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Energy methods for reaction-diffusion problems

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ABSTRACT

ENERGY METHODS FOR REACTION-DIFFUSION PROBLEMS

by
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Nonlinear reaction-diffusion equations arise in many areas of applied sciences such as combustion modeling, population dynamics, chemical kinetics, etc. A fundamental problem in the studies of these equations is to understand the long time behavior of solutions of the associated Cauchy problem. These kinds of questions were originally studied in the context of combustion modeling.

For suitable nonlinearity and a monotone increasing one-parameter family of initial data starting with zero data, small values of the parameter lead to extinction, whereas large values of the parameter may lead to spreading, i.e., the solution converging locally uniformly to a positive spatially independent stable steady state. A natural question is the existence of the threshold set of the parameters for which neither extinction nor spreading occurs. Even in one space dimension, this long standing question concerning threshold phenomena is far from trivial.

Recent results show that if the initial data are compactly supported, then there exists a sharp transition between extinction and spreading, i.e., the threshold set contains only one point. However, these results rely in an essential way on compactly supported initial data assumption and only give limited information about the solutions when spreading occurs.

In this dissertation, energy methods based on the gradient flow structure of reaction-diffusion equations are developed. The long time behavior of solutions of the Cauchy problem for nonlinear reaction-diffusion equations in one space dimension with the nonlinearity of bistable, ignition or monostable type is analyzed. For symmetric decreasing initial data in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the convergence results for the

considered equations are studied, and the existence of a one-to-one relation between the long time behavior of the solution and the limit value of its energy is proved.

In addition, by employing a weighted energy functional, a mathematical description of the equivalence between spreading and propagation of the solutions of the considered equations is given. More precisely, if spreading occurs, then the leading and the trailing edge of the solution propagate faster than some constant speed. Therefore, if the solution spreads, it also propagates. Furthermore, for a monotone family of symmetric decreasing initial data, there exists a sharp threshold between propagation and extinction.

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To my family.

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INTRODUCTION

1.1 Reaction-diffusion Equations

Reaction-diffusion equations arise as models in many different contexts such as combustion modeling [8,46,47], population dynamics [38], and modeling of other physical and chemical systems [24,33,34]. A typical reaction-diffusion problem can be written as

$$u_t(\vec{x},t) = \Delta u(\vec{x},t) + f(u(\vec{x},t),\vec{x},t), \quad \vec{x} \in \Omega \subseteq \mathbb{R}^n, \ t > 0, \tag{1.1}$$

with boundary conditions, for example, the Dirichlet boundary condition

$$u(\vec{x},t) = 0, \quad \vec{x} \in \partial\Omega, \ t > 0, \tag{1.2}$$

or the Neumann boundary condition

$$\nabla u(\vec{x},t) \cdot \vec{n}(\vec{x}) = 0, \quad \vec{x} \in \partial\Omega, \ \ t > 0, \tag{1.3}$$

and initial condition

$$u(\vec{x},0) = \phi(\vec{x}), \quad \vec{x} \in \Omega. \tag{1.4}$$

The corresponding stationary equation is as follows (assuming f does not explicitly depend on t)

$$\Delta u(\vec{x}) + f(u(\vec{x}),\vec{x}) = 0, \quad \vec{x} \in \Omega \subset \mathbb{R}^n, \tag{1.5}$$

with the same boundary condition as for equation (1.1). The solutions of equation (1.5) are time-independent solutions of equation (1.1) and are also called stationary solutions or steady states.

To understand how to obtain equation (1.1) from physical models, a problem of population dynamics is considered. To study the behavior of the population of a given species in region Ω, for any point $\vec{x} \in \Omega$ and time $t \geq 0$, the population density $p(\vec{x}, t)$ needs to be defined. In the study of particle systems, if the number of particles is large, then a stochastic particle system can be reduced to a deterministic reaction-diffusion equation [2, 9, 40]. The population density $p(\vec{x}, t)$ may be defined as follows:

$$p(\vec{x}, t) = \lim_{N \to \infty} \frac{\text{number of alive organisms in } B(\vec{x}, r_N) \text{ at time } t}{N|B(\vec{x}, r_N)|}, \tag{1.6}$$

where $B(\vec{x}, r)$ is the n-dimensional ball with center $\vec{x}$ and radius r, $|B(\vec{x}, r)|$ is the volume of $B(\vec{x}, r)$, N is the total number of organisms in the unit ball $B(\vec{x}, 1)$, and r_N is chosen as $r_N = N^{-\frac{1}{2n}}$. As $N \to \infty$, $B(\vec{x}, r_N)$ is macroscopically small, because $B(\vec{x}, r_N) \sim N^{-\frac{1}{2}} \to 0$ whereas if the distribution of organisms is sufficiently smooth in $B(\vec{x}, 1)$, then the number of alive organisms in $B(\vec{x}, r_N)$ has order $N|(B(\vec{x}, r_N))| \sim N^{\frac{1}{2}} \to \infty$, which means $B(\vec{x}, r)$ is also microscopically large. Therefore, (1.6) gives a proper definition of population density.

Once population density is defined, a rigorous derivation of equation (1.1) may be performed based on an underlying stochastic mechanism. Here, instead, a phenomenological derivation of the reaction-diffusion equation is given. The flux $\vec{J}(\vec{x}, t)$ of the population density is assumed to satisfy the *Fick's law*, i.e., it is proportional to the gradient of the population density. More precisely,

$$\vec{J}(\vec{x}, t) = -D \nabla p(\vec{x}, t), \tag{1.7}$$

where $D = D(\vec{x})$ is called the diffusion coefficient. The population of organisms also changes by birth and death. By analogy with chemical problems, the rate f of change of the population density due to birth and death may be called the *reaction rate*. In general, f depends on location $\vec{x}$, time t, and $p(\vec{x}, t)$ itself, i.e., $f = f(p(\vec{x}, t), \vec{x}, t)$.

By the conservation law, for any region $\Lambda \in \Omega$, the following relation holds:

$$\frac{d}{dt}\int_{\Lambda} p(\vec{x},t)d\vec{x} = -\int_{\partial\Lambda} \vec{J}(\vec{x},t)\cdot\vec{n}(\vec{x})dS + \int_{\Lambda} f(p(\vec{x},t),\vec{x},t)d\vec{x}. \tag{1.8}$$

From the divergence theorem, and since the region Λ is arbitrary, the following partial differential equation, therefore, holds:

$$\frac{\partial}{\partial t}p(\vec{x},t) = \nabla\cdot(D(\vec{x})\nabla p(\vec{x},t)) + f(p(\vec{x},t),\vec{x},t), \tag{1.9}$$

with boundary and initial conditions above. If the diffusion coefficient is independent of $\vec{x}$, i.e., $D(\vec{x}) \equiv D$, then the equation (1.9) can be simplified to

$$\frac{\partial}{\partial t}p(\vec{x},t) = D\Delta p(\vec{x},t)) + f(p(\vec{x},t),\vec{x},t). \tag{1.10}$$

Moreover, the coefficient D can be chosen as $D = 1$ by rescaling in $\vec{x}$, so that (1.1) is derived.

Mathematically, the reaction rate $f(p(x,t),x,t)$ is often nonlinear in the variable $p(x,t)$. In many models, e.g. the model of population dynamics mentioned above, the reaction rate only depends on the population density itself, i.e., $f = f(p)$ [38]. Since the population density is nonnegative, the domain of the function f is $[0,\infty)$, i.e., $f : [0,\infty) \to \mathbb{R}$, with a natural assumption $f(0) = 0$ (no reaction when the density is 0). Moveover, most models give f some continuity, so that $f(u)$ is often considered as a locally Lipschitz continuous function, or a C^1 function.

A typical example of such a nonlinearity is when the reaction rate f is the logistic function with respect to the population density p [17]. More precisely

$$f = f(p) = Kp(1-\frac{p}{P}), \tag{1.11}$$

where K is a positive constant, and $P > 0$ is the so-called *carrying capacity*, i.e., the maximum possible population density. After rescaling, equation (1.10) can be simplified to the Fisher-KPP equation

$$u_t = \Delta u + u(1 - u). \tag{1.12}$$

In one space dimensional case, for every wave speed $c \geq c^* = 2$, the Fisher-KPP equation (1.12) admits traveling wave solutions $u(x, t) = \bar{u}(x - ct)$. There is a classical result by Kolmogorov, Petrowskii and Piscunov [25] that the large time behavior of the solution to the Fisher-KPP equation with Heaviside initial data is a traveling wave, see Figure 1.1.

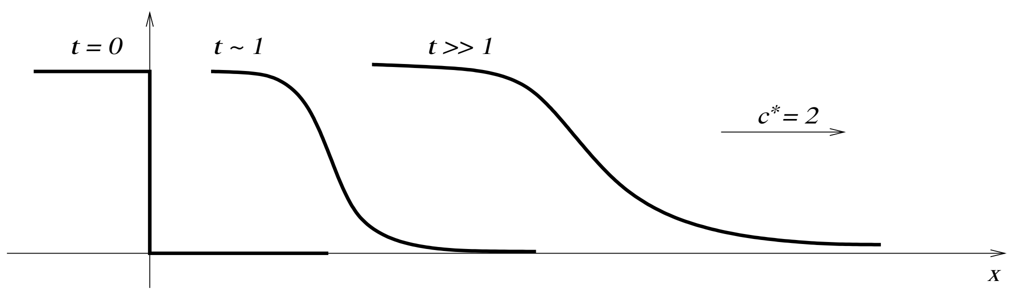

Figure 1.1 Qualitative behavior of solution of the Fisher-KPP equation in one space dimension for the initial data in the form of the Heaviside function. The solution converges in shape to a traveling wave solution with speed $c^* = 2$ as $t \to \infty$.

Note that carrying capacity can be always taken as $u = 1$ by rescaling, so that the nonlinearity $f(u)$ can always be assumed to satisfy:

$$f \in C^1([0, \infty), \mathbb{R}), \quad f(0) = f(1) = 0, \quad f(u) < 0 \text{ for } u > 1. \tag{1.13}$$

Note that the precise structure of nonlinearity $f(u)$ is essentially important only on the interval $(0, 1)$. The reason $f(u) < 0$ for $u > 1$ is that the population density declines whenever it is above the carrying capacity. Mathematically, $u = 1$ is either a stable or a semistable steady state of the population dynamics.

In this dissertation, two different types of nonlinearities $f(u)$ from the population dynamics mentioned above and satisfying condition (1.13) are studied. They are the *monostable nonlinearity* and the *bistable nonlinearity*.

For the Fisher-KPP nonlinearity, the *per capita growth rate* $f(u)/u$ reaches its maximum at $u = 0$, i.e., $f'(0) \geq f(u)/u$ for any $u \in (0,1)$. If the maximum of the per capita growth rate does not occur at $u = 0$, then it is referred to as the *Allee effect* in population biology (see e.g. [45] for further discussion). In particular, if the per capita growth rate is smaller than its maximum but still positive for small density, then the growth pattern is called *weak Allee effect*. A typical example is the generalized Fisher nonlinearity:

$$f(u) = u^p(1-u), \quad p > 1. \tag{1.14}$$

A type of nonlinearity $f(u)$, which is always positive on $(0,1)$, including the Fisher-KPP type and some weak Allee effects such as the generalized Fisher nonlinearity, is called the monostable nonlinearity. In other words, a monostable nonlinearity obeys $f(u) \in C^1([0,\infty),\mathbb{R})$ and

$$f(0) = f(1) = 0, \quad f(u) \begin{cases} > 0, & \text{in } (0,1), \\ < 0, & \text{in } (1,\infty). \end{cases} \tag{1.15}$$

Some biological populations exhibit *strong Allee effect*, i.e., the population declines if it falls below a threshold level. A typical example is the following Nagumo nonlinearity [32, 39],

$$f(p) = Kp(1-\frac{p}{P})(\frac{p}{P_c}-1), \tag{1.16}$$

where $P > 0$ is the carrying capacity, and $P_c \in (0,P)$ is the threshold level. In one space dimension, after rescaling, equation (1.10) can be simplified to

$$u_t = u_{xx} + u(1-u)(u-\theta_0), \tag{1.17}$$

where $0 < \theta_0 < 1$. Note that equation (1.17) has exact traveling wave solutions

$$u(x,t) = \frac{1}{2}(1 + \tanh(\pm\frac{\sqrt{2}}{4}(x - ct)), \tag{1.18}$$

where $c = \mp(1 - 2\theta_0)/\sqrt{2}$.

