path would then read as follows:

$$A[x_N(t)] = \int_{t_0}^{t_f} L[\vec{x}_N(t), \dot{\vec{x}}_N(t)]dt$$

or,

$$A[x(t) + \varepsilon\eta(t)] = \int_{t_0}^{t_f} L[\vec{x}(t) + \varepsilon\vec{\eta}(t), \dot{\vec{x}}(t) + \varepsilon\dot{\vec{\eta}}(t)]dt \tag{1.3}$$

The *change in the action* is given by:

$$A[x_N(t)] - A[x(t)] = \int_{t_0}^{t_f} [L\{\vec{x}_N(t), \dot{\vec{x}}_N(t)\} - L\{\vec{x}(t), \dot{\vec{x}}(t)\}]dt \tag{1.4}$$

where $\begin{aligned} x_N(t) &= \vec{x}(t) + \varepsilon\vec{\eta}(t) \\ \dot{x}_N(t) &= \dot{\vec{x}}(t) + \varepsilon\dot{\vec{\eta}}(t) \end{aligned}$

In the limit that $\varepsilon \rightarrow 0$, we define the rate of change of the action $\left.\frac{\delta A}{\delta x}\right|_\eta$ for a given path x(t) and given test function $\eta(t)$ as

$$\lim_{\varepsilon \to 0} \frac{A[x_N(t)] - A[x(t)]}{\varepsilon} = \frac{\int_{t_0}^{t_f} [L\{\vec{x}_N(t), \dot{\vec{x}}_N(t)\} - L\{\vec{x}(t), \dot{\vec{x}}(t)\}]dt}{\varepsilon} \tag{1.5}$$

$$\equiv \left.\frac{\delta A}{\delta x}\right|_\eta$$

Nature chooses an x(t) that is critical, in the sense that $\left.\frac{\delta A}{\delta x}\right|_\eta = 0$, no matter what $\eta(t)$ is.

That is, for a path x(t) that is critical, the rate of change of the action is 0 no matter what path nearby to it is chosen.

Expanding $L[\vec{x}+\varepsilon\vec{\eta},\dot{\vec{x}}+\varepsilon\dot{\vec{\eta}}]$ in a Taylor series about $(x,\dot{x})$ gives the following

$$\left.\frac{\delta A}{\delta x}\right|_{\eta} = \int_{t_0}^{t_f}\left[\left\{L(x,\dot{x})+\frac{\partial L}{\partial x}\left(\frac{\varepsilon\eta(t)}{\varepsilon}\right)+\frac{\partial L}{\partial \dot{x}}\left(\frac{\varepsilon\dot{\eta}(t)}{\varepsilon}\right)+\frac{\partial^2 L}{\partial x^2}\left(\frac{\varepsilon^2\eta^2(t)}{\varepsilon}\right)+\frac{\partial^2 L}{\partial \dot{x}^2}\left(\frac{\varepsilon^2\dot{\eta}^2(t)}{\varepsilon}\right)+(O\varepsilon^2)\right\}\right.$$
$$\left.-L(x,\dot{x})\right]dt$$

$$\left.\frac{\delta A}{\delta x}\right|_{\eta} = \int_{t_0}^{t_f}\left[\frac{\partial L}{\partial x}\eta+\frac{\partial L}{\partial \dot{x}}\dot{\eta}\right]dt \tag{1.6}$$

Integrating Equation (1.6) by parts, we get

$$\int_{t_0}^{t_f}\left[\frac{\partial L}{\partial \dot{x}}\dot{\eta}\right]dt = \underbrace{\left.\frac{\partial L}{\partial \dot{x}}\eta\right|_{t_0}^{t_f}}_{P} - \int_{t_0}^{t_f}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\eta \tag{1.7}$$

Since $\eta(t)=0$ for $t=t_0$ and $t=t_f$, $P=0$ in Equation (1.7).

Hence,

$$\frac{\delta A}{\delta x} = \int_{t_0}^{t_f}\left[\frac{\partial L}{\partial x}\eta-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\eta\right]dt \tag{1.8}$$

Equation (1.8) can also be written as

$$\frac{\delta A}{\delta x} = \int_{t_0}^{t_f}\underbrace{\left[\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]}_{R}\underbrace{\eta}_{S}dt$$

The equation above is a dot product between R and S. For a critical path x(t), $R\cdot S=0$.

The only way that can be satisfied for all choices of $\eta(t)$, is to have $R=0$.

Hence, we get

$$\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=0 \tag{1.9}$$

or, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$, which is the *Lagrange equation of motion.* (1.10)

1.2.3.2 For a system of N particles. Consider a system of N particles. Particle 1 follows a path $x_1(t)$, particle 2 follows path $x_2(t)$ ..... path $x_N(t)$ for the N^{th} particle. $x(t)$ then, is a vector path consisting of 'N' particle paths.

$$\vec{x}(t)=[x_1(t),x_2(t),.....,x_N(t)]$$

The velocity vector is given by

$$\dot{\vec{x}}(t)=[\dot{x}_1(t),\dot{x}_2(t),.....,\dot{x}_N(t)]$$

The action for a system of 'N' particles is given by

$$A[X(t)]=\int_{t_0}^{t_f} L[\vec{x}(t),\dot{\vec{x}}(t)]dt \tag{1.11}$$

The test function is given by

$$\vec{\eta}(t)=[\eta_1(t),\eta_2(t),.....,\eta_N(t)]$$

Scaling the test function by ε gives us

$$\varepsilon\vec{\eta}(t)=\varepsilon[\eta_1(t),\eta_2(t),.....,\eta_N(t)]$$

Hence, for small ε a new nearby path to X(t) is defined by

$$\vec{X}_N=\vec{X}+\varepsilon\vec{\eta}$$

The action for the old path is given by

$$A[\vec{X}(t)]=\int_{t_0}^{t_f} L[\vec{x}(t),\dot{\vec{x}}(t)]dt \tag{1.12}$$

The action for the new path is

$$A[\vec{X}(t)+\varepsilon\vec{\eta}(t)] = \int_{t_0}^{t_f} L[(\vec{x}(t)+\varepsilon\vec{\eta}(t)),(\dot{\vec{x}}(t)+\varepsilon\dot{\vec{\eta}}(t))]dt \tag{1.13}$$

Again, expanding $L(\vec{x}+\varepsilon\vec{\eta},\dot{\vec{x}}+\varepsilon\dot{\vec{\eta}})$ in a taylor series about $L(X,\dot{X})$ gives

$$\begin{aligned} A[\vec{X}(t)+\varepsilon\vec{\eta}(t)] - A[\vec{X}(t)] &= \int_{t_0}^{t_f} [L\{(\vec{x}(t)+\varepsilon\vec{\eta}(t)),(\dot{\vec{x}}(t)+\varepsilon\dot{\vec{\eta}}(t))\} - L\{\vec{x}(t),\dot{\vec{x}}(t)\}]dt \\ &= \int_{t_0}^{t_f} [\{L(\vec{x},\dot{\vec{x}}) + \frac{\partial L}{\partial x_1}\varepsilon\eta_1 + ... + \frac{\partial L}{\partial x_N}\varepsilon\eta_N + \frac{\partial L}{\partial \dot{x}_1}\varepsilon\dot{\eta}_1 + ... \\ &\quad + \frac{\partial L}{\partial \dot{x}_N}\varepsilon\dot{\eta}_N + \frac{\partial^2 L}{\partial x_1^2}\varepsilon^2\eta_1^2 + ... + \frac{\partial^2 L}{\partial x_N^2}\varepsilon^2\eta_N^2 + \frac{\partial^2 L}{\partial \dot{x}_1^2}\varepsilon^2\dot{\eta}_1^2 \\ &\quad + ... + \frac{\partial^2 L}{\partial \dot{x}_N^2}\varepsilon^2\dot{\eta}_N^2 + O(\varepsilon^3)\} - L(\vec{x},\dot{\vec{x}})]dt \end{aligned} \tag{1.14}$$

Dividing by ε and taking the limit as $\varepsilon \to 0$, we get

$$\begin{aligned} \lim_{\varepsilon\to 0}\frac{A[\vec{X}(t)+\varepsilon\vec{\eta}(t)]-A[\vec{X}(t)]}{\varepsilon} &= \int_{t_0}^{t_f} \nabla_x L \bullet \eta + \nabla_{\dot{x}} L \bullet \dot{\eta}\, dt = \sum_i \int_{t_0}^{t_f} \frac{\partial L}{\partial x_i}\eta_i + \frac{\partial L}{\partial \dot{x}_i}\dot{\eta}_i\, dt \\ &\equiv \left.\frac{\delta A}{\delta x}\right|_{\eta} \end{aligned} \tag{1.15}$$

Integrating by parts

$$\int_{t_0}^{t_f}\left[\frac{\partial L}{\partial \dot{x}}\dot{\eta}\right]dt = \left.\frac{\partial L}{\partial \dot{x}}\eta\right|_{t_0}^{t_f} - \int_{t_0}^{t_f}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\eta$$

Since $\eta(t) = 0$ at times t_0 and t_f, we get

$$\left.\frac{\delta A}{\delta x}\right|_{\eta} = \int_{t_0}^{t_f}\left[\frac{\partial L}{\partial x}\eta - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\eta\right]dt \tag{1.16}$$

Equation (1.16) can also be written as

$$\frac{\delta A}{\delta x} = \int_{t_0}^{t_f}\left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\eta dt$$

For a critical point, $\frac{\delta A}{\delta x} = 0$, for all choices of η , i.e,

$$\left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\cdot\eta = 0 \quad \forall\eta$$

$$\therefore \quad \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

or,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \qquad \text{where } i = 1,\ldots\ldots\ldots\ldots,N \tag{1.17}$$

Equation (1.17) is the *Lagrange Equation of Motion* for a system of N particles or an N degree of freedom system.