The bistable nonlinearity, which is used to describe some strong Allee effects, has exactly one zero in $(0, 1)$, i.e., $f(u) \in C^1([0, \infty), \mathbb{R})$,

$$f(0) = f(\theta_0) = f(1) = 0, \quad f(u) \begin{cases} < 0, & \text{in } (0, \theta_0) \cup (1, \infty), \\ > 0, & \text{in } (\theta_0, 1), \end{cases} \tag{1.19}$$

where $\theta_0 \in (0, 1)$. Introduce

$$V(u) := -\int_0^u f(s)ds. \tag{1.20}$$

Then the sign of $V(1)$ is an important factor to determine the structure of solutions of the reaction-diffusion equations with bistable nonlinearities. The essential difference is that equation (1.1) with bistable nonlinearity has localized radial stationary solutions if $V(1) < 0$ [6].

Another nonlinearity studied in the dissertation is from combustion modeling. There *ignition temperature* is introduced to avoid the so-called *cold boundary difficulty* [5]. The associated nonlinearity is called *ignition nonlinearity*, satisfying $f(u) \in C^1([0, \infty), \mathbb{R})$,

$$f(u) \begin{cases} = 0, & \text{in } [0, \theta_0] \cup \{1\}, \\ > 0, & \text{in } (\theta_0, 1), \\ < 0, & \text{in } (1, \infty), \end{cases} \tag{1.21}$$

where $\theta_0 \in (0, 1)$.

In this dissertation, the nonlinear reaction-diffusion equations with the three types of nonlinearities defined above are studied. These nonlinearities are shown in Figure 1.2.

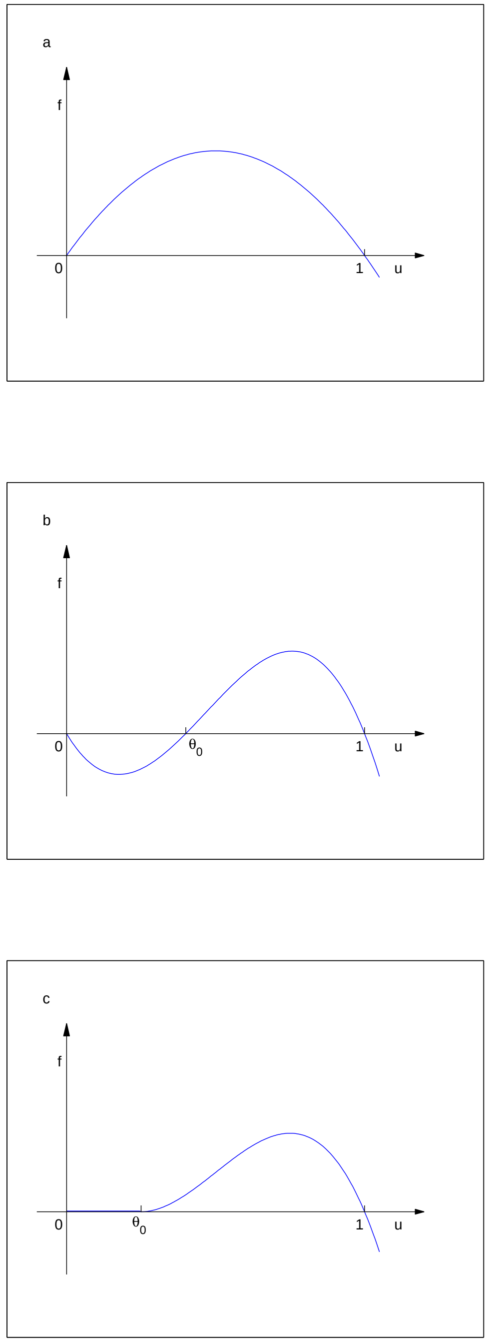

Figure 1.2 The three types of nonlinearities considered in this dissertation. (a) Monostable nonlinearity. (b) Bistable nonlinearity. (c) Ignition nonlinearity.

1.2 Cauchy Problem and Threshold Phenomena

In this dissertation, the following Cauchy problem for one-dimensional nonlinear reaction-diffusion equation with localized initial condition is considered:

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R},\ t > 0, \tag{1.22}$$

$$u(x,0) = \phi(x) \in L^2(\mathbb{R}), \quad x \in \mathbb{R}. \tag{1.23}$$

The nonlinearity $f(u)$ can be one of the three types mentioned above, or in general, $f(u)$ satisfies condition (1.13).

The corresponding stationary equation for equation (1.22) is

$$v''(x) + f(v(x)) = 0, \quad x \in \mathbb{R}. \tag{1.24}$$

For bistable nonlinearity with $V(1) < 0$, and, say, $f'(0) < 0$, the phase portrait looks like the Figure 1.3. Equation (1.24) has, therefore, a unique symmetric decreasing "*bump*" solution with maximum θ^*, where $\theta^* \in (0,1)$ is a root of $V(u)$, i.e., $V(\theta^*) = 0$, see Figure 1.4.

An important question associated with the Cauchy problem is the long-time behavior of the solution, i.e., the asymptotic behavior of $u(x,t)$ as $t \to \infty$.

In this dissertation, the localized initial data, e.g. initial value $\phi(x)$ in (1.23), are focused on. For different localized initial data, the solution of nonlinear reaction-diffusion equation (1.22) with the considered nonlinearities can exhibit quite different long time behaviors.

Since $u = 0$ and $u = 1$ are solutions of the stationary equation (1.24), one possible long-time behavior of solution of equation (1.22) is *extinction*, i.e., $\lim\limits_{t\to\infty} u(x,t) = 0$ uniformly in $\mathbb{R}$. Another possible behavior of the solution is *propagation*, i.e., $\lim\limits_{t\to\infty} u(x,t) = 1$ locally uniformly in $\mathbb{R}$ and, moreover, $\lim\limits_{t\to\infty} u(x+ct,t) = 1$ locally uniformly for all sufficiently small $c \in \mathbb{R}$.

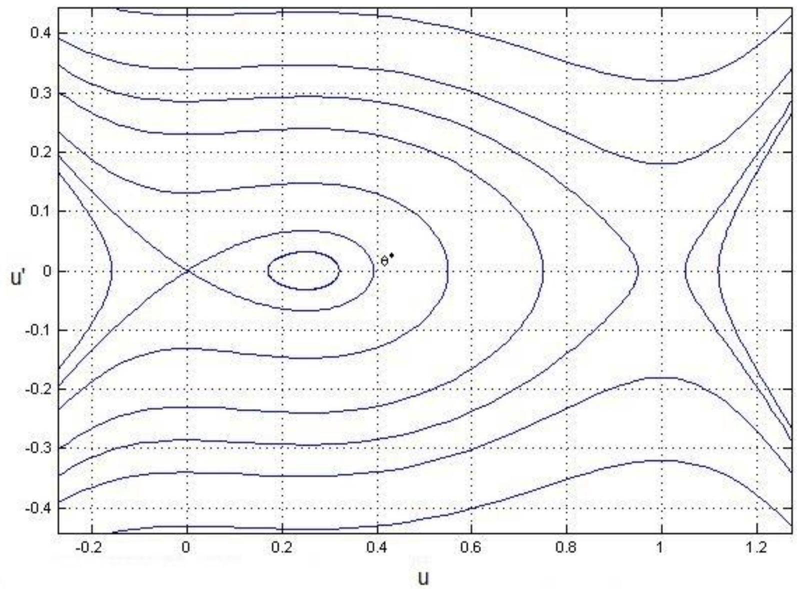

Figure 1.3 The phase portrait for the cubic nonlinearity $f(u) = u(1-u)(u-1/4)$ obtained numerically.

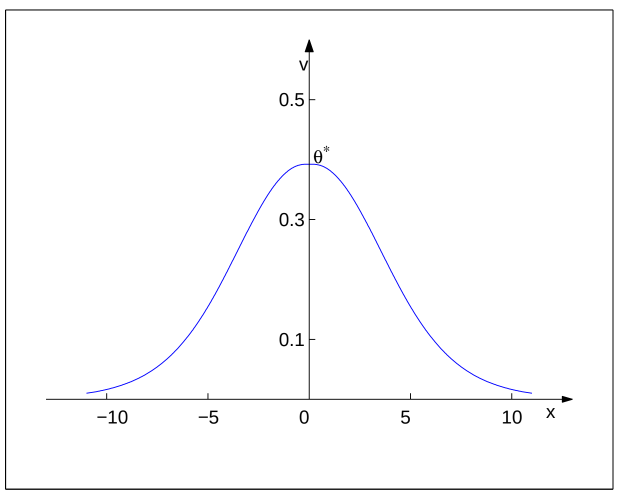

Figure 1.4 Qualitative form of bump solution for $f(u) = u(1-u)(u-1/4)$.

The following simple example illustrates the above possibilities. An infinite pipe filled with combustible gas mixture is heated by a heat source at $x = 0$ with constant temperature T_b. The initial temperature in the pipe is the ambient temperature T_0. Incorporating heat loss into the pipe wall, one has a half-line problem [44]:

$$\begin{cases} c\rho\frac{\partial}{\partial t}T(x,t) &= \kappa\frac{\partial^2}{\partial x^2}T(x,t) + Q(T) - \alpha(T(x,t) - T_0), \quad x > 0,\ t > 0, \\ T(0,t) &= T_b, \quad t > 0, \\ T(x,0) &= T_0, \quad x > 0, \end{cases} \tag{1.25}$$

where ρ is the mixture density, c is the specific heat of the mixture, κ is the thermal conductivity of the mixture, α is the rate of heat loss, and $Q(T)$ is the heat release rate. $Q(T)$ is nonnegative and supported on $[T_i, T_c]$, where T_i is the ignition temperature, which is above T_0, and $Q(T)$ incorporates the effect of the fuel consumption for $T \geq T_c$. Moreover, $Q(T)$ is assumed to have a single maximum at $T = T_m \in (T_i, T_c)$, and $Q(T) - \alpha(T - T_0)$ is assumed to have a unique root $T = T_r \in (T_i, T_m)$. Then $Q(T) - \alpha(T - T_0)$ has another root $T_R > T_r$.

By taking

$$u = \frac{T - T_0}{T_R - T_0}, \quad \lambda = \frac{T_b - T_0}{T_R - T_0}, \tag{1.26}$$

after rescaling one has

$$\begin{cases} u_t(x,t) &= u_{xx}(x,t) + f(u(x,t)), \quad x > 0,\ t > 0, \\ u(0,t) &= \lambda, \quad t > 0, \\ u(x,0) &= 0, \quad x > 0. \end{cases} \tag{1.27}$$

By the assumptions on $Q(T)$, the nonlinearity $f(u)$ can be assumed to be bistable. Under an additional assumption $V(1) < 0$, which is always true for sufficiently small α, the following result concerns the long-time behavior of solutions of equation (1.27).

Theorem 1.1. *Denote by $u(x,t)$ the solution of equation (1.27) on $\mathbb{R}^+ \times \mathbb{R}^+$. Then*

$$\lim_{t\to\infty} u(x,t) = w(x) \tag{1.28}$$

locally uniformly in $\mathbb{R}^+$, where $w(x)$ is the unique positive solution of

$$w'' + f(w) = 0, \quad x > 0, \tag{1.29}$$

satisfying

$$\lim_{x\to 0} w(x) = \lambda, \quad \lim_{x\to\infty} w(x) = \begin{cases} 0, & \textit{for } 0 \le \lambda \le \theta^*, \\ 1, & \textit{for } \lambda > \theta^*. \end{cases} \tag{1.30}$$

Proof. In the spirit of [3, Proposition 2.2], $u(x,t)$ is increasing on $\mathbb{R}^+ \times \mathbb{R}^+$. Moreover, as $t \to \infty$, it converges to the smallest stationary solution $w(x)$ with boundary condition $w(0) = \lambda$.

For $\lambda \le \theta^*$, $v(x)$ (restricted in $\mathbb{R}^+$) defined in (1.24) is a supersolution of equation (1.27), so that $w(x) \le v(x)$. In addition, $\lim\limits_{x\to\infty} w(x) = 0$. For $\lambda > \theta^*$, by phase plane argument, the only positive stationary solution $w(x)$ has asymptotic behavior $\lim\limits_{x\to\infty} w(x) = 1$. $\square$

From Theorem 1.1, a source with sufficiently high temperature will generate a propagating flame front, while insufficient heating will fail to ignite a flame. In addition, there is a sharp transition at $\lambda = \theta^*$, see Figure 1.5 and 1.6.

As in the work of Du and Matano [10], in this dissertation, an increasing one-parameter family of initial conditions ϕ_λ, $\lambda > 0$ is considered. The family satisfies conditions in (1.23), with $\lim\limits_{\lambda\to 0} \phi_\lambda \equiv 0$, and the map $\lambda \mapsto \phi_\lambda$ is increasing and continuous in the $L^2(\mathbb{R})$ norm. An additional technical assumption is required: for any $\lambda > 0$, $\phi_\lambda(x)$ is a symmetric decreasing function of x:

(SD) The initial condition $\phi(x)$ in (1.23) is symmetric decreasing, i.e., if $\phi(-x) = \phi(x)$ and $\phi(x)$ is non-increasing for every $x > 0$.

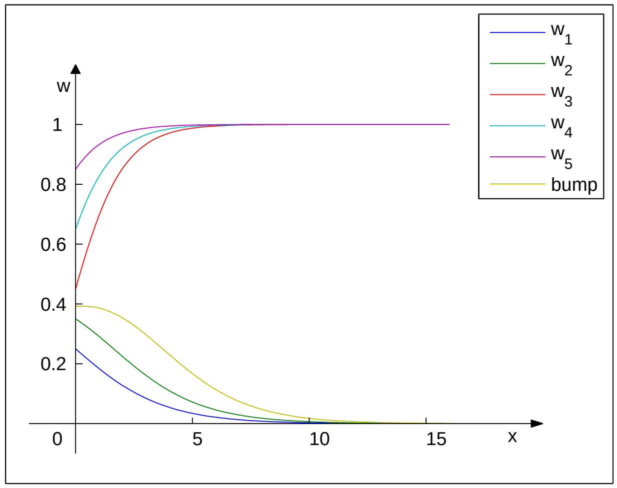

Figure 1.5 Long time behavior of solutions of equation (1.27) for the bistable nonlinearity $f(u) = u(1-u)(u-1/4)$ with different boundary conditions.

This assumption allows us to avoid a possible long-time behavior consisting of a bump solution slowly moving off to infinity, which was pointed out for some related problems [13]. In the case of bistable and ignition nonlinearities, it is easy to show that if the parameter λ is small enough, then extinction occurs. It is interesting to know if propagation can occur when λ is large. And a more interesting question is: does there exist any long time behavior of solution, which is neither extinction, nor propagation, for intermediate values of λ? On the other hand, for monostable nonlinearities it is known that propagation occurs for any $\lambda > 0$ if $f'(0) > 0$ [3], or even when $f(u) \sim u^p$ for small u, when $p \leq p_c$, where $p_c = 3$ is the Fujita exponent in one space dimension (see e.g. [4,43]). Nevertheless, the question of long-time behavior is also non-trivial for $p > p_c$ and has not been treated so far.

The following conclusions are proved: for bistable and ignition nonlinearities, that if propagation occurs at some value of $\lambda > 0$, then there is a value of $\lambda = \lambda^* > 0$ which serves as a sharp threshold between propagation for $\lambda > \lambda^*$ and extinction for

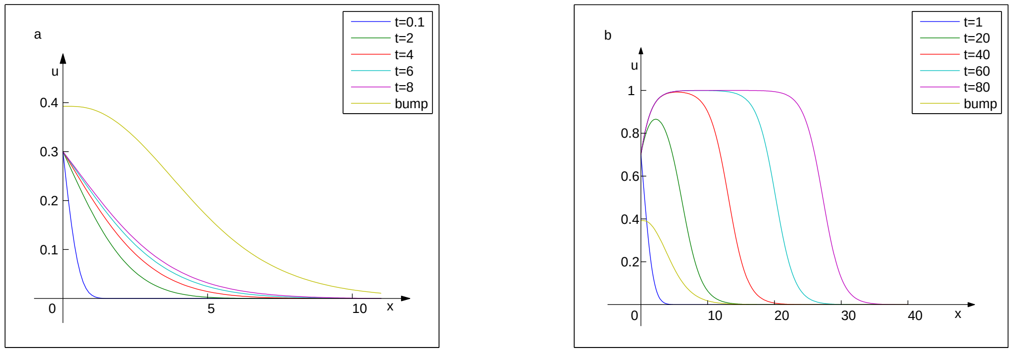

Figure 1.6 Results of numerical solutions of equation (1.27) at different times for $f(u) = u(1-u)(u-1/4)$. (a) Small boundary condition $u(0,t) < \theta^*$ leads to uniform convergence to a stationary solution below the bump solution. (b) Large boundary condition $u(0,t) > \theta^*$ leads to locally uniform convergence to a stationary solution above the bump solution.

$\lambda < \lambda^*$. The behavior of solution at $\lambda = \lambda^*$ is also characterized, thus generalizing the result of Zlatoš to the considered class of data. And for monostable nonlinearities which are supercritical with respect to the Fujita exponent, if propagation occurs at some value of $\lambda > 0$, then the existence of a value $\lambda^* > 0$, which serves as a sharp threshold between propagation for $\lambda > \lambda^*$ and extinction at $\lambda \leq \lambda^*$, is proved. Note that in this case propagation and extinction exhaust the list of possible long-time behaviors of solutions. In addition, a new sufficient condition for propagation is obtained, which can be easily verified. Note that with minor modifications, many of these conclusions still hold if $f(u)$ is only locally Lipschitz.

1.3 Outline of Dissertation

Below is the outline of this dissertation. Chapter 2 is an overview of existing results. In Chapter 3 some background results related to the variational structure of the considered problem is introduced. In Chapter 4 the bistable nonlinearities are discussed. The convergence result related to bistable nonlinearities with $V(1) > 0$ is studied by weak convergence methods; the problems related to bistable nonlinearities with $V(1) < 0$ are studied by energy methods. The convergence result, the one-to-one relation, and the sharp threshold result are proved there. Monostable nonlinearities are studied in Chapter 5. Ignition nonlinearities are studied in Chapter 6. Finally, Chapter 7 contains some concluding remarks and suggestions for future work.

CHAPTER 2

OVERVIEW OF EXISTING RESULTS

2.1 Work on Problems with Compactly Supported Initial Data

The first mathematical studies of extinction and propagation in the models of deflagration flames was by Kanel' [23]. He studied the long time behavior of solution of equation (1.22) with ignition nonlinearity (1.21) and initial data in the form of the characterstical function of an interval, i.e., $\phi(x) = \chi_{[-L,L]}(x)$. He proved existence of two lengths L_0 and L_1 such that extinction occurs when $L < L_0$, and propagation occurs when $L > L_1$, i.e.,

$$\lim_{t\to\infty} u(x,t) = 0 \text{ uniformly in } x \in \mathbb{R}, \quad \text{if } L < L_0, \tag{2.1}$$

$$\lim_{t\to\infty} u(x,t) = 1 \text{ locally uniformly in } x \in \mathbb{R}, \quad \text{if } L > L_1. \tag{2.2}$$

Obviously, $L_0 \leq L_1$.

Moreover, Kanel' proved the *hair-trigger effect*, i.e., for monostable nonlinearities with non-degeneracy assumption $f'(0) > 0$, if the initial data $\phi(x)$ is nonnegative and strictly positive on an interval, then $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly in $\mathbb{R}$.

Aronson and Weinberger [3] extended Kanel's results to bistable nonlinearity (1.19) with $V(1) < 0$. Indeed, they proved that for a disturbance larger than a specific compactly supported subsolution, the state $u = 0$ is unstable, i.e., $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly in $\mathbb{R}$ if the initial condition is larger than this specific compactly supported subsolution. In addition, if the disturbance is not sufficiently large on a large enough interval, then extinction occurs. Numercial results in Figures 2.1 and 2.2 give an illustration of the conclusions by Aronson and Weinberger in the case of the bistable nonlinearity $f(u) = u(1-u)(u-1/4)$.

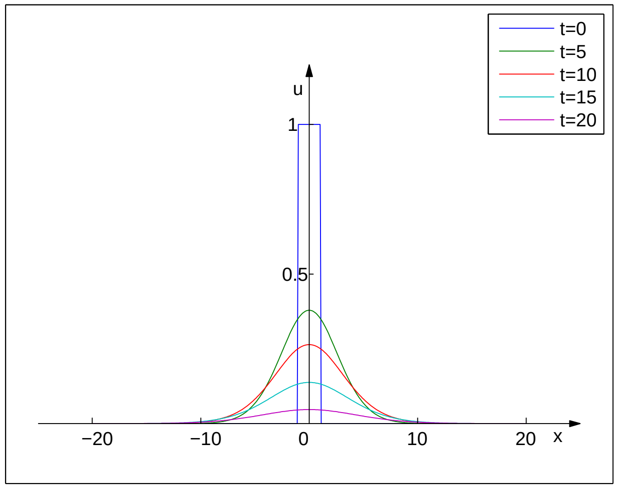

Figure 2.1 Result of the numerical solution of equation (1.22) for the bistable nonlinearity $f(u) = u(1-u)(u-1/4)$. Small initial data $\phi(x) = \chi_{[-1,1]}(x)$ leads to extinction.

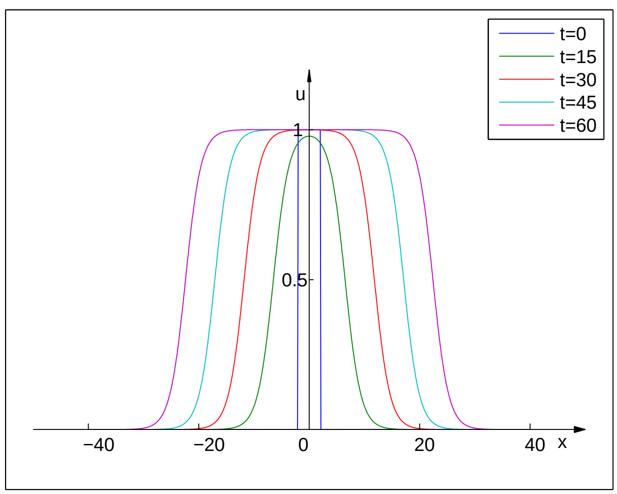

Figure 2.2 Result of the numerical solution of equation (1.22) for the bistable nonlinearity $f(u) = u(1-u)(u-1/4)$. Large initial data $\phi(x) = \chi_{[-2,2]}(x)$ leads to propagation.