CHAPTER 2

BOUNDARY METHOD: A DIFFERENT MODELLING APPROACH

2.1 Rationale

The *forward* and *inverse* methods respectively require either forces or the actual motion as input to solve the equations of motion. Since muscle forces are not generally known for a new movement pattern the forward method is not a practical method for finding new movement patterns that solve a desired motor task. The inverse method does not solve for motion patterns since they are input to the method. These two methods, therefore, lack the ability to solve for new movement patterns to accomplish a particular task. The *boundary method* approach has the advantage of being able to solve for both new motions *and* the net muscular joint forces required to produce those motions.

2.2 Method Description

Although muscles are required to initiate movement, once initiated, movement can and generally does continue without further muscular activity because of the momentum initially imparted to the segments, and their interaction both with the gravitational field and with each other. The motion of the system and its parts only stops because of the presence of dissipative forces such as friction and joint viscosity that eventually removes the macroscopic mechanical energy initially imparted to the system.

Macroscopic motion which is imparted to a system by the sudden injection of energy into it is sometimes called *ballistic* because it is similar to what happens to a projectile shot out of a cannon. After the initial explosive force that generates the

projectile's initial velocity it continues to move towards the target by the action of gravity alone. If the projectile is composed of many internal parts or segments, each rigid but connected by movable joints then these segments will also continue to move relative to each other as the system moves towards its target. If the initial velocity of each of its segments is given, the motion of the system and of each of its parts is relatively easy to calculate, since the only external force acting on the system is gravity. Frictional or viscous forces can also be included if they are believed to play an important role.

Reaction forces at the points of segment contact, however, are only needed if a free body Newtonian approach is used to solve for the motion of the segments. If an energy based method such as Lagrangian or Hamiltonian mechanics is used then it is not required to know joint reaction forces to solve for the motion since these arise as forces of constraint that simply maintain the postulated rigidity of the segments.

The main point is that one does not need to know the muscle forces during a ballistic motion because they are not active *except to give the segments their starting velocity* at the beginning of the movement. Of course, it would be difficult in a complicated system of joints and segments to know what the correct initial velocity of each of the segments should be in order that the system might arrive from the action of gravity alone in a desired target configuration at a specified target time.

If a ballistic solution exists we can solve the dynamical equations of motion either as an initial value problem or as a two-point boundary value problem. The input required to solve an initial value problem is the system's initial position and its initial velocity. In a two-point boundary value problem the initial velocity of the system is not used as input for the motion solution but is replaced by knowledge of the desired target position of the

system. The two-points in a two-point boundary value problem are the initial and target points. In a system consisting of many segments these two points really represent the configuration of the system at each of these times. The coordinates of each point, for example, could be represented by the location of each segment's center of mass or for planar motion by the angle that the beginning of each segment makes relative to its horizontal position. Instead of asking to find the motion of a projectile given its initial position and velocity, a two-point boundary value problem solves for what the initial velocity must be in order that the resulting motion solution hits the target configuration in a specified time.

Ballistic motions, if they exist, are often quite efficient because muscles need act only at the beginning of a movement. Gravity and the spontaneous transfer of momentum between segments will accomplish the rest. Such ballistic motions however would often require great skill to perform especially if they occur over a long duration. This is because small errors in the initial velocities of the segments magnify into large errors in the target configuration as the duration of the ballistic movement increases.

What happens if no solution exists that moves the body for a designated time from a given starting configuration to a desired target configuration using muscles only to deliver the initial segment velocities? Unlike many initial value problems, the solution of a boundary value problem is not guaranteed to either exist or even be unique if it does exist. Perhaps if a single ballistic solution does not exist, the motion could be successfully performed if two ballistic phases were allowed. One ballistic phase is used to get to an important intermediate target configuration. This configuration serves as the initial configuration of a second ballistic phase where muscle activity is again permitted to

actively change the momentum of some or all of the segments to complete the motor task in the second ballistic phase.

The *boundary method* approach for this two ballistic phase motion will consist of three input configurations (initial, intermediate and target) and the target time will be the sum of the two times to complete each phase separately. Both ballistic phases are solved as separate and independent two-point boundary value problems. The entire motion will now consist of the two ballistic solutions contiguously pieced together. The complete motion will be continuous since the end configuration of the first ballistic phase solution is the beginning configuration of the second ballistic phase. Since both phases are solved separately and independently the final segment velocities at the end of the first ballistic phase solution will not, in general, match the initial velocities that are given by the solution of the second phase.

The method, therefore, generates as output the sudden change in segment velocity that is required to be delivered by the muscles at the crucial intermediate configuration in order to complete the task in the designated time. If there are several two phase ballistic solutions each obtained with a different intermediate configuration then each would represent a different possible movement strategy that solves the desired motor task. Since the solution of a boundary value problem is not guaranteed to be unique, there may be several single phase ballistic solution strategies as well to choose from. Different motion solution strategies will predict that muscles deliver different amounts of mechanical energy to the system and one of many possible selection criteria could be to search for that strategy that minimizes muscular effort, assuming, of course, that this can be defined in some reasonable way.

Of course, one need not restrict the search for motion strategies that are limited to one or two ballistic phases. If more ballistic phases than two are permitted then more intermediate configurations must be specified as input. This results in more times when muscle activity occurs. In the limit of a large number of phases, the boundary method approaches a type of (stable) inverse dynamics since essentially the input consists of a given motion and the output consists of the large number of impulsive forces that are required to produce it. At the other extreme, a single ballistic phase solution is a type of forward dynamic solution where the forces are known but no muscle force terms are used in the equations of motion since the motion is assumed to be ballistic.

In summary, the Boundary Method breaks any movement task into a finite and adjustable number of separate contiguous phases. Each phase is solved as an independent ballistic two-point boundary value problem using Lagrangian Mechanics. The generated output will always be the continuous motion obtained by piecing together the independent two-point boundary value solutions of the separate but contiguous ballistic phases. In addition, the generated output will also consist of a predicted series of impulsive "hammer blows" that represent the net impulsive muscle forces that are needed to suddenly change the segment velocities at the beginning of each contiguous phase. The Boundary Method yields muscular activity at a finite but adjustable number of discrete times during the motion.

CHAPTER 3

EQUATIONS OF MOTION

3.1 Equations of Motion for a Double Pendulum in 3 Dimensions

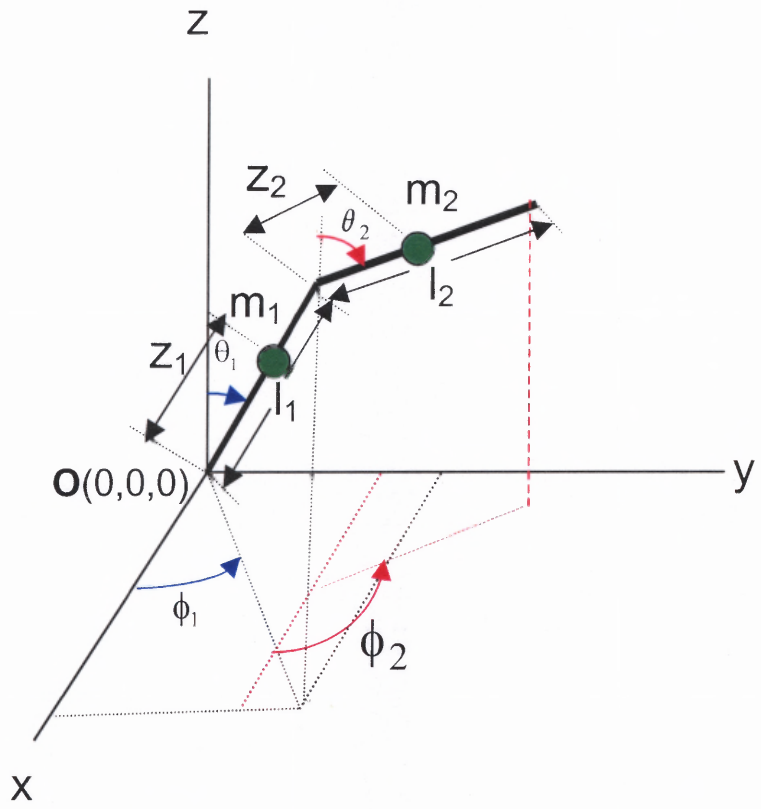


Figure 3.1 Double pendulum in a cartesian coordinate system.

Consider a double pendulum attached end-to-end as shown in figure 3.1. Point **O**(0,0,0) identifies the origin of the system. The first pendulum has length l_1 and the second has a length l_2. Mass points m_1 and m_2 are located at distances z_1 and $(l_1 + z_2)$ respectively, from the origin. θ_1 (theta 1) and θ_2 (theta 2) are the angles which pendulum 1 and pendulum 2 respectively, make with the positive z axis. The projection of

pendulum 1 and pendulum 2 onto the x-y plane, makes an angle ϕ_1 and ϕ_2 respectively, with the positive x axis.

Hence, the system has *four degrees of freedom* as defined by the four angles.

Relation Matrix:

The relation matrix **R** shows the relationship of connection between segments of the system. Each row of **R** represents a point mass of the system, and has information about the path from the origin of the system to the point mass. Each column represents a segment, and has the information about the usage of the segment for every path to point masses. For a system with S segments and P point masses, the relation matrix R has the form as

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1S} \\ r_{21} & r_{22} & \cdots & r_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ r_{P1} & r_{P2} & \cdots & r_{PS} \end{bmatrix} \tag{3.1}$$

The path from the origin to the p-th point mass may consist of many segments. Let us assume that the p-th point mass is on the i-th segment of the system. Then, the i-th segment is the last segment of the path, and all other segments of the path are called forefather segments of the i-th segment.

Using the above information each element r_{PS} of **R** can be determined as follows:

$$r_{ps} = \begin{cases} z_p & \text{if } s = i, \\ L_s & \text{if the } s\text{-}th \text{ segment is a forefather segment of the } i\text{-}th \text{ segment,} \\ 0 & \text{otherwise.} \end{cases}$$

where

z_p : mass center of the *p-th* particle from the joint with its forefather segment

L_s : length of the *s-th* segment

$p = 1, 2, \ldots P$ (*P*: total number of particles)

$s = 1, 2, \ldots S$ (*S*: total number of segment).

Relation Matrix for the Double Pendulum:

The relation matrix **R** for the double pendulum with two point masses m_1 and m_2 at length l_1 and (l_1+l_2) from the origin respectively, is given by

$$R = \begin{bmatrix} z_1 & 0 \\ l_1 & z_2 \end{bmatrix}_{2\times 2} \tag{3.2}$$

x-, y-, and z- coordinates for a system of s- segments and p- mass points:

After the relation matrix is determined, the general x, y and z- coordinates of point masses can be written as follows:

$$x=[x_1\ x_2\ \ldots\ x_P]^T = R\cdot[(\cos\phi_1\sin\theta_1)\ \ (\cos\phi_2\sin\theta_2)\ \ \ldots\ \ (\cos\phi_S\sin\theta_S)]^T \tag{3.3}$$

$$y=[y_1\ y_2\ \ldots\ y_P]^T = R\cdot[(\sin\phi_1\sin\theta_1)\ \ (\sin\phi_2\sin\theta_2)\ \ \ldots\ \ (\sin\phi_S\sin\theta_S)]^T \tag{3.4}$$

$$z=[z_1\ z_2\ \ldots\ z_P]^T = R\cdot[(\cos\theta_1)\ \ (\cos\theta_2)\ \ \ldots\ \ (\cos\theta_S)]^T \tag{3.5}$$

where

θ_S : Angle of the s-th segment with respect to the positive z-direction.

ϕ_S: Angle of the s-th segment with respect to the positive x-direction.

Double pendulum coordinates:

The coordinates of the two mass points $m_1(x_1,y_1,z_1)$ and $m_2(x_2,y_2,z_2)$ are given by

$$x=[x_1 \;\; x_2]^T = R\cdot[(\cos\phi_1\sin\theta_1) \;\; (\cos\phi_2\sin\theta_2)]^T \tag{3.6}$$

$$y=[y_1 \;\; y_2]^T = R\cdot[(\sin\phi_1\sin\theta_1) \;\; (\sin\phi_2\sin\theta_2)]^T \tag{3.7}$$

$$z=[z_1 \;\; z_2]^T = R\cdot[(\cos\theta_1) \;\; (\cos\theta_2)]^T \tag{3.8}$$

Velocity of components in the x- and y- and z- direction :

From the x-, y- and z- coordinate equations, the x- and y- and z- component of the velocity of *p* point masses can be determined as follows

$$\dot{x}=[\dot{x}_1 \;\; \dot{x}_2 \;\; \ldots \;\; \dot{x}_P]^T = R\cdot CSX\cdot V \tag{3.9}$$

$$\dot{y}=[\dot{y}_1 \;\; \dot{y}_2 \;\; \ldots \;\; \dot{y}_P]^T = R\cdot CSY\cdot V \tag{3.10}$$

$$\dot{z}=[\dot{z}_1 \;\; \dot{z}_2 \;\; \ldots \;\; \dot{z}_P]^T = -R\cdot CSZ\cdot V \tag{3.11}$$

where

$$V=\left[\dot{\theta}_1 \;\; \dot{\theta}_2 \;\; \cdots \;\; \dot{\theta}_s \middle| \dot{\phi}_1 \;\; \dot{\phi}_2 \;\; \cdots \;\; \dot{\phi}_s\right]^T_{1\times 2s}$$

$$CSX=\left[\begin{array}{cccc|cccc} \cos\phi_1\cos\theta_1 & 0 & \ldots & 0 & -\sin\phi_1\sin\theta_1 & 0 & \ldots & 0 \\ 0 & \cos\phi_2\cos\theta_2 & \cdots & 0 & 0 & -\sin\phi_2\sin\theta_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \cos\phi_s\cos\theta_s & 0 & 0 & \cdots & -\sin\phi_s\sin\theta_s \end{array}\right]$$

$$CSY = \left[\begin{array}{cccc|cccc} \sin\phi_1\cos\theta_1 & 0 & \dots & 0 & \cos\phi_1\sin\theta_1 & 0 & \dots & 0 \\ 0 & \sin\phi_2\cos\theta_2 & \dots & 0 & 0 & \cos\phi_2\sin\theta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sin\phi_s\cos\theta_s & 0 & 0 & \dots & \cos\phi_s\sin\theta_s \end{array}\right]$$

$$CSY = \left[\begin{array}{cccc|cccc} \sin\theta_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \sin\theta_2 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sin\theta_s & 0 & 0 & \dots & 0 \end{array}\right]$$

Double pendulum velocity components in the x, y, and z directions:

The velocity components for point masses $m_1(\dot{x}_1, \dot{y}_1, \dot{z}_1)$ and $m_2(\dot{x}_2, \dot{y}_2, \dot{z}_2)$ are given by

$$[\dot{x}_1 \quad \dot{x}_2]^T = R \cdot CSX \cdot V \tag{3.12}$$

$$[\dot{y}_1 \quad \dot{y}_2]^T = R \cdot CSY \cdot V \tag{3.13}$$

$$[\dot{z}_1 \quad \dot{z}_2]^T = R \cdot CSZ \cdot V \tag{3.14}$$

where

$$V = \left[\dot{\theta}_1 \quad \dot{\theta}_2 \quad \dot{\phi}_1 \quad \dot{\phi}_2\right]^T_{1\times 4}$$

$$CSX = \left[\begin{array}{cc|cc} \cos\phi_1\cos\theta_1 & 0 & -\sin\phi_1\sin\theta_1 & 0 \\ 0 & \cos\phi_2\cos\theta_2 & 0 & -\sin\phi_2\sin\theta_2 \end{array}\right]_{2\times 4}$$

$$CSY = \left[\begin{array}{cc|cc} \sin\phi_1\cos\theta_1 & 0 & \cos\phi_1\sin\theta_1 & 0 \\ 0 & \sin\phi_2\cos\theta_2 & 0 & \cos\phi_2\sin\theta_2 \end{array}\right]_{2\times 4}$$

$$CSZ = \left[\begin{array}{cc|cc} \sin\theta_1 & 0 & 0 & 0 \\ 0 & \sin\theta_2 & 0 & 0 \end{array}\right]_{2\times 4}$$

and, $$R = \begin{bmatrix} z_1 & 0 \\ l_1 & z_2 \end{bmatrix}_{2\times 2}$$

Hence, the *velocity components* of the system under consideration are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} z_1(\cos\phi_1 \cos\theta_1\dot{\theta}_1 - \sin\phi_1 \sin\theta_1\dot{\phi}_1) \\ l_1(\cos\phi_1 \cos\theta_1\dot{\theta}_1 - \sin\phi_1 \sin\theta_1\dot{\phi}_1) + z_2(\cos\phi_2 \cos\theta_2\dot{\theta}_2 - \sin\phi_2 \sin\theta_2\dot{\phi}_2) \end{bmatrix} \quad (3.15)$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} z_1(\sin\phi_1 \cos\theta_1\dot{\theta}_1 + \cos\phi_1 \sin\theta_1\dot{\phi}_1) \\ l_1(\sin\phi_1 \cos\theta_1\dot{\theta}_1 + \cos\phi_1 \sin\theta_1\dot{\phi}_1) + z_2(\sin\phi_2 \cos\theta_2\dot{\theta}_2 + \cos\phi_2 \sin\theta_2\dot{\phi}_2) \end{bmatrix} \quad (3.16)$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = - \begin{bmatrix} z_1(\sin\theta_1\dot{\theta}_1) \\ l_1(\sin\theta_1\dot{\theta}_1) + z_2(\sin\theta_2\dot{\theta}_2) \end{bmatrix} \quad (3.17)$$