The higher dimensional extentions of [23] and [3] have been done also by Aronson Weinberger [4], with a higher dimensional statement of the hair-trigger effect.

A natural question left from the one-dimensional works by Kanel' and Aronson and Weinberger is if L_0 equals L_1. Furthermore, if $L_0 = L_1$, then what is the long time behavior of solution of equation (1.22) corresponding to the initial condition $\phi(x) = \chi_{[-L_0,L_0]}(x)$?

Zlatoš [48] proved that for both bistable nonlinearity and ignition nonlinearity, with the initial data in the form of the characteristic functions $\phi(x) = \chi_{[-L,L]}(x)$, there is a sharp threshold between extinction and propagation, i.e., indeed $L_0 = L_1$. Moreover, he determined the precise behavior at the threshold.

For ignition nonlinearity, and f non-decreasing on $[\theta_0, \theta_0 + \delta]$ with some $\delta > 0$, Zlatoš proved that there exists $L_0 > 0$ such that

$$\lim_{t\to\infty} u(x,t) = \begin{cases} 0, & \text{uniformly in } \mathbb{R}, \quad \text{for } L < L_0, \\ \theta_0, & \text{uniformly in } \mathbb{R}, \quad \text{for } L = L_0, \\ 1, & \text{locally uniformly in } \mathbb{R}, \quad \text{for } L > L_0. \end{cases} \tag{2.3}$$

In particular, the monostable nonlinearity was considered by Zlatoš as a special case of ignition nonlinearity with $\theta_0 = 0$. For the special case, even without the nondecreasing assumption on $[\theta_0, \theta_0 + \delta]$, the above sharp transition result still holds as follows. There exists $L_0 \geq 0$ such that

$$\lim_{t\to\infty} u(x,t) = \begin{cases} 0, & \text{uniformly in } \mathbb{R}, \quad \text{for } L \leq L_0, \\ 1, & \text{locally uniformly in } \mathbb{R}, \quad \text{for } L > L_0. \end{cases} \tag{2.4}$$

In addition, $L_0 = 0$ if $f(u) \geq cu^p$ for $p < 3$ and all sufficiently small u, while $L_0 > 0$ if $f(u) \leq cu^p$ for $p > 3$ and all sufficiently small u.

For bistable nonlinearity with $V(1) < 0$, Zlatoš proved that there exists $L_0 > 0$ such that

$$\lim_{t\to\infty} u(x,t) = \begin{cases} 0, & \text{uniformly in } \mathbb{R}, \quad \text{for } L < L_0, \\ v(x), & \text{uniformly in } \mathbb{R}, \quad \text{for } L = L_0, \\ 1, & \text{locally uniformly in } \mathbb{R}, \quad \text{for } L > L_0, \end{cases} \tag{2.5}$$

where $v(x)$ is the bump solution.

Du and Matano [10] studied the long time behavior problem with general compactly supported initial data. By using the so-called zero-counting argument [1, 30], they proved that the following convergence results:

Assume that $f : [0,\infty) \to \mathbb{R}$ is a locally Lipschitz continuous function satisfying $f(0) = 0$, and $\phi(x) \in L^\infty(\mathbb{R})$ is nonnegative and compactly supported. Suppose that the solution $u(x,t)$ of equation (1.22) is globally defined for $t \geq 0$ and remains bounded as $t \to \infty$. Then $u(x,t)$ converges to a stationary solution as $t \to \infty$ locally uniformly in $\mathbb{R}$. In addition, the limit is either a constant solution or a symmetrically decreasing stationary solution.

Moreover, Du and Matano proved that for the bistable nonlinearity (1.19) with $V(1) < 0$, as $t \to \infty$, the solution $u(x,t)$ can only converge to 0 uniformly, the bump solution v uniformly, or 1 locally uniformly; whereas for the ignition nonlinearity (1.21), as $t \to \infty$, the solution $u(x,t)$ can only converge to 0 uniformly, θ_0 locally uniformly, or 1 locally uniformly.

To study the sharp transition between extinction and propagation, Du and Matano considered a one-parameter family of initial data $\phi_\lambda(x)$ $(\lambda > 0)$ satisfying the assumptions (P1) through (P3) below, where

$$X^+ = \{\phi \in L^\infty(\mathbb{R}) : \ \phi \geq 0 \text{ is compactly supported}\} :$$

(P1) $\phi_\lambda \in X^+$ for every $\lambda > 0$, and map $\lambda \mapsto \phi_\lambda$ is continuous from $\mathbb{R}_+$ to $L^1(\mathbb{R})$;

(P2) if $0 < \lambda_1 < \lambda_2$, then $\phi_{\lambda_1} \le \phi_{\lambda_2}$ and $\phi_{\lambda_1} \ne \phi_{\lambda_2}$ in the a.e. sense;

(P3) $\lim_{\lambda\to 0} \phi_\lambda(x) = 0$ a.e. in $\mathbb{R}$.

Denoted by $u_\lambda(x,t)$ the solution of equation (1.22) corresponding to the initial data $u_\lambda(x,0) = \phi_\lambda(x)$, the results by Du and Matano on the long time behavior of $u_\lambda(x,t)$ are as follows.

For bistable nonlinearity with $V(1) < 0$, one of the following holds:

(a) $\lim_{t\to\infty} u(x,t) = 0$ uniformly in $\mathbb{R}$ for every $\lambda > 0$;

(b) there exists $\lambda^* > 0$ and x_0 on the support of $\phi(x)$ such that

$$\lim_{t\to\infty} u_\lambda(x,t) = \begin{cases} 0, & \text{uniformly in } \mathbb{R}, & \text{for } 0 < \lambda < \lambda^*, \\ v(x-x_0), & \text{uniformly in } \mathbb{R}, & \text{for } \lambda = \lambda^*, \\ 1, & \text{locally uniformly in } \mathbb{R}, & \text{for } \lambda > \lambda^*, \end{cases} \tag{2.6}$$

where $v(x)$ is the bump solution of the stationary equation (1.24).

For ignition nonlinearity, if there exists $\delta_0 > 0$, such that $f(s)$ is non-decreasing in $(\theta_0, \theta_0 + \delta_0)$, then one of the following holds:

(a) $\lim_{t\to\infty} u(x,t) = 0$ uniformly in $\mathbb{R}$ for every $\lambda > 0$;

(b) there exists $\lambda^* > 0$ such that

$$\lim_{t\to\infty} u_\lambda(x,t) = \begin{cases} 0, & \text{uniformly in } \mathbb{R}, & \text{for } 0 < \lambda < \lambda^*, \\ \theta_0, & \text{locally uniformly in } \mathbb{R}, & \text{for } \lambda = \lambda^*, \\ 1, & \text{locally uniformly in } \mathbb{R}, & \text{for } \lambda > \lambda^*, \end{cases} \tag{2.7}$$

Note that the above results by Zlatoš, and by Du and Matano rely on one-dimensional techniques. For higher dimensional problems, Poláčik [42] studied threshold solutions and sharp transitions for a class of nonautonomous parabolic equations on $\mathbb{R}^n$, whose initial data are still compactly supported. Problems with bistable nonlinearities in a generalized setting ($f = f(u,t)$) are considered, and the proof of instability of the threshold solutions is based on exponential separations and principal Floquet bundles for linear parabolic equations.

As was pointed out by Matano [31], all the above works on sharp threshold behavior between ignition and extinction crucially rely on the assumption of the data being compactly supported (or rapidly decaying) and, therefore, may not be applied to data that lie in the natural function spaces, such as, e.g., L^2. The purpose of this dissertation is to provide such an extension to symmetric decreasing $L^2(\mathbb{R})$ initial data in the context of the problem originally considered by Kanel'. To achieve this goal, the advantage of the gradient flow structure of the considered equation is taken. In addition, energy-based methods are developed, which are quite different from those used in the above works.

Moreover, when $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly. the works by Zlatoš, and by Du and Matano give little information on the nature of convergence. More precisely, though the authors called this behavior "propagation", in general, they did not prove that the level sets of solutions actually propagate, i.e., move faster than a constant speed $c > 0$. In this dissertation, under the symmetric decreasing hypothesis, it is proved that the behavior $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly implies $\lim_{t\to\infty} u(x \pm ct, t) = 1$ for any sufficiently small c. Therefore, propagation in the sense described in the introduction indeed occurs.

2.2 Previous Work on Energy Methods

There is a brief overview of previous works of the long time behavior of solutions of nonlinear reaction-diffusion equations by energy-based methods. Some techniques used in this dissertaion are similar to the approaches in those works.

Fife [15] studied the long time behavior of solutions with bistable nonlinearities and $V(1) < 0$. He assumed that the differentiable initial data $\phi(x)$ cross θ_0 at most twice, and also that f has continuous derivatives of sufficiently high order. With non-degeneracy assumptions $f'(0) < 0$ and $f'(1) < 0$, Fife proved that $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly, if and only if the associated energy $E[u(x,t)]$ is unbounded from below. Moreover, if $E[u(x,t)]$ is bounded from below, then the solution $u(x,t)$ either converges to 0 uniformly, or converges to the bump $v(x)$ uniformly, as $t \to \infty$. His idea will be used in this dissertation for problems with different assumptions.

For the Nagumo equation with $H^1(\mathbb{R})$ initial data, Flores [18] proved that $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly, provided the associated energy $E[u(x,t)]$ is unbounded from below. The key point of his proof is the exact multiplicity of the related elliptic equation with cubic nonlinearity.

Feireisl et al. [13, 14] studied the long time behavior in a different way. They considered the solution $u(x,t)$ of equation (1.22) with bistable nonlinearity, $V(1) < 0$ under non-degeneracy assumption $f'(0) < 0$. If $\|u(\cdot,t)\|_{H^1(\mathbb{R})}$ is bounded in time, then the associated energy is also bounded. With the help of the estimate of dissipation rate for the energy, by concentration compactness principle [27] there exists an increasing time sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$, such that $u(\cdot,t)$ converges to a sum of finite bump solutions in $H^1(\mathbb{R})$ on the sequence $\{t_n\}$, where at most one of the bumps is localized, and the distance between each bump goes to infinity as $n \to \infty$. More precisely, there exists an integer $m \geq 0$ and functions $y_k(t)$, $1 \leq k \leq m$ (if $m \geq 1$) satisfying

$$\begin{aligned} \lim_{n\to\infty} |y_k(t_n)| &= \infty, \;\; 1 \leq k \leq m, \\ \lim_{n\to\infty} |y_k(t_n) - y_l(t_n)| &= \infty, \;\; 1 \leq k, \; l \leq m, \; k \neq l, \end{aligned} \tag{2.8}$$

such that

$$\lim_{n\to\infty} \|u(x,t_n) - w - \sum_{k=1}^{m} v(x - y_k(t_n))\|_{H^1(\mathbb{R})} = 0, \tag{2.9}$$

where $v(x)$ is the bump solution, w is either 0 or $v(x - x_0)$, for some $x_0 \in \mathbb{R}$.