Potential Energy:

The total potential energy of a system can be calculated by adding up potential energies of all point masses of the system.

$$\begin{aligned} P &= \sum_{p=1}^{P} g m_p z_p \\ &= g z^T \boldsymbol{M_D} [1 \quad \dots \quad 1]^T_{1\times P} \\ &= g[\cos\theta_1 \quad \cos\theta_2 \quad \dots \quad \cos\theta_s] \cdot R^T \cdot \boldsymbol{M_D} \cdot [1 \quad \dots \quad 1]^{\ T}_{1\times P} \\ &= g[\cos\theta_1 \quad \cos\theta_2 \quad \dots \quad \cos\theta_s] \cdot [P_i]_{S\times 1} \end{aligned}$$

$$\therefore \quad P = g \sum_{i=1}^{P} P_i \cos\theta_i \quad (3.18)$$

where, $\boldsymbol{M}_D = \begin{bmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & m_p \end{bmatrix}$

Potential Energy for the double pendulum:

Potential energy for the system of two point masses m_1 and m_2 is given by

$$P = g\sum_{i=1}^{2} P_i \cos\theta_i$$

Here P_i is calculated to be

$$\begin{aligned} P_i &= R^T_{P\times S} \cdot \boldsymbol{M}_{P\times P} \cdot [1 \quad 1]^T_{1\times P} \\ &= \begin{bmatrix} z_1 & l_1 \\ 0 & z_2 \end{bmatrix}_{S\times P} \cdot \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}_{P\times P} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{P\times 1} \\ \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} &= \begin{bmatrix} (m_1 z_1 + m_2 l_1) \\ m_2 z_2 \end{bmatrix} \end{aligned}$$

$$\therefore P = g[P_1 \cos\theta_1 + P_2 \cos\theta_2]$$

Hence the *final form for the potential energy* of a double pendulum with two point masses is

$$\mathbf{P} = g[(m_1 z_1 + m_2 l_1)\cos\theta_1 + m_2 z_2 \cos\theta_2] \tag{3.19}$$

Kinetic Energy:

Adding up kinetic energies of all point masses of a system gives the total kinetic energy of the system. When the mass and the velocity of the p-th point mass are m_p and v_p, respectively, the kinetic energy of the point mass is given as $e_p = (1/2)m_P \left\| v_P^2 \right\|$.

If $\left\|v_P^2\right\|$ is substituted by $(\dot{x}_P^2 + \dot{y}_P^2 + \dot{z}_P^2)$, the total kinetic **K** of a system with *P* mass points can be written as follows:

$K = \frac{1}{2}\sum_{p=1}^{P} m_p(\dot{x}_P^2 + \dot{y}_P^2 + \dot{z}_P^2)$, where p is the number of point masses in the system

$$
\begin{aligned}
K &= \frac{1}{2}(\dot{x}^T \cdot M \cdot \dot{x} + \dot{y}^T \cdot M \cdot \dot{y} + \dot{z}^T \cdot M \cdot \dot{z}) \\
&= \frac{1}{2}(V^T \cdot CSX^T \cdot \mathbf{R}^T \cdot \boldsymbol{M_D} \cdot \mathbf{R} \cdot CSX \cdot V + V^T \cdot CSY^T \cdot \mathbf{R}^T \cdot \boldsymbol{M_D} \cdot \mathbf{R} \cdot CSY \cdot V + \\
&\quad V^T \cdot CSZ^T \cdot \mathbf{R}^T \cdot \boldsymbol{M_D} \cdot \mathbf{R} \cdot CSZ \cdot V) \\
&= \frac{1}{2}(V^T \cdot CSX^T \cdot \mathbf{C} \cdot CSX \cdot V + V^T \cdot CSY^T \cdot \mathbf{C} \cdot CSY \cdot V + V^T \cdot CSZ^T \cdot \mathbf{C} \cdot CSZ \cdot V) \\
&= \frac{1}{2} \cdot V^T \cdot (CSX^T \cdot \mathbf{C} \cdot CSX + CSY^T \cdot \mathbf{C} \cdot CSY + CSZ^T \cdot \mathbf{C} \cdot CSZ) \cdot V
\end{aligned}
$$

where,

$$
\begin{aligned}
\mathbf{C} &= \mathbf{R}^T \cdot \boldsymbol{M_D} \cdot \mathbf{R} \\
\mathbf{C_{ij}} &= [c_{ij}]_{S \times S}
\end{aligned}
\tag{3.20}
$$

Then the *total kinetic energy* can be written in a compact form as:

$$\mathbf{K} = \frac{1}{2} \cdot (V^T \cdot \mathbf{M} \cdot V) \tag{3.21}$$

where

$$V = \left[\dot{\theta}_1 \quad \dot{\theta}_2 \quad \cdots \quad \dot{\theta}_s \middle| \dot{\phi}_1 \quad \dot{\phi}_2 \quad \cdots \quad \dot{\phi}_s\right]^T_{1 \times 2s}$$

and **M** is a $2s \times 2s$ symmetric matrix that can be represented as a 2×2 block matrix

$$\mathbf{M} = \begin{bmatrix} [M_{11}] & [M_{12}] \\ [M_{21}] & [M_{22}] \end{bmatrix} \tag{3.22}$$

where each block matrix M_{ij} (i = 1,2; j = 1,2) is an $s \times s$ whose elements are given by,

$$M_{11}(i,j) = C_{ij}[\cos\theta_i \cos\theta_j \cos(\phi_i - \phi_j) + \sin\theta_i \sin\theta_j]$$
$$M_{12}(i,j) = C_{ij}[\cos\theta_i \sin\theta_j \sin(\phi_i - \phi_j)]$$
$$M_{21}(i,j) = C_{ij}[\sin\theta_i \cos\theta_j \sin(\phi_j - \phi_i)]$$
$$M_{22}(i,j) = C_{ij}[\sin\theta_i \sin\theta_j \cos(\phi_i - \phi_j)]$$

and i & j go from $1 \rightarrow s$ (s = 2 for the double pendulum), where s is the number of segments in the system.

Block matrices $M_{11}(i,j)$ and $M_{22}(i,j)$ are symmetric in the sense that:

$$M_{11}(i,j) = M_{11}(j,i) \quad \textit{and} \quad M_{22}(i,j) = M_{22}(j,i) \tag{3.23}$$

Again, $M_{12}(i,j)$ and $M_{21}(i,j)$ are related as follows:

$$M_{12}(i,j) = M_{21}(j,i) \tag{3.24}$$

Results (3.23) and (3.24) imply that

$M_{IJ}(i,j) = M_{JI}(j,i)$, which makes M *symmetric*, that is,

$$M(i,j) = M(j,i) \tag{3.25}$$

Kinetic Energy for the double pendulum:

Generalised form of Kinetic Energy reads as

$$\mathbf{K} = \frac{1}{2} \cdot (V^T \cdot \mathbf{M} \cdot V)$$

For a system of two pendulum with two point masses

$$V = \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{\phi}_1 & \dot{\phi}_2 \end{bmatrix} \tag{3.26}$$

$$M = \begin{bmatrix} [M_{11}]_{s\times s} & [M_{12}]_{s\times s} \\ [M_{21}]_{s\times s} & [M_{22}]_{s\times s} \end{bmatrix}_{s\times s} \quad \text{where } s = 2 \tag{3.27}$$

$$C_{ij} = R^{T}_{S\times P} \cdot \boldsymbol{M}_{\boldsymbol{D}_{P\times P}} \cdot R_{P\times S}$$

$$= \begin{bmatrix} z_1 & l_1 \\ 0 & z_2 \end{bmatrix} \cdot \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \cdot \begin{bmatrix} z_1 & 0 \\ l_1 & z_2 \end{bmatrix}$$

$$C_{ij} = \begin{bmatrix} (m_1 z_1^2 + m_2 l_1^2) & m_2 l_1 z_2 \\ m_2 l_1 z_2 & m_2 z_2^2 \end{bmatrix}_{s\times s} \tag{3.28}$$

The submatrices M_{11}, M_{12}, M_{21}, and M_{22} are calculated in the following manner:

$$M_{11}(i,j) = C_{ij}\{\cos\theta_i \cos\theta_j \cos(\phi_i - \phi_j) + \sin\theta_i \sin\theta_j\}$$

$$M_{11} = \begin{bmatrix} (m_1 z_1^2 + m_2 l_1^2) & m_2 l_1 z_2 \{\cos\theta_1 \cos\theta_2 \cos(\phi_1 - \phi_2) + \sin\theta_1 \sin\theta_2\} \\ m_2 l_1 z_2 \{\cos\theta_1 \cos\theta_2 \cos(\phi_1 - \phi_2) + \sin\theta_1 \sin\theta_2\} & m_2 z_2^2 \end{bmatrix}_{2\times 2} \tag{3.29}$$

$$M_{12}(i,j) = C_{ij}\{\cos\theta_i \sin\theta_j \sin(\phi_i - \phi_j)\}$$

$$M_{12} = \begin{bmatrix} 0 & m_2 l_1 z_2 \{\cos\theta_1 \sin\theta_2 \sin(\phi_1 - \phi_2)\} \\ m_2 l_1 z_2 \{\cos\theta_2 \sin\theta_1 \sin(\phi_2 - \phi_1)\} & 0 \end{bmatrix}_{2\times 2} \tag{3.30}$$

$$M_{21}(i,j) = C_{ij}\{\cos\theta_j \sin\theta_i \sin(\phi_j - \phi_i)\}$$

$$M_{21} = \begin{bmatrix} 0 & m_2 l_1 z_2 \{\cos\theta_2 \sin\theta_1 \sin(\phi_2 - \phi_1)\} \\ m_2 l_1 z_2 \{\cos\theta_1 \sin\theta_2 \sin(\phi_1 - \phi_2)\} & 0 \end{bmatrix}_{2\times 2} \tag{3.31}$$

$$M_{22}(i,j) = C_{ij}\{\sin\theta_i \sin\theta_j \cos(\phi_i - \phi_j)\}$$

$$M_{22} = \begin{bmatrix} (m_1 z_1^2 + m_2 l_1^2)\sin^2\theta_1 & m_2 l_1 z_2 \{\sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)\} \\ m_2 l_1 z_2 \{\sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1)\} & m_2 z_2^2 \sin^2\theta_2 \end{bmatrix}_{2\times 2} \tag{3.32}$$

Hence, the Kinetic Energy for the double pendulum is given by

$$\therefore \mathbf{K} = \frac{1}{2}\cdot(V^T \cdot \mathbf{M} \cdot V) = \frac{1}{2}\begin{bmatrix}\dot{\theta}_1 & \dot{\theta}_2 & \dot{\phi}_1 & \dot{\phi}_2\end{bmatrix}\cdot\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\cdot\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\phi}_1\\ \dot{\phi}_2\end{bmatrix} \tag{3.33}$$

The Lagrangian:

The Lagrangian is defined as

$$L(X, V) = K(X, V) - P(X)$$

$$L = \frac{1}{2}\cdot(V^T \cdot \mathbf{M} \cdot V) - P(X)$$

The Lagrangian for the double pendulum is given by:

$$L(X, V) = [\frac{1}{2}\cdot(V^T \cdot \mathbf{M} \cdot V)] - g[(m_1 z_1 + m_2 l_1)\cos\theta_1 + m_2 z_2 \cos\theta_2] \tag{3.34}$$

Equations of Motion:

For a conservative system the i^{th} equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial V_i}\right) - \frac{\partial L}{\partial X_i} = 0 \quad \text{where } i = 1.....2s \text{ (} s \text{ is the number of segments in the system)}$$

$$\frac{d}{dt}\left(\frac{\partial (K-P)}{\partial V_i}\right) - \frac{\partial (K-P)}{\partial X_i} = 0$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial V_i}\right) - \frac{\partial K}{\partial X_i} = -\frac{\partial P}{\partial X_i}$$

$$M_{ij}(X)\cdot \dot{V}_i = -\frac{\partial P}{\partial x_i} - \left[\frac{d}{dt}\{M_{ij}(X)\}\cdot V_i - \frac{\partial K}{\partial x_i}\right] \tag{3.35}$$

The i^{th} equation of motion in a matrix-vector format is given by:

$$\left[M_{ij}(X)\right]\cdot\dot{V} = -\nabla\vec{P}(X) - \vec{S}(X,V) \tag{3.36}$$

$$\text{where } \vec{S}(X,V) = \left[\frac{d}{dt}\{M(X)\}\cdot\vec{V} - \nabla\vec{K}\right]$$

$$\left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right]\cdot\begin{pmatrix}\ddot{\theta}\\ -\\ \ddot{\phi}\end{pmatrix} = -\begin{pmatrix}\dfrac{\partial P}{\partial\theta_s}\\ ---\\ \dfrac{\partial P}{\partial\phi_s}\end{pmatrix} - \begin{pmatrix}S_\theta(i)\\ ---\\ S_\phi(i)\end{pmatrix} \tag{3.37}$$

Hence, the general form for the equations of motion is as follows:

$$M_{11}\ddot{\theta} + M_{12}\ddot{\phi} = \nabla_\theta P - S_\theta \tag{3.38}$$

$$M_{21}\ddot{\theta} + M_{22}\ddot{\phi} = \nabla_\phi P - S_\phi \tag{3.39}$$

$$\text{where } \begin{cases}\theta = (\theta_1, \theta_2, \ldots\ldots\ldots, \theta_S)\\ \phi = (\phi_1, \phi_2, \ldots\ldots\ldots, \phi_S)\end{cases}$$

A system of two pendulums with four degrees of freedom is governed by the four equations of motion as shown below:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = 0 \tag{3.40}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) - \frac{\partial L}{\partial\theta_2} = 0 \tag{3.41}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}_1}\right) - \frac{\partial L}{\partial\dot{\phi}_1} = 0 \tag{3.42}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}_2}\right) - \frac{\partial L}{\partial\dot{\phi}_2} = 0 \tag{3.43}$$

Hence, the form of the *equations of motion for the double pendulum* system is as follows:

$$M_{11}\ddot{\theta} + M_{12}\ddot{\phi} = \nabla_{\theta}P - S_{\theta} \tag{3.44}$$

$$M_{21}\ddot{\theta} + M_{22}\ddot{\phi} = \nabla_{\phi}P - S_{\phi} \tag{3.45}$$

where $\begin{cases}\theta = (\theta_1, \theta_2) \\ \phi = (\phi_1, \phi_2)\end{cases}$

The matrix-vector format is given by:

$$\left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] \cdot \begin{pmatrix} \ddot{\theta} \\ -- \\ \ddot{\phi} \end{pmatrix} = -\begin{pmatrix} \frac{\partial P}{\partial \theta} \\ ----- \\ \frac{\partial P}{\partial \phi} \end{pmatrix} - \begin{pmatrix} S_{\theta} \\ ---- \\ S_{\phi} \end{pmatrix} \tag{3.46}$$

where

$$\begin{pmatrix} \frac{\partial P}{\partial \theta_1} \\ \frac{\partial P}{\partial \theta_2} \\ \frac{\partial P}{\partial \phi_1} \\ \frac{\partial P}{\partial \phi_2} \end{pmatrix} = \begin{pmatrix} -g(m_1 z_1 + m_2 l_1)\sin\theta_1 \\ -g m_2 z_2 \sin\theta_2 \\ 0 \\ 0 \end{pmatrix} \tag{3.47}$$

$$S_{\theta_1} = \begin{bmatrix} m_2 l_1 z_2 \dot{\theta}_2^2 \{\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)\} - (m_1 z_1^2 + m_2 l_1^2)\dot{\phi}_1^2 \{\sin\theta_1 \cos\theta_1\} \\ -m_2 l_1 z_2 \dot{\phi}_2^2 \{\cos\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)\} + 2m_2 l_1 z_2 \dot{\theta}_2 \dot{\phi}_2 \{\cos\theta_1 \cos\theta_2 \sin(\phi_1 - \phi_2)\} \end{bmatrix} \tag{3.48}$$

$$S_{\theta_2} = \begin{bmatrix} m_2 l_1 z_2 \dot{\theta}_1^2 \{\cos\theta_1 \sin\theta_2 - \cos\theta_2 \sin\theta_1 \cos(\phi_1 - \phi_2)\} - m_2 l_1 z_2 \dot{\phi}_1^2 \{\sin\theta_1 \cos\theta_2 \cos(\phi_1 - \phi_2)\} \\ -m_2 z_2^2 \dot{\phi}_2^2 \{\sin\theta_2 \cos\theta_2\} - 2m_2 l_1 z_2 \dot{\theta}_1 \dot{\phi}_1 \{\cos\theta_1 \cos\theta_2 \sin(\phi_1 - \phi_2)\} \end{bmatrix} \tag{3.49}$$

$$S_{\phi_1} = \begin{bmatrix} m_2 l_1 z_2 \dot{\phi}_2^2 \{\sin\theta_1 \sin\theta_2 \sin(\phi_1 - \phi_2)\} - m_2 l_1 z_2 \dot{\theta}_2^2 \{\sin\theta_1 \sin\theta_2 \sin(\phi_2 - \phi_1)\} \\ +(m_1 z_1^2 + m_2 l_1^2)\dot{\theta}_1 \dot{\phi}_1 \sin 2\theta_1 + 2m_2 l_1 z_2 \dot{\theta}_2 \dot{\phi}_2 \{\sin\theta_1 \cos\theta_2 \cos(\phi_1 - \phi_2)\} \end{bmatrix} \quad (3.50)$$

$$S_{\phi_2} = \begin{bmatrix} -m_2 l_1 z_2 \dot{\phi}_1^2 \{\sin\theta_1 \sin\theta_2 \sin(\phi_1 - \phi_2)\} - m_2 l_1 z_2 \dot{\theta}_1^2 \{\sin\theta_1 \sin\theta_2 \sin(\phi_1 - \phi_2)\} \\ +m_2 z_2^2 \dot{\theta}_2 \dot{\phi}_2 \{\sin 2\theta_2\} + 2m_2 l_1 z_2 \dot{\theta}_1 \dot{\phi}_1 \{\cos\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)\} \end{bmatrix} \quad (3.51)$$