For compactly supported initial data, since $u(x,t)$ uniformly decays outside the support of its initial value as $|x| \to \infty$, it is obvious that $m = 0$, and the convergence behavior is clear. However, for general L^2 initial data, the number m of "moving bumps" is not a priori known. Perhaps there exists a "moving bump" (if $m \geq 1$), which implies that $u(x,t)$ converges to either 0 or the bump solution v locally uniformly, but does not converges to it uniformly in $\mathbb{R}$.

For half-line problem on $x > 0$ with Direchlet boundary condition $u(0,t) = 0$, Feireisl et al. [12] proved that the convergence to a "moving bump" can really occur, i.e.,

$$\lim_{t \to \infty} \|u(x,t) - v(x - y(t))\|_{H^1(\mathbb{R}^+)} = 0, \tag{2.10}$$

where $\lim_{t\to\infty} y(t) = \infty$. It can be found as the threshold behavior of solutions with initial data $\lambda \bar{u}$, where the nonnegative function $\bar{u} \in H_0^1(\mathbb{R}^+)$ has exactly one maximum at $x_0 > 0$.

CHAPTER 3

PRELIMINARIES

In this chapter, some mathematical preliminaries are reviewed.

First, existence of classical solutions for (1.22) with initial data satisfying (1.23) is well known [19]. In view of (1.13), these solutions are positive, uniformly bounded and, hence, global in time. Furthermore, it is well known that the derivatives $u_t(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$ of the solution of (1.22) can be estimated in the uniform norm in terms of u itself. More precisely, the uniform boundedness of $|u|$ in the half-space $t > 0$ controls the boundedness of $|u_t|$, $|u_x|$ and $|u_{xx}|$ in the half-space $t \geq T$ for any $T > 0$ (see, e.g. [16, 19]). This boundedness will be referred as "standard parabolic regularity" in the rest of this dissertation. For the purposes of this dissertation, however, it will also need a suitable existence theory for solutions in integral norms that measure, in some sense, the rate of the decay of solutions as $x \to \pm\infty$. This is because the *energy* functional, defined as

$$E[u] := \int_{\mathbb{R}} \left(\frac{1}{2} u_x^2 + V(u) \right) dx, \qquad V(u) := -\int_0^u f(s) ds, \tag{3.1}$$

will be used in this dissertation. Clearly, this functional is well-defined for any nonnegative $u \in H^1(\mathbb{R})$, and it is class C^1 in this class. Similarly, for a given $c > 0$ the exponentially weighted functional Φ_c associated with (3.1) is defined as

$$\Phi_c[u] := \int_{\mathbb{R}} e^{cx} \left(\frac{1}{2} u_x^2 + V(u) \right) dx, \tag{3.2}$$

which is well-defined for nonnegative L^∞ functions in the exponentially weighted Sobolev space $H_c^1(\mathbb{R})$ with the norm

$$\|u\|_{H_c^1}^2 := \|u\|_{L_c^2}^2 + \|u_x\|_{L_c^2}^2, \qquad \|u\|_{L_c^2}^2 := \int_{\mathbb{R}} e^{cx} u^2 dx. \tag{3.3}$$

Similarly, the space $H_c^2(\mathbb{R})$ can be defined as the space of functions whose first derivatives belong to $H_c^1(\mathbb{R})$.

The following proposition guarantees existence and regularity properties of solutions of (1.22) in both the usual and the exponentially weighted Sobolev spaces.

Proposition 3.1. *Under (1.13), there exists a unique solution* $u \in C_1^2(\mathbb{R} \times (0,\infty)) \cap L^\infty(\mathbb{R} \times (0,\infty)))$ *satisfying (1.22) and (1.23) (using the notations from [11]), with*

$$u \in C([0,\infty); L^2(\mathbb{R})) \cap C((0,\infty); H^2(\mathbb{R}))$$

and $u_t \in C((0,\infty); H^1(\mathbb{R}))$. *Furthermore, if there exists* $c > 0$ *such that the initial condition* $\phi(x) \in L_c^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, *then the solution of (1.22) and (1.23) satisfies*

$$u \in C([0,\infty); L_c^2(\mathbb{R})) \cap C((0,\infty); H_c^2(\mathbb{R})),$$

with $u_t \in C((0,\infty); H_c^1(\mathbb{R}))$. *In addition, small variations of the initial data in* $L^2(\mathbb{R})$ *result in small changes of solution in* $H^1(\mathbb{R})$ *at any* $t > 0$.

Proof. Follows from the arguments in the proof of [37, Proposition 3.1] based on the approach of [29], taking into consideration that by (1.13) the function $\bar{u}(x,t) = \max\{1, \|\phi\|_{L^\infty(\mathbb{R})}\}$ is a universal supersolution for the considered problem. □

Remark 3.2. *Note that Proposition 3.1 does not require hypothesis (SD). However, under (SD), the solution* $u(x,t)$ *is a symmetric decreasing function of* x *for all* $t > 0$.

In view of Proposition 3.1, by direct calculation the well-known identity related to the energy dissipation rate for the solutions of (1.22) is valid for all $t > 0$:

$$\frac{dE}{dt}[u(\cdot,t)] = -\int_{\mathbb{R}} u_t^2(x,t)\,dx. \tag{3.4}$$

In fact, the basic reason for (3.4) is the fact that (1.22) is a gradient flow in L^2 generated by E. Similarly, as was first pointed out in [35], equation (1.22) written in the reference frame moving with an arbitrary speed $c > 0$ is a gradient flow in L^2_c generated by Φ_c. More precisely, defining $\tilde{u}(x,t) := u(x+ct,t)$, which solves

$$\tilde{u}_t = \tilde{u}_{xx} + c\tilde{u}_x + f(\tilde{u}), \tag{3.5}$$

it is easy to see with the help of Proposition 3.1 that an identity similar to (3.4) holds for Φ_c:

$$\frac{d\Phi_c}{dt}[\tilde{u}(\cdot,t)] = -\int_{\mathbb{R}} e^{cx}\tilde{u}_t^2(x,t)dx. \tag{3.6}$$

In particular, both $E[u(\cdot,t)]$ and $\Phi_c[\tilde{u}(\cdot,t)]$ are well defined and are non-increasing in t for all $t > 0$. Also note that non-trivial fixed points of (3.5) are *variational traveling waves*, i.e., solutions that propagate with constant speed $c > 0$ invading the equilibrium $u = 0$ and belong to $H^1_c(\mathbb{R})$ [36]. Furthermore, as was shown in [36], for sufficiently rapidly decaying front-like initial data the propagation speed associated with the leading edge of the solution (see the next paragraph for the definition) is determined by the special variational traveling wave solutions which are *minimizers* of Φ_c for some unique speed $c = c^\dagger > 0$. In the context of the nonlinearities considered in this dissertation, the following proposition gives existence, uniqueness and several properties of these minimizers (follows directly from [36, Theorem 3.3]; in fact, under these assumptions they are the only variational traveling waves, see [37, Corollary 3.4]).

Proposition 3.3. *Let f satisfy* (1.13)*, let $f'(0) \leq 0$, and let $u_0 = 1$ be the unique zero of f such that $\int_0^{u_0} f(u)du < 0$. Then there exists $c^\dagger > 0$ and a unique (up to translation) positive traveling wave solution $u(x,t) = \bar{u}(x - c^\dagger t)$ of* (1.22) *such that $\bar{u}(+\infty) = 0$, $\bar{u}(-\infty) = 1$, $\bar{u}' < 0$, and $\bar{u}$ minimizes Φ_c with $c = c^\dagger$.*

Turning back to the question of propagation, for a given $\delta > 0$, the *leading edge* $R_\delta(t)$ of the solution $u(x,t)$ of (1.22) is defined as

$$R_\delta(t) := \sup\{x \in \mathbb{R} : u(x,t) \geq \delta\}. \tag{3.7}$$

If the set $\{x \in \mathbb{R} : u(x,t) \geq \delta\} = \varnothing$, then $R_\delta(t) := -\infty$. Then, as follows from [36, Theorem 5.8], under the assumptions of Proposition 3.3 for every $\phi \in L^2_c(\mathbb{R})$ with some $c > c^\dagger$, $\phi(x) \in [0,1]$ for all $x \in \mathbb{R}$, and $\lim_{x \to -\infty} \phi(x) = 1$ the leading edge $R_\delta(t)$ propagates asymptotically with speed $c^\dagger$ for sufficiently small $\delta > 0$. Similarly, the same conclusion holds for the initial data obeying (1.23), provided that $\phi \in L^2_c(\mathbb{R})$ with some $c > c^\dagger$ and $u(x,t) \to 1$ as $t \to \infty$ locally uniformly in $x \in \mathbb{R}$ [36, Corollary 5.9]. In fact, a stronger conclusion can be made, which implies that the latter condition is equivalent to the stronger notion of propagation presented in the introduction, extending the results of Aronson and Weinberger [3, Theorem 4.5] to the considered class of nonlinearities.

Proposition 3.4. *Under the assumptions of Proposition 3.3, let ϕ satisfy* (1.23) *and assume that $u(x,t) \to 1$ as $t \to \infty$ locally uniformly in $x \in \mathbb{R}$. Then for every $\delta_0 \in (0,1)$ and every $c \in (0, c^\dagger)$, where $c^\dagger$ is the same as in Proposition 3.3, there exists $T \geq 0$ such that $R_\delta(t) \geq ct$ for every $t \geq T$ and every $\delta \in (0, \delta_0]$.*

Proof. Minimizers of Φ_c among $u \in X$ is considered, where X consists of all functions in $H^1_c(\mathbb{R})$ with values in $[0,1]$ that vanish for all $x > 0$. It can be shown that a non-trivial minimizer $\bar{u}_c \in X$ of Φ_c exists for all $c \in (0, c^\dagger)$. Indeed, by the argument in the proof of [36, Proposition 5.5], $\inf_{u \in X} \Phi_c[u] < 0$ for any $c \in (0, c^\dagger)$. In addition, by boundedness of $u \in X$, Φ_c is coercive on X. Existence of a minimizer then follows from weak sequential lower semicontinuity of Φ_c on X (see [28, Lemma 5.3]). Furthermore, by [28, Corollary 6.8], which can be easily seen to be applicable to $\bar{u}_c$, $\bar{u}_c(x) \to 1$ as $x \to -\infty$.