3.2 Equations of Motion for a 3D Single Pendulum Derived as a Check for the 3D Double Pendulum Equations

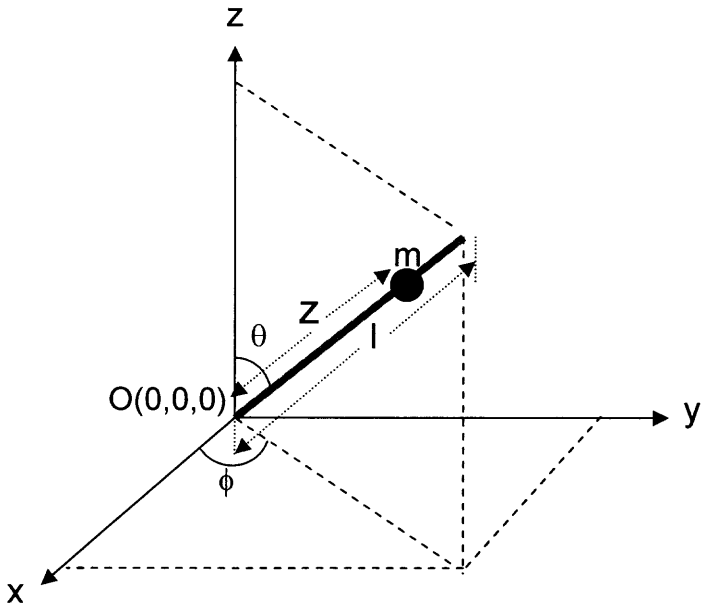


Figure 3.2 Single pendulum in a cartesian coordinate system.

Figure 3.2 shows a single pendulum with *length l* and a *mass m* in a *cartesian coordinate system*. The mass 'm' located at a distance 'z' from the origin, has coordinates (x,y,z). The pendulum makes an angle (theta) θ with the positive z – axis. Its projection onto the XY plane makes an angle (phi) ϕ with the positive direction of x – axis.

The above system has *two degress of freedom* given by the two angles: theta and phi. At any given point in time, the position of the pendulum can be determined if the angles θ and ϕ are specified.

The equations of motion governing the system in 3D can be determined using the Lagrangian method.

Relation Matrix:

The relation matrix **R** shows the relationship of connection between segments of the system, where each row of **R** represents a point mass of the system, and has the information about the path from the origin of the system to the point mass. Each column represents a segment, and has the information about the usage of the segment for every path tp point masses. For this particular system, with *one segment* and *one point mass*, the relation matrix **R** has the form:

$$\mathbf{R} = [\,\mathbf{z}\,] \tag{3.52}$$

After determining the relation matrix, the x- and y- and z- coordinates of a point mass can be written as follows:

$$. = R \cdot [\cos\phi \sin\theta] \tag{3.53}$$

$$y = R \cdot [\sin\phi \sin\theta] \tag{3.54}$$

$$z = R \cdot [\cos\theta] \tag{3.55}$$

From Equation (3.53), (3.54) and (3.55), the x- and y- and z- components of velocity can be determined as follows:

$$\dot{x} = R \cdot [\cos\phi\cos\theta \quad -\sin\phi\sin\theta] \cdot \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix} \tag{3.56}$$

$$\dot{y} = R \cdot [\sin\phi\cos\theta \quad \cos\phi\sin\theta] \cdot \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix} \tag{3.57}$$

$$\dot{z} = R \cdot [-\sin\theta] \cdot [\dot{\theta}] \tag{3.58}$$

Potential Energy:

The potential energy for the system is given by:

$$P = mgz$$

$$P = mgR \cdot [\cos\theta]$$

$$P = mgz \cdot [\cos\theta] \tag{3.59}$$

Kinetic Energy:

The total kinetic is given as $K = [\frac{1}{2} \cdot m \cdot v^2]$ .The velocity term, v^2 , is determined by adding up the velocities in x-, y- and z- directions.

$$K = \frac{1}{2} m v^2$$

$$= \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$K = \frac{1}{2} \cdot (\dot{\theta}^T \cdot M_{11} \cdot \dot{\theta} + \dot{\theta}^T \cdot M_{12} \cdot \dot{\phi} + \dot{\phi}^T \cdot M_{21} \cdot \dot{\theta} + \dot{\phi}^T \cdot M_{22} \cdot \dot{\phi})$$

$$K = \frac{1}{2} \cdot \begin{bmatrix} \dot{\theta} & \dot{\phi} \end{bmatrix} \cdot \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] \cdot \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix} \tag{3.60}$$

where, $\mathbf{M}_{11}$, $\mathbf{M}_{12}$, $\mathbf{M}_{21}$ and $\mathbf{M}_{22}$ are block matrices of the Mass matrix **M,** each with dimension $\mathbf{s} \times \mathbf{s}$. Here, *s* is the *number of segments* in the system, which in this case is one. Hence, the mass matrix **M** is a 2×2 matrix.

Again, each block matrix is given by the following equations:

$$M_{11}(i,j) = C_{ij} \cdot [\cos\theta_i \cos\theta_j \cos(\phi_i - \phi_j) + \sin\theta_i \sin\theta_j]$$

$$M_{12}(i,j) = C_{ij} \cdot [\cos\theta_i \sin\theta_j \sin(\phi_i - \phi_j)]$$

$$M_{21}(i,j) = C_{ij} \cdot [\cos\theta_j \sin\theta_i \sin(\phi_j - \phi_i)]$$

$$M_{22}(i,j) = C_{ij} \cdot [\sin\theta_i \sin\theta_j \cos(\phi_i - \phi_j)]$$

For this particular case, i and j are one.

$$C_{ij} = R^T \cdot \boldsymbol{M_D} \cdot R$$

Let us define a matrix $\boldsymbol{M_D}$, which is a $\mathbf{p} \times \mathbf{p}$, where *p* is the *number of point masses* in the system.

Hence $\boldsymbol{M_D}$ for this example is given by:

$$\boldsymbol{M_D} = [m] \tag{3.61}$$

$$\text{Hence, } C = mz^2 \tag{3.62}$$

$$\therefore \quad M_{11} = mz^2$$
$$M_{12} = M_{21} = 0$$
$$M_{22} = mz^2 \cdot [\sin^2\theta]$$

The expression for the Kinetic energy is given by:

$$K = \frac{1}{2}[\dot{\theta} \quad \dot{\phi}] \cdot \begin{bmatrix} mz^2 & 0 \\ 0 & mz^2 \sin^2\theta \end{bmatrix} \cdot \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix} \tag{3.63}$$

The Lagrangian is given by: $L = K - P$

The equations of motion are given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial V}\right) - \left(\frac{\partial L}{\partial X}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial V}\right) - \frac{\partial K}{\partial X} = -\frac{\partial P}{\partial X}$$

Hence, the equation of motion in a vector – matrix format is given by:

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \cdot \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = -\begin{bmatrix} \frac{\partial P}{\partial \theta} \\ \frac{\partial P}{\partial \phi} \end{bmatrix} - \begin{bmatrix} S_\theta \\ S_\phi \end{bmatrix} \tag{3.64}$$

The equations of motion for a 3D single pendulum are given by:

$$\begin{bmatrix} mz^2 & 0 \\ 0 & mz^2 \sin^2\theta \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = -\begin{bmatrix} -mgz\sin\theta \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \cdot mz^2 \cdot \dot{\phi}^2 \cdot (\sin 2\theta) \\ mz^2 \cdot \dot{\theta}\dot{\phi} \cdot (\sin 2\theta) \end{bmatrix} \tag{3.65}$$

CHAPTER 4

NUMERICAL METHODS FOR SOLVING THE EQUATIONS OF MOTION

Given the equations of motion, one needs to solve them numerically to solve for motion problems. For purposes of this dissertation, two methods are discussed; the Initial Value Problem and the two point Boundary Value Problem.

4.1 Initial Value Problem

The input to an initial value problem (IVP) is the system's initial position and initial velocity. That is, we are given

$$X(t_0) = X_0$$
$$V(t_0) = V_0$$

Given these two side conditions it is possible to solve the dynamical equations for the motion X(t). These dynamical equations of motion take on a general form $M \cdot A = \mathcal{F}$,i.e,

$$M(X) \cdot \dot{V} = \mathcal{F}(X, V) = \mathcal{F} \tag{4.1}$$

The generalized mass 'M' is a function of position only, whereas the force term $\mathcal{F}$ is a function of both position *and* velocity. Since X_0 and V_0 are known at time t = 0, we could solve for $\dot{V}_0$ (acceleration) as follows,

$$M(X_0) \cdot \dot{V}_0 = \mathcal{F}(X_0, V_0)$$

$$\therefore \dot{V}_0 = M(X_0)^{-1} \cdot \mathcal{F}(X_0, V_0) * \tag{4.2}$$

*Numerically, the inverse of $M(X_0)$ is not calculated, since it is inefficient to do so. In practice, an 'LU Decomposition' of M is used to solve for $\dot{V}_o$. See Section (4.2) for explanation.

To obtain the position (X) and velocity (V) at the next time step ' Δt ', *Euler's Method* can be used as shown below,

$$X(t + \Delta t) = X(t) + V(t) \cdot \Delta t \tag{4.3}$$

$$V(t + \Delta t) = V(t) + \dot{V}(t) \cdot \Delta t \tag{4.4}$$

where $\dot{V}(t)$ is the rate of change of velocity at time 't' (acceleration) and

V(t) is the velocity at time 't'

Hence, the position and velocity for the next time step is obtained, which can be used to solve for the acceleration $\dot{V}$ at that time step. Iterations over Equations (4.2), (4.3) and (4.4), solves for an approximation of the motion X(t) (as well as V(t) and $\dot{V}(t)$)at discrete time steps of Δt .This approximation becomes more accurate as the time step is decreased. It should be sufficiently small to give accurate results. But here there is a trade –off between accuracy and computational iterations. As Δt gets smaller, the number of iterations increases which effectively reduces computation speed.

Often a higher order accuracy method such as the *Runge Kutta* method is used to iterate over time't'. The advantage here is that the time step ' Δt ' can be made large, reducing the number of iterations involved in the computation.

4.2 Solving the Ax = b Problem

The matrix shorthand Ax = b, is a common Linear Algebra problem which solves for vector 'x' given matrix 'A' and vector 'b'. One way to solve this problem could be to solve for the inverse of 'A' and multiply it by 'b' to get $\vec{x}$, i.e,

$$x = (A^{-1}).b$$

But as stated in Section (4.1), this method is not very efficient, especially if 'N' (the number of unknowns) is very large. Instead of finding the inverse, what is done numerically is called the 'LU Decomposition' of matrix 'A'.

Gaussian elimination allows the matrix 'A' to be written as a product of two matrices,

$$L \cdot U = A \tag{4.5}$$

where 'L' is the *lower triangular* matrix (has elements only on the diagonal and below) and 'U' is upper triangular (has elements only on the diagonal and above). For the case of a 4×4 matrix 'A', for example, equation (4.5) would look like:

$$\begin{bmatrix} \alpha_{11} & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{22} & 0 & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix} \cdot \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ 0 & \beta_{22} & \beta_{23} & \beta_{24} \\ 0 & 0 & \beta_{33} & \beta_{34} \\ 0 & 0 & 0 & \beta_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \tag{4.6}$$

A decomposition such as (4.5) can be used to solve the Linear set

$$A \cdot x = (L \cdot U) \cdot x = L \cdot (U \cdot x) = b \tag{4.7}$$

by first solving for the vector y such that

$$L \cdot y = b \tag{4.8}$$

and then solving the upper triangular system

$$U \cdot x = y \tag{4.9}$$

for the desired unknown x.

The advantage of breaking up one linear set into two successive sets is that the solution of a triangular set of equations is quite trivial.

Thus, Equation (4.8) can be solved by *forward substitution*

$$\downarrow \begin{pmatrix} \alpha_{11} & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{N1} & \alpha_{N2} & \cdots & \alpha_{NN} \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} \tag{4.10}$$

while Equation (4.9) can be solved by *back substitution*

$$\uparrow \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1N} \\ 0 & \beta_{22} & \cdots & \beta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_{NN} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} \tag{4.11}$$

Equation (4.11) gives a solution for the vector $\vec{x}$.

Going back to Equation (4.1) we have

$$M(X_0) \cdot \dot{V}_0 = \mathcal{F}(X_0, V_0)$$

This is also an Ax = b problem where

$$A = M(X)$$

$$b = \mathcal{F}(X, V)$$

and $x = \dot{V}$

From Equation (4.5) we have that

$$L \cdot U = A$$

i.e. $M(X) = L \cdot U$ (4.12)

Since M(x) is a symmetric matrix,

$$L = U^T \quad (4.13)$$

or $U = L^T$ (4.14)

From Equations (4.12), (4.13) and (4.14),

$$M(X) = L \cdot L^T \quad (4.15)$$

$$M(X) \cdot \dot{V} = \mathcal{F}$$

$$L \cdot L^T \cdot \dot{V} = \mathcal{F} \quad (4.16)$$

Let $L^T \cdot \dot{V} = \vec{C}$ (4.17)

$$\therefore L \cdot C = \mathcal{F} \quad (4.18)$$

Hence, Equation (4.18) can be solved for 'C' by '*forward substitution.*'

Again, by Equation (4.17)

$C = L^T \cdot \dot{V}$, we solve for $\dot{V}$ by '*back substitution.*'

4.3 Two point Boundary Value Problem

Unlike an initial value problem, the input to a two point boundary value problem (BVP) is the desired initial (X_0) and target (X_T) configuration of the system, and a time period (T). Given this, the method searches for an initial velocity that can take the system from

the initial configuration to the target configuration in time 'T.' In the shooting method for solving the BVP, we start by taking a guess at this initial velocity and solve an Initial Value Problem to arrive at a configuration X(T). The guess may or may not take the system to the target configuration. If it is not the right initial velocity, the solution might overshoot or undershoot the desired configuration. Both cases will generate an 'error' by which the solution misses the target (X_T).

The error is given by

$$\vec{e}(V_0) = \vec{X}(T) - \vec{X}_T \tag{4.19}$$

where $\vec{V}_0$ = initial guess

$\vec{X}(T)$ = Configuration of the system for guess $\vec{V}_0$

$\vec{X}_T$ = Target Configuration

This error is used to guess at a *new velocity* by using *Newton's approach* as shown below

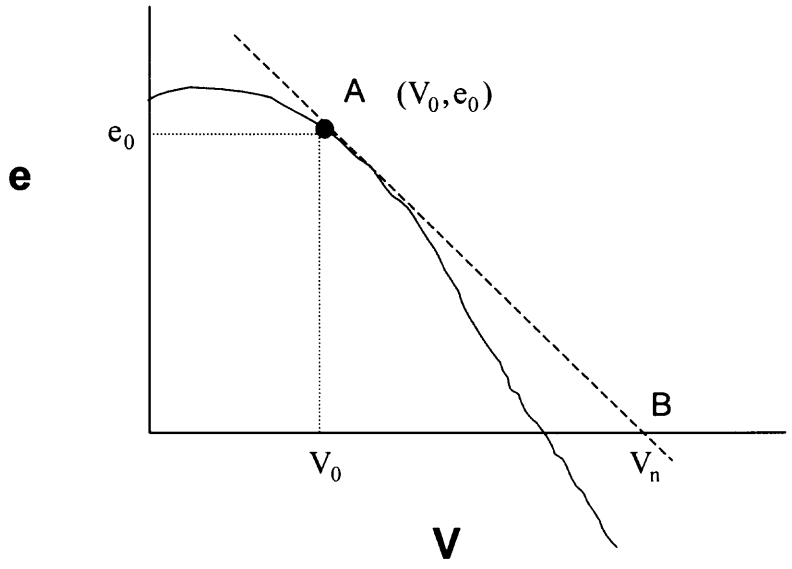


Figure 4.1 Error-velocity curve.

In Figure 4.1, e_0 is the error for the old velocity (V_0) guess. Using Newton's approach, a straight line is drawn at point A. This line intersects the x-axis at point B which corresponds to velocity V_n which represents the next initial velocity guess. For this velocity, the error would come out to be zero if $e(\vec{V}_0)$ were linear (which it is not) . This fact enables us to find V_n and use it as our next guess.

The equation for the straight line is given by

$$\left.\frac{de}{dv}\right|_{V_0} = \frac{0-\vec{e}(V_0)}{V_n - V_0} \tag{4.20}$$

$$\text{Let } \vec{V}_n - \vec{V}_0 = \delta\vec{V} \tag{4.21}$$

$$\therefore -\vec{e}(V_0) = \left.\frac{de}{dV}\right|_{\vec{V}_0} \cdot \delta\vec{V} \tag{4.22}$$

Equation (4.22) is also an Ax = b problem (see Section 4.2), where we are solving for $\delta\vec{V}$.