Similarly, for large enough $R > 0$ there exists a non-trivial minimizer $\bar{u}_c^R \in X_R$ of Φ_c, where X_R is a subset of X with all functions vanishing for $x < -R$ as well. These are stationary solutions of (3.5) with Dirichlet boundary conditions at $x = 0$ and $x = -R$, and by strong maximum principle, $\bar{u}_c^R < 1$ is obtained. Furthermore, if $R_n \to \infty$, then $\{\bar{u}_c^{R_n}\}$ constitute a minimizing sequence for Φ_c in X and, in view of the continuity of $\int_{-\infty}^0 e^{cx} V(u) dx$ with respect to the weak convergence in $H_c^1(\mathbb{R})$, the convergence $\bar{u}_c^{R_n} \to \bar{u}_c$ strongly in $H_c^1(\mathbb{R})$ and, by Sobolev imbedding, also locally uniformly. In particular, $\|\bar{u}_c^{R_n}\|_{L^\infty(\mathbb{R})} \to 1$ as $n \to \infty$. The proof is then completed by using $\bar{u}_c^{R_n}$ with a large enough n depending on δ_0 as a subsolution after a sufficiently long time t. □

Remark 3.5. *If in Proposition 3.4 one also has $\phi \in L_c^2(\mathbb{R})$ for some $c > c^\dagger$, then by [36, Proposition 5.2] for every $\delta_0 > 0$ and every $c' > c^\dagger$ there exists $T \geq 0$ such that $R_\delta(t) < c' t$ for every $\delta \geq \delta_0$, implying that $c^\dagger$ is the sharp propagation velocity for the level sets in the above sense. The same conclusion also holds for the "trailing edge", i.e., the leading edge defined using $u(-x,t)$ instead of $u(x,t)$, indicating the formation of a pair of counter-propagating fronts with speed $c^\dagger$.*

Remark 3.6. *Under hypothesis (SD), the conclusion of Proposition 3.4 clearly implies propagation in the sense defined in the introduction.*

The difficult part in applying Proposition 3.4 is to establish that $u(x,t) \to 1$ locally uniformly in $x \in \mathbb{R}$ as $t \to \infty$ for a given initial condition $\phi(x)$. In the absence of such a result, a weaker notion of propagation of the leading edge analyzed in [35] can be still appealed to. Following [35], the solution $u(x,t)$ of (1.22) and (1.23) is called *wave-like*, if there exist constants $c > 0$ and $T \geq 0$ such that $\phi \in L_c^2(\mathbb{R})$ and $\Phi_c[u(\cdot, T)] < 0$. Note that by monotonicity of $\Phi_c[\tilde{u}(\cdot,t)]$ and the fact that $\Phi_c[u(\cdot,t)] = e^{c^2 t}\Phi_c[\tilde{u}(\cdot,t)]$, it follows that for a wave-like solution $\Phi_c[u(\cdot,t)] < 0$ is ture for all $t \geq T$ as well. This fact allows to obtain an important characterization of the leading edge dynamics for wave-like solutions which is intimately related to the gradient descent structure of (3.5). In view of the "hair-trigger effect" discussed in the introduction in the case when $u = 0$ is linearly unstable [3], it is only necessary to consider the nonlinearities satisfying $f'(0) \leq 0$.

Proposition 3.7. *Let f satisfy* (1.13)*, let $f'(0) \leq 0$, and let $u(x,t)$ be a wave-like solution of (1.22) and* (1.23)*. Then there exists a constant $\delta_0 > 0$ such that*

$$V(u) \geq -\frac{1}{8}c^2u^2 \qquad \forall 0 \leq u \leq \delta_0, \tag{3.8}$$

and

$$\max_{x\in\mathbb{R}} u(x,t) \geq \delta_0, \tag{3.9}$$

for $t \geq T$. Furthermore, there exists $R_0 \in \mathbb{R}$ such that for every $\delta \in (0, \delta_0]$,

$$R_\delta(t) \geq ct + R_0 \tag{3.10}$$

is true for all $t \geq T$.

Proof. The statement is a direct consequence of [35, Proposition 4.10 and Theorem 4.11], which remain valid under the assumptions above in view of Proposition 3.1. $\square$

One of the goals of the analysis in the next chapters will be to show that under further assumptions on the nonlinearities and hypothesis (SD) propagation in the sense of Proposition 3.7 implies propagation in the sense of Proposition 3.4.

A key ingredient of the proofs that allows people to efficiently use variational methods and to go from sequential limits to full limits as $t \to \infty$ without much information about the limit states relies on an interesting observation regarding uniform Hölder continuity of the solutions of (1.22) with bounded energy. This result is stated in the following proposition. In addition, a more general result is available in $\mathbb{R}^N$ (it will be discussed in more detail elsewhere).

Proposition 3.8. *Suppose that ϕ satisfies* (1.23) *and f satisfies* (1.13). *If $E[u(\cdot,t)]$ is bounded from below, then $u(x,\cdot) \in C^{1/4}([T,\infty))$ for each $x \in \mathbb{R}$ and each $T > 0$. Moreover, the corresponding Hölder constant of $u(x,t)$ converges to 0 as $T \to \infty$ uniformly in x.*

Proof. Using (3.4), for any $x_0 \in \mathbb{R}$ and $t_2 > t_1 \geq T$ one has the following estimate:

$$\begin{aligned}
\int_{x_0}^{x_0+1} |u(x,t_2) - u(x,t_1)| dx &\leq \int_{t_1}^{t_2} \int_{x_0}^{x_0+1} |u_t(x,t)| dx dt \\
&\leq \sqrt{t_2 - t_1} \left(\int_{t_1}^{t_2} \int_{x_0}^{x_0+1} u_t^2(x,t) dx dt \right)^{1/2} \\
&\leq \sqrt{t_2 - t_1} \left(\int_{T}^{\infty} \int_{\mathbb{R}} u_t^2(x,t) dx dt \right)^{1/2} \\
&= \sqrt{(E[u(\cdot,T)] - E_\infty)(t_2 - t_1)}, \qquad (3.11)
\end{aligned}$$

where E_∞ is denoted by $\lim\limits_{t\to\infty} E[u(\cdot,t)]$.

On the other hand, by standard parabolic regularity there exists $M > 0$ such that

$$\|u_x(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq M, \quad \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq M \qquad \forall t \geq T, \qquad (3.12)$$

Without loss of generality, $u(x_0,t_2) - u(x_0,t_1) \in [0,M]$ can be assumed. Then, for every $x \in I$, where

$$I := [x_0, x_0 + \frac{u(x_0,t_2) - u(x_0,t_1)}{2M}], \qquad |I| < 1, \qquad (3.13)$$

one has the following inequality:

$$u(x,t_2) \geq u(x_0,t_2) - M(x - x_0) \geq u(x_0,t_1) + M(x - x_0) \geq u(x,t_1),\ x \in I. \quad (3.14)$$

This implies that

$$\begin{aligned}
\int_{x_0}^{x_0+1} |u(x,t_2) - u(x,t_1)| dx &\geq \int_I (u(x_0,t_2) - u(x_0,t_1) - 2M(x - x_0)) dx \\
&= \frac{|u(x_0,t_2) - u(x_0,t_1)|^2}{4M}. \qquad (3.15)
\end{aligned}$$

Then the following estimate can be obtained:

$$|u(x_0,t_2)-u(x_0,t_1)| \le 2\sqrt{M}(E[u(\cdot,T)]-E_\infty)^{1/4}(t_2-t_1)^{1/4}, \tag{3.16}$$

i.e., $u(x,\cdot)\in C^{1/4}([T,\infty))$ by the arbitrariness of x_0. Moreover, the limit of the Hölder constant is

$$\lim_{T\to\infty} 2\sqrt{M}(E[u(\cdot,T)]-E_\infty)^{1/4}=0, \tag{3.17}$$

which completes the proof. □

CHAPTER 4

BISTABLE NONLINEARITY

4.1 General Setting

In this chapter, the bistable nonlinearity is studied, i.e., $f(u)$ satisfies condition (1.19). As discussed in the introduction, the sign of $V(u)$ (defined in (1.20)) at $u = 1$ determines different qualitative behaviors of the solution. This is explained in more detail below.

If the $u = 0$ equilibrium is more energetically favorable than the $u = 1$ equilibrium, i.e., if

$$V(1) = -\int_0^1 f(s)ds < 0, \tag{4.1}$$

then there exists a bump solution of equation (1.24), see Figure 1.4.

If the $u = 1$ equilibrium is more energetically favorable than the $u = 0$ equilibrium, i.e., if

$$V(1) = -\int_0^1 f(s)ds > 0, \tag{4.2}$$

then there is no localized stationary solution of equation (1.24).

Note that the case

$$V(1) = -\int_0^1 f(s)ds = 0, \tag{4.3}$$

is called *balanced bistable nonlinearity*. There is no localized stationary solution in this case, too. However, there exists heteroclinic connections, which are monotone solutions converging to 0 and 1 as $x \to \pm\infty$ [16].

Actually, for bistable nonlinearity propagation (in the sense defined in the introduction) is only possible in the case of condition (4.1), at least for L^2 initial data. Further discussion of this case will be given in the last section of this chapter.

Indeed, if either (4.2) or (4.3) holds, then $V(u) \geq 0$ holds for all $u \geq 0$ and, therefore, $R_\delta \leq ct$ for any $\delta > 0$, any $c > 0$ and large enough t, at least for all $\phi \in L^2_c(\mathbb{R})$ by [36, Proposition 5.2]. Furthermore, if $V(1) > 0$ and $f'(0) < 0$ (the latter condition is not essential and will be replaced by a weaker non-degeneracy condition introduced later), then the energy functional in (3.1) is coercive in $H^1(\mathbb{R})$, and so it is not difficult to see that every solution of (1.22) and (1.23) converges uniformly to zero, implying extinction for all initial data.

Thus the only case in which the situation may be subtle is the balanced bistable nonlinearity, in which spreading, i.e., sublinear behavior of the leading edge with time, namely $R_\delta(t) \to \infty$ as $t \to \infty$, but $R_\delta(t) = o(t)$ for some $\delta > 0$, cannot be excluded a priori, even for exponentially decaying initial data. The analysis of the balanced case is beyond the scope of the present dissertation.

A kind of weak non-degeneracy assumption that $f(u) \simeq -ku^p$ for some $p \geq 1$ and $k > 0$ as $u \to 0$ is required in this chapter. More precisely, it is assumed that

$$f'(u) \leq 0 \quad \text{for all } u \in [0, \theta_1], \text{ for some } \theta_1 > 0, \tag{4.4}$$

and

$$\lim_{u \to 0} \frac{f(u)}{u^p} = -k \quad \text{for some } p \geq 1 \text{ and } k > 0. \tag{4.5}$$

Note that (4.4) and (4.5) are automatically satisfied for the generic non-degenerate case when $f'(0) < 0$.

4.2 The Case $V(1) > 0$

In this section, bistable nonlinearities (1.19) with condition (4.2) are focused on. In this case there is only one root of $V(u)$: $u = 0$. This implies that once $u > 0$, then $V(u) > 0$ is always true. By the arguments above, $E[u(\cdot, t)] \geq 0$ for any $t > 0$.

Theorem 4.1. *Let f satisfy conditions (1.19), (4.1), (4.4), and (4.5). Let $\phi(x)$ satisfy condition (1.23). Then $\lim_{t\to\infty} u(x,t) = 0$ uniformly in $\mathbb{R}$.*

Note that the conclusion of Theorem 4.1 is expected from the general theory of monotone semiflows [41]. The proof follows via a sequence of lemmas. In those lemmas, $D^1(\mathbb{R})$ is used to denote the space of functions equipped with the norm $\|u\|_{D^1} = \|u_x\|_{L^2}$ (see [26, Chapter 8] for a rigorous definition). In addition, the Banach space $L^{p+1} \cap D^1$ with the norm $\|u\|_{L^{p+1}\cap D^1} = \|u\|_{L^{p+1}} + \|u_x\|_{L^2}$ is introduced.

Lemma 4.1. *There exists an increasing sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n \to \infty$ such that*

1. $u_t(x,t_n) \to 0$, in $L^2(\mathbb{R})$;

2. $u(x,t_n) \rightharpoonup v$, in $L^{p+1}(\mathbb{R}) \cap D^1(\mathbb{R})$.

Proof. By Proposition 3.1 and (3.4), E is non-increasing in time. In addition, for any $T > 1$,

$$0 \le E[u(\cdot,2T)] \le E[u(\cdot,T)] \le E[u(\cdot,1)] < \infty. \tag{4.6}$$

Then the following equality holds:

$$E[u(\cdot,T)] - E[u(\cdot,2T)] = \int_T^{2T} \frac{dE[u(\cdot,t)]}{dt} dt = \int_T^{2T} \int_{\mathbb{R}} u_t^2(x,t)dxdt. \tag{4.7}$$

From the continuity in time of $\|u_t(x,t)\|_{L^2(\mathbb{R})}$, there exists $t_T \in [T, 2T]$ such that

$$\frac{E[u(\cdot,T)] - E[u(\cdot,2T)]}{T} = \int_{\mathbb{R}} u_t^2(x,t_T)dx. \tag{4.8}$$

As $T \to \infty$, $\int_{\mathbb{R}} u_t^2(x,t_T)dx \to 0$. It implies that there exists a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n \to \infty$ such that $u_t(x,t_n)$ converges to 0 in $L^2(\mathbb{R})$, i.e., the first assertion holds. It is only necessary to find a subsequence of $\{t_n\}$ satisfying the second assertion.

Since $V(u) > 0$ for all $u > 0$, by condition (4.5), coercivity of $V(u)$ follows:

$$V(u) \ge Cu^{p+1}, \tag{4.9}$$

where C is a positive constant. Then one has

$$E[u(\cdot,t)] \geq \int_{\mathbb{R}} \left(\frac{1}{2}u_x^2(x,t) + Cu^{p+1}(x,t)\right) dx \geq \min\left\{\frac{1}{2}, C\right\} (\|u\|_{L^{p+1}}^{p+1} + \|u_x\|_{L^2}^2). \tag{4.10}$$

Recalling that $E[u(x,t)]$ is non-increasing in time, there exists $M > 0$ such that

$$\|u(\cdot,t)\|_{L^{p+1}\cap D^1} \leq M,\ \forall t \geq 1. \tag{4.11}$$

Since $u(x,t_n)$ is a bounded sequence in $L^{p+1}(\mathbb{R}) \cap D^1(\mathbb{R})$, it has a weakly convergent subsequence, with weak limit v. This proves the second assertion. □

Lemma 4.2. *v in Lemma 4.1 is a classical solution of the stationary equation.*

Proof. From equation (1.22) at $t = t_n$, with $\{t_n\}$ chosen in Lemma 4.1, the following equation holds:

$$\frac{\partial u}{\partial t}(x,t_n) = \frac{\partial^2 u}{\partial x^2}(x,t_n) + f(u(x,t_n)). \tag{4.12}$$

Multiplying equation (4.12) by a test function $\phi \in C_c^\infty(\mathbb{R})$ and integrating over $\mathbb{R}$, one obtains

$$\int_{\mathbb{R}} \phi(x)\frac{\partial u}{\partial t}(x,t_n)dx = -\int_{\mathbb{R}} \frac{d\phi(x)}{dx}\frac{\partial u}{\partial x}(x,t_n)dx + \int_{\mathbb{R}} \phi(x)f(u(x,t_n))dx. \tag{4.13}$$

According to the first assertion of Lemma 4.1, it is known that $\int_{\mathbb{R}} \phi\frac{\partial u}{\partial t}(x,t_n)dx$ converges to 0. Then from the second assertion of Lemma 4.1, the continuity of $f(u)$, and Sobolev imbedding, the following equation holds:

$$-\int_{\mathbb{R}} \phi' v' dx + \int_{\mathbb{R}} \phi f(v)dx = 0. \tag{4.14}$$

It is the weak form of the stationary equation. Then by standard regularity arguments [7,20], v is a classical solution. □

Proposition 4.3. *There exists an increasing sequence $\{t_n\}$, $t_n \to \infty$ such that $u(x,t_n) \to 0$ in $L^\infty(\mathbb{R})$.*

Proof. At first, since the only stationary solution in $L^{p+1}(\mathbb{R}) \cap D^1(\mathbb{R})$ is $v = 0$, one has $u(x, t_n) \rightharpoonup 0$ in $L^{p+1}(\mathbb{R}) \cap D^1(\mathbb{R})$. Then for every t_n, the following translation is taken:

$$x_n = \max\{y \in \mathbb{R} : |u(y, t_n)| = \max_{x \in \mathbb{R}} |u(x, t_n)|\}.$$

Since $u(x, t_n) \in L^{p+1}(\mathbb{R}) \cap D^1(\mathbb{R})$, x_n is well-defined. Denote the translation operator τ_y, $y \in \mathbb{R}$ as $\tau_y(g(x)) = g(x - y)$. Then denote by $w_n = \tau_{x_n}(u(x, t_n))$. Since $u(x, t_n) \in L^{p+1}(\mathbb{R}) \cap D^1(\mathbb{R})$, $w_n \in L^{p+1}(\mathbb{R}) \cap D^1(\mathbb{R})$ also holds, and there is a subsequence $\{w_n\}$ such that $w_n \rightharpoonup w$ in $L^{p+1}(\mathbb{R}) \cap D^1(\mathbb{R})$.

Multiplying equation (1.22) for $\tau_{x_n} u(x, t)$ by a test function ϕ at $t = t_n$, the following equation holds:

$$\phi(x)(\tau_{x_n} u(x, t_n))_t = ((\tau_{x_n} u(x, t_n))_{xx} + f(\tau_{x_n} u(x, t_n)))\phi(x). \tag{4.15}$$

Since $u_t(x, t_n) \to 0$ in $L^2(\mathbb{R})$, by Lemma 4.1, one obtains $(\tau_{x_n} u(x, t_n))_t \to 0$ in $L^2(\mathbb{R})$. Integrating equation (4.15) and taking $n \to \infty$, by using $\tau_{x_n} u(x, t_n) = w_n \rightharpoonup w$, the following equation can be obtained:

$$-\int_{\mathbb{R}} \phi' w' dx + \int_{\mathbb{R}} \phi f(w) dx = 0. \tag{4.16}$$

It means that w is also a weak solution of the stationary equation. So $w = 0$.

Since $w_n \in L^{p+1}(\mathbb{R}) \cap D^1(\mathbb{R})$, by Sobolev imbedding, $w_n \rightharpoonup 0$ implies, upon extraction of a subsequence, that $w_n \to 0$ at $x = 0$. By the definition of w_n, one then has $w_n \to 0$ in $L^\infty(\mathbb{R})$. $\square$

Note that the convergence result above proves Theorem 4.1. Indeed, for any $\varepsilon > 0$, there exists $T = T(\varepsilon)$ such that $\max_{x \in \mathbb{R}} u(x, T) < \varepsilon$. In addition, for $\varepsilon < \theta_0$, $\bar{u} \equiv \varepsilon$ is a supersolution, which implies that $\lim_{t \to \infty} u(x, t) = 0$ uniformly in $\mathbb{R}$.

4.3 The Case $V(1) < 0$

In this section, bistable nonlinearities (1.19) with condition (4.1) are studied. There exist two roots of $V(u)$: $u = 0$, $u = \theta^* \in (0,1)$, and possibly a third root $u = \theta^\diamond > 1$. However, $\limsup_{t\to\infty} \|u(x,t)\|_{L^\infty(\mathbb{R})} \leq 1$ by (1.13), so without loss of generality, in the latter case $\|\phi\|_{L^\infty(\mathbb{R})} < \theta^\diamond$ can be assumed. This implies that once $u > \theta^*$, then $V(u) < 0$ is always true.

It is well known that under the assumptions (1.24) possesses "bump" solutions, i.e., classical positive solutions of (1.24) that vanish at infinity. After a suitable translation, these solutions are known to be symmetric decreasing and unique (see, e.g., [6, Theorem 5]). In the following proposition the properties of the bump solution, which are needed for the analysis, are summarized.

Proposition 4.4. *Let f satisfy conditions (1.19), (4.1), (4.4), and (4.5), and let $v \in C^2(\mathbb{R})$ be the unique positive symmetric decreasing solution of (1.24). Then*

1. *$v(0) = \theta^*$ and $E_0 := E[v] > 0$.*

2. *If $f'(0) < 0$, then $v(x), v'(x), v''(x) \sim e^{-\mu|x|}$ for $\mu = \sqrt{|f'(0)|}$ as $|x| \to \infty$.*

3. *If $f'(0) = 0$, then $v(x) \sim |x|^{-\frac{2}{p-1}}$, $v'(x) \sim |x|^{-\frac{p+1}{p-1}}$, $v''(x) \sim |x|^{-\frac{2p}{p-1}}$ as $|x| \to \infty$.*

4. *$f'(v(\cdot)) \in L^1(\mathbb{R})$ and $v' \in H^1(\mathbb{R})$*

Proof. The fact that $v(0) = \theta^*$ follows from [6, Theorem 5]. Integrating (1.24) once, $|v'| = \sqrt{2V(v)}$ can be obtained, where by the previous result the constant of integration is zero. Upon second integration,

$$|x| = \int_v^{\theta^*} \frac{du}{\sqrt{2V(u)}} \tag{4.17}$$

is obtained. The proof then follows by a careful analysis of the singularity in the integral in (4.17) to establish the decay of the solution. Once the decay is known, the rest of the statements follows straightforwardly. $\square$

The main theorems in this chapter are about the following convergence and equivalence conclusions.

Theorem 4.2. *Let f satisfy conditions (1.19), (4.1), (4.4), and (4.5). Let $\phi(x)$ satisfy condition (1.23) and hypothesis (SD). Then one of the following holds.*

1. $\lim_{t\to\infty} u(x,t) = 1$ *locally uniformly in* $\mathbb{R}$,

2. $\lim_{t\to\infty} u(x,t) = v(x)$ *uniformly in* $\mathbb{R}$,

3. $\lim_{t\to\infty} u(x,t) = 0$ *uniformly in* $\mathbb{R}$.

Theorem 4.2 will be proved together with establishing the following one-to-one relation between the long time behavior of the solutions and those of their energy E.

Theorem 4.3. *Under the same assumptions as in Theorem 4.2, the following three alternatives hold:*

1. $\lim\limits_{t\to\infty} u(x,t) = 1$ *locally uniformly in* $\mathbb{R} \Leftrightarrow \lim\limits_{t\to\infty} E[u(\cdot,t)] = -\infty$.

2. $\lim\limits_{t\to\infty} u(x,t) = v(x)$ *uniformly in* $\mathbb{R} \Leftrightarrow \lim\limits_{t\to\infty} E[u(\cdot,t)] = E_0$.

3. $\lim\limits_{t\to\infty} u(x,t) = 0$ *uniformly in* $\mathbb{R} \Leftrightarrow \lim\limits_{t\to\infty} E[u(\cdot,t)] = 0$.

The strategy of the proof is as follows. The limit behaviors of the energy in Theorem 4.3 will be proved to be the only possible ones. At first, if $E[u(\cdot,t)]$ is not bounded from below, then u converges to 1 locally uniformly. In addition, the reverse also holds. Then for bounded from below $E[u(\cdot,t)]$, the solution $u(x,t)$ converges to either 0 or $v(x)$. Finally, the convergence of $u(x,t)$ to 0 or $v(x)$ implies the corresponding convergence of energy.

$E[u(\cdot,t)]$ is assumed that it is unbounded from below at first. In this case, for cubic nonlinearity Flores proved in [18] that $\lim\limits_{t\to\infty} u(x,t) = 1$ locally uniformly by constructing a proper subsolution. Under (SD), the following stronger conclusion will be proved: if there exists $T \geq 0$ such that $E[u(\cdot,T)] < 0$, then propagation occurs, in the sense defined in the introduction. Throughout the rest of this chapter, the assumptions of the above theorems are always assumed to be satisfied, and $u(x,t)$ always refers to the solutions of (1.22) and (1.23).

Lemma 4.5. *Suppose there exists $c_0 > 0$ such that $\phi(x) \in H^1_{c_0}(\mathbb{R})$. If there exists $T \geq 0$ such that $E[u(\cdot,T)] < 0$, then $u(x,t)$ is wave-like.*

Proof. First observe that if $\phi(x) \in H^1_{c_0}(\mathbb{R})$, then $u(\cdot,T) \in H^1(\mathbb{R}) \cap H^1_{c_0}(\mathbb{R})$. Then for any small $\varepsilon > 0$, if $E[u(x,T)] = -\varepsilon < 0$ there exists $L > 0$ such that $V(u(x,T)) \geq 0$ for $|x| \geq L$, and

$$\int_{\{x \leq -L\}} \left(\frac{1}{2}u_x^2(x,T) + V(u(x,T))\right) dx < \frac{\varepsilon}{4}, \tag{4.18}$$

$$\int_{\{x \geq L\}} e^{c_0 x} \left(\frac{1}{2}u_x^2(x,T) + V(u(x,T))\right) dx < \frac{\varepsilon}{4}. \tag{4.19}$$

Note that if a smaller positive c is used to instead c_0 in the above inequality, the inequality still holds. And by the definition of L, the following equation hold:

$$\int_{\{|x| < L\}} \left(\frac{1}{2}u_x^2(x,T) + V(u(x,T))\right) dx < -\varepsilon. \tag{4.20}$$

So there is a sufficiently small $c \in (0, c_0)$ such that

$$\int_{\{|x| < L\}} e^{cx} \left(\frac{1}{2}u_x^2(x,T) + V(u(x,T))\right) dx < -\frac{\varepsilon}{2}, \tag{4.21}$$

and

$$\Phi_c[u(\cdot,T)] = \int_{\mathbb{R}} e^{cx} \left(\frac{1}{2}u_x^2(x,T) + V(u(x,T))\right) dx < 0. \tag{4.22}$$

So u is wave-like. $\square$

Then for symmetric decreasing solutions and bistable nonlinearities, it will be shown that the wave-like property also implies propagation in the sense of the introduction.

Lemma 4.6. *Suppose that $u(x,t)$ is wave-like. Then $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly in $\mathbb{R}$.*

Proof. In view of the definition of θ^*, there exists $\delta_0 > \theta^*$ in Proposition 3.7. Therefore, by that proposition

$$R_{\theta^*}(t) > \frac{ct}{2}, \tag{4.23}$$

for sufficiently large t. Then, because $u(x,t)$ is symmetric decreasing, for any $L > 0$ there exists $T_L > 0$ such that $u(x,t) > \theta^*$ on the interval $[-L, L]$, for any $t \geq T_L$. Now, consider $\underline{u}(x,t)$ solving (1.22) with $\underline{u}(x, T_L) = \theta^*$ for all $x \in (-L, L)$ and $\underline{u}(\pm L, t) = \theta^*$ for all $t > T_L$. Since by the assumption on the nonlinearity the function $\underline{u}(x, T_L)$ is a strict subsolution, in the spirit of [3, Proposition 2.2], $\underline{u}(\cdot, t) \to v_L$ uniformly on $[-L, L]$ can be proved, where v_L solves (1.24) with $v_L(\pm L) = \theta^*$. Then, by comparison principle,

$$v_L \leq \liminf_{t\to\infty} u(\cdot,t) \leq \limsup_{t\to\infty} u(\cdot,t) \leq 1 \qquad \text{uniformly in } [-L, L] \tag{4.24}$$

is obtained. Also, by standard elliptic estimates, $v_L \to \bar{v}$ locally uniformly as $L \to \infty$, where $\bar{v}$ solves (1.24) in the whole of $\mathbb{R}$. Since by construction $\bar{v} \geq \theta^*$, the only possibility is $\bar{v} = 1$. Then, passing to the limit in (4.24), the result is obtained. $\square$

A truncation argument is used in next lemma to extend the conclusion of Lemma 4.6 to solutions that are not necessarily lying in any exponentially weighted Sobolev space, but have negative energy at some time $T \geq 0$.

Lemma 4.7. *Suppose that there exists $T \geq 0$ such that $E[u(\cdot, T)] < 0$, then $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly in $\mathbb{R}$.*

Proof. For any $L > 0$, a cutoff function $\varphi_L(x) = \eta(|x|/L)$ can be constructed, where η is a non-increasing $C^\infty(\mathbb{R})$ function such that $\eta(x) = 1$ for $x < 1$, and $\eta(x) = 0$ for $x > 2$. Let $\hat{\phi}(x; L) = \varphi_L(x) u(x, T)$, so that $\hat{\phi}(x; L) \to u(x, T)$ in $H^1(\mathbb{R})$ as $L \to \infty$.

By the assumption and continuity of E, there exists a sufficiently large $L = L_0$, such that $E[\hat{\phi}(x; L_0)] < 0$. Note that $\hat{\phi}(x; L_0)$ is a compactly supported function, so it lies in $H_c^1(\mathbb{R})$ for any $c > 0$. Now consider the solution $\hat{u}(x,t)$ which satisfies (1.22) with initial condition $\hat{u}(x,0) = \hat{\phi}(x; L_0)$. From Lemma 4.6, it is known that $\lim_{t\to\infty} \hat{u}(x,t) = 1$ locally uniformly in $\mathbb{R}$. Because $u(x, t+T) \geq \hat{u}(x,t)$, the lemma is proved by comparison principle. $\square$

An obvious corollary to the above lemma is the following.

Corollary 4.8. *Suppose that* $\lim_{t\to\infty} E[u(\cdot,t)] = -\infty$, *then* $\lim_{t\to\infty} u(x,t) = 1$ *locally uniformly in* $\mathbb{R}$.

The next lemma provides a sufficient condition for propagation, which, in particular, yields a conclusion converse to that of Corollary 4.8.

Lemma 4.9. *Suppose that* $\lim_{t\to\infty} u(x,t) = 1$ *locally uniformly in* $\mathbb{R}$, *then* $\lim_{t\to\infty} E[u(\cdot,t)] = -\infty$.

Proof. It is argued by contradiction. Suppose that $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly in $\mathbb{R}$ and $E[u(\cdot,t)]$ is bounded below. Then for any $L > 0$, a cutoff function $\kappa_L(x) = \eta(|x| - L)$, can be constructed, where η is defined in the proof of Lemma 4.7. For any $L > 0$, $\kappa_L(x) = 1$ for $|x| < L+1$, $\kappa(x) = 0$ for $|x| > L+2$, and $|\kappa_L'(x)|$ is bounded. Since $u(x,t)$ is symmetric decreasing, κ_L, κ_L' are both bounded, and u, u_x are both bounded for all $t \geq 1$ by standard parabolic regularity. Then, for $\tilde{u}_L(x,t) := \kappa_L(x)u(x,t)$ with any $t \geq 1$ the following energy estimate holds:

$$
\begin{aligned}
E[\tilde{u}_L(x,t)] = 2\int_0^{L+1} V(u)dx + \int_0^{L+1} u_x^2 dx \\
+ \int_{L+1}^{L+2} \left\{ \left(\frac{\partial(\kappa_L u)}{\partial x} \right)^2 + 2V(\kappa_L u) \right\} dx \leq 2\int_0^{L+1} V(u)dx + C, \qquad (4.25)
\end{aligned}
$$

where the constant C is independent of L. Since $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly in $\mathbb{R}$, for every $L_0 > 0$ satisfying $(L_0+1)V(1) < -C$, constant $t_0 > 0$ can be chosen, such that $V(u(x,t_0)) < V(1)/2 < 0$ for any $x \in (-L_0-1, L_0+1)$. This implies that $\tilde{\phi}(x;L_0) = \kappa_{L_0}(x)u(x,t_0)$ satisfies $E[\tilde{\phi}(x;L_0)] < 0$.

Note that $\tilde{\phi}(x;L_0)$ is a compactly supported function, so it lies in $H_c^1(\mathbb{R})$ for any $c > 0$. Now consider the solution $\tilde{u}(x,t)$ that satisfies (1.22), with the initial condition $\tilde{u}(x,0) = \tilde{\phi}(x;L_0)$. By Proposition 3.7, Lemma 4.5, and the fact that

$$u(x,t+t_0) \geq \tilde{u}(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{4.26}$$

there exists $c > 0$ such that for any $t > t_0$,

$$R_{\theta^*} > c(t-t_0) + R_0, \tag{4.27}$$

for some constant $R_0 \in \mathbb{R}$. Moreover, there exists constant $T_0 > 0$ such that

$$u(x,t+t_0) \geq \theta^*, \tag{4.28}$$

for any $t > T_0$ and $|x| \leq ct/2$.

On the other hand, by (3.4) there exists a sufficiently large $t_\alpha \geq 0$ such that

$$\int_{t_\alpha}^{\infty} \int_{\mathbb{R}} u_t^2(x,t)dxdt < \alpha^2, \tag{4.29}$$

for every $\alpha > 0$. Taking $\alpha = \theta_0\sqrt{c}/9$, $t_1 > \max\{t_0, t_\alpha\}$, $x_1 = R_{\theta_0/2}(t_1)$, also taking $T > T_0$ such that $x_1 < cT/4$, and $t_2 = t_1 + T$, $x_2 = x_1 + cT$, then the following inequality holds by Cauchy-Schwarz inequality:

$$\begin{aligned} \int_{t_1}^{t_2}\int_{x_1}^{x_2} |u_t(x,t)|dxdt &\leq \sqrt{(x_2-x_1)(t_2-t_1)}\left(\int_{t_1}^{t_2}\int_{x_1}^{x_2} u_t^2(x,t)dxdt\right)^{1/2} \\ &\leq \sqrt{c}T\left(\int_{t_\alpha}^{\infty}\int_{\mathbb{R}} u_t^2(x,t)dxdt\right)^{1/2} \\ &\leq \frac{cT\theta_0}{9}. \end{aligned} \tag{4.30}$$

At the same time, by the definition of x_1, one has $0 < x_1 < cT/4$. Then the following inequality holds:

$$\begin{aligned} \int_{t_1}^{t_2}\int_{x_1}^{x_2} |u_t(x,t)|dxdt &\geq \int_{cT/4}^{cT/2}\left(\int_{t_1}^{t_2} |u_t(x,t)|dt\right)dx \\ &\geq \int_{cT/4}^{cT/2}(u(x,t_2)-u(x,t_1))dx. \end{aligned} \tag{4.31}$$

Since $t_2 > T > T_0$, $u(x,t_2) \geq \theta^* > \theta_0$ holds for $x \in (cT/4, cT/2)$. In addition, by the definitions of x_1 and T, $u(x,t_1) < \theta_0/2$ holds for $x \in (cT/4, cT/2)$. So that

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} |u_t(x,t)|dxdt \geq \frac{cT\theta_0}{8}, \tag{4.32}$$

which contradicts (4.30). $\square$

Note that the first equivalence in Theorem 4.3 is proved. Indeed, a stronger corollary is obtained.

Corollary 4.10. $\lim_{t\to\infty} u(x,t) = 1$ *locally uniformly in* $\mathbb{R}$, *if and only if there exists* $T \geq 0$ *such that* $\lim_{t\to\infty} E[u(\cdot,T)] < 0$.

Then the case that $E[u(\cdot,t)]$ bounded from below is considered. By Lemmas 4.7 and 4.9, boundedness of $E[u(\cdot,t)]$ implies $\lim_{t\to\infty} E[u(\cdot,t)] \geq 0$. In addition, it will be proved that either $\lim_{t\to\infty} u(x,t) = 0$ uniformly in $\mathbb{R}$, or $\lim_{t\to\infty} u(x,t) = v(x)$ uniformly in $\mathbb{R}$ in this case. The idea of the proof is due to Fife [15, Lemma 10]. However, Fife's arguments will be refined under the weaker assumptions