$\left.\frac{de}{dV}\right|_{\vec{V}_0}$ is the Jacobian Matrix of $N \times N$ or $2s \times 2s$ dimensions, where 'N' is the number of degrees of freedom and '*s*' is the number of segments of the system. The elements of the Jacobian are given by

$$J_{ij} = \left.\frac{\partial e_i}{\partial V_j}\right|_{V_0} \quad \text{where i \& j go from 1 } \rightarrow \text{ N} \tag{4.23}$$

In Equation (4.23), $\frac{\partial e_i}{\partial V_j}$ is the rate of change of error when $\vec{V}_0$ is changed by a small amount Δv . Once the Jacobian (see Section 4.3.1) is constructed, it is used to solve for $\delta\vec{V}$.

$$\delta\vec{V} = \vec{V}_n - \vec{V}_0$$

$$\therefore V_n = \delta\vec{V} + \vec{V}_0 \tag{4.24}$$

Equation (4.24) solves for V_n, the new guess for velocity. In general, for every new velocity guess, 2s+1 initial value problems are solved. One IVP for the initial guess and 2s IVP's for constructing the Jacobian matrix.

4.3.1 Solving for the Jacobian

The elements of the Jacobian matrix give the rate of change of error for a small change Δv in each independent coordinate direction. If there are 'N' unknowns then the Jacobian is a $N \times N$ matrix.

Let the number of unknowns be 2. Hence N = 2.

$$\vec{V} = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

Let the initial guess for the velocity be given by

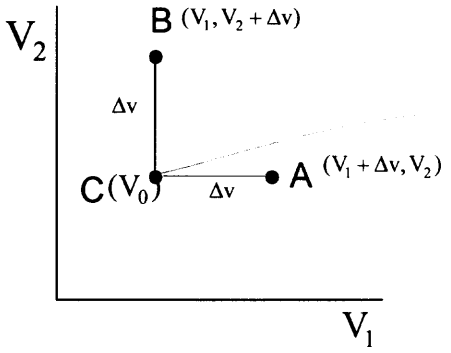
$$\vec{V}_0 = \begin{bmatrix} V_{0_1} & V_{0_2} \end{bmatrix} \tag{4.25}$$

An IVP is solved to arrive at configuration $X_0(T)$. The error is given by

$$\vec{e}(V_0) = \vec{X}_0(T) - \vec{X}_T \tag{4.26}$$

By Newton's approach,

$$-\vec{e}(V_0) = \delta\vec{V} \cdot \left.\frac{de}{dv}\right|_{\vec{V}_0} \tag{4.27}$$



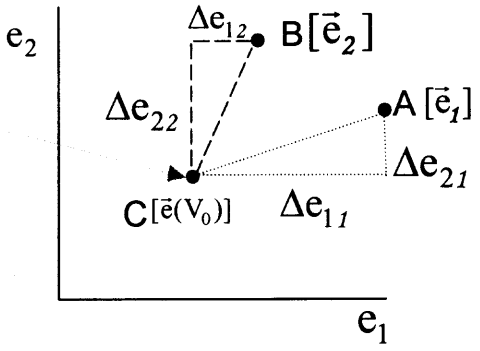


Figure 4.2 Independent velocity directions.

Figure 4.3 Error for the velocity.

In Figure 4.2 and Figure 4.3, point C represents the initial velocity guess and the resulting error, respectively. Let us move Δv in direction V_1, then the coordinates for point A are given by $(V_1 + \Delta v, V_2)$ (Figure 4.2). An IVP is solved to arrive at error $[\vec{e}_1]$ as shown in Figure 4.3(point A). Change in error in direction V_1 is given by

$$\Delta e_1 = \vec{e}_0 - \vec{e}_1$$

$$\Delta e_1 = \begin{bmatrix} \Delta e_{11} \\ \Delta e_{21} \end{bmatrix} \tag{4.28}$$

Similarly, let us move Δv in direction V_2, then the coordinates for point B are given by $(V_1, V_2 + \Delta v)$ (Figure 4.2). Again, an IVP is solved to arrive at error $[\vec{e}_2]$ as shown in Figure 4.3(point B). Change in error in direction V_2 is given by

$$\Delta e_2 = \vec{e}_0 - \vec{e}_2$$

$$\Delta e_2 = \begin{bmatrix} \Delta e_{12} \\ \Delta e_{22} \end{bmatrix} \tag{4.29}$$

Hence, the Jacobian, which in this case will be a 2×2 matrix, is given by

$$J_{ij} = \begin{bmatrix} \frac{\Delta e_{11}}{\Delta v_1} & \frac{\Delta e_{12}}{\Delta v_2} \\ \frac{\Delta e_{21}}{\Delta v_1} & \frac{\Delta e_{22}}{\Delta v_2} \end{bmatrix} \tag{4.30}$$

4.4 Equations of Motion with Constraints

Motion for many systems in practice is constrained. This section, discusses the solution of equations of motion with constraints. A constraint can often be written in the form

$$g(X) = 0$$

$$\text{or} \qquad G(X) = 0 \tag{4.31}$$

where X is the position variable

If there are 'k' such constraint equations, G is a $k\times1$ vector

$$G = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_k \end{pmatrix}_{k\times1} \tag{4.32}$$

Again, $G(X) = 0$

i.e. $G(X(t)) = 0$

Applying the chain rule, we get

$$\nabla G \cdot V(t) = 0 \tag{4.33}$$

i.e. $$\sum_i \frac{\partial G}{\partial X_i} \cdot V_i = \nabla G \cdot V = 0 \tag{4.34}$$

Again, applying the product rule, we get

$$\frac{d}{dt}(\nabla G \cdot V) = 0$$

$$\left(\nabla G \cdot \dot{V}\right) + \frac{d}{dt}\left(\frac{\nabla G(X)}{\partial X_i}\right) \cdot V = 0 \tag{4.35}$$

i.e. $$\left(\nabla G \cdot \dot{V}\right) + V^T \cdot H \cdot V = 0$$

or, $$\left(\nabla G \cdot \dot{V}\right) = -V^T \cdot H \cdot V \tag{4.36}$$

Hence, the *equations of motion with constraints* can be written as

$$M\ddot{X} - \nabla G^T \cdot \lambda = \mathcal{F} \tag{4.37}$$

and $$\nabla G \cdot \ddot{X} = -V^T \cdot \mathbf{H} \cdot V \tag{4.38}$$

Equation $M\ddot{X} - \nabla G^T \cdot \lambda = \mathcal{F}$ can be written as

$$M\ddot{X} = \mathcal{F} + \nabla G^T \cdot \lambda \tag{4.39}$$

$(\nabla G^T \cdot \lambda)$ can be thought of as *constraint forces*. Just like the unconstrained case, the equations of motion are solved numerically as an Ax = b problem. For the constrained case, besides solving for the acceleration, the equations of motion also solve for $\lambda(t)$, the Lagrange multiplier at each time step.

For an 'N' degree of freedom system and 'k' constraints, the matrix vector format for the equations of motion read as follows

$$G(X) = 0$$

$$\left[\begin{array}{c|c} M^{*}_{N\times N} & -(\nabla G)^{T}_{N\times k} \\ \hline \nabla G_{k\times N} & 0 \end{array}\right]_{(N+k)\times(N+k)} \left[\begin{array}{c} \ddot{X}_{N\times 1} \\ \hline \lambda_{k\times 1} \end{array}\right]_{(N+k)\times 1} = \left[\begin{array}{c} F_{N\times 1} \\ \hline (-V^{T}\cdot \mathbf{H}\cdot V)_{k\times 1} \end{array}\right]_{(N+k)\times 1} \tag{4.40}$$

where

$$\nabla_i G = \frac{\partial G}{\partial X_i}, \quad \mathbf{H} = \frac{\partial^2 G}{\partial X_i \partial X_j}$$

CHAPTER 5

CONCLUSIONS

A 3 dimensional double pendulum system with four degrees of freedom has been developed in this thesis, which can be solved numerically using the 2 point boundary value method. The equations of motion with constraints have also been discussed for applications that have a constrained motion. A computer code to solve the equations of motion for a double pendulum in 3D has been started. The code uses the numerical methods outlined in this thesis to solve the Ax = b problem. Once the simulation is set up, this model can be used to imitate the shoulder joint, where the clavicle and humerus are modeled as a coupled pendulum in 3 dimensions. But unlike this model, which has unrestricted movement, the human shoulder has physiological constraints which restrict its movement. The joint structure of the shoulder plus the tendons and muscles that insert into it restrict freedom of movement of the joint. Hence, for practical solutions to the equations, we need to develop constraint equations that are the natural constraints present in the shoulder. Future work would include developing these constraints and using them to solve the equations of motion.

Once a working model has been developed, it can be used to solve for human motion problems and provide an insight into new movement patterns that exploit momentum and gravity so that they are more efficient. These more efficient movement patterns are also less likely to produce shoulder injury because they would avoid energy dissipating collisions with the shoulder constraint surface. This might be particularly

important in sport applications such as in baseball throwing and pitching where high velocities are used and shoulder injury is common.

The mathematical model developed is general enough to be extended to solve for a system of 'N' coupled pendulums. It can be used to model other segments of the human body. Future research would also include modeling joint rotation, which is an important and interesting property of the joints. The research undertaken in this thesis has a lot of future potential in solving human motion problems.

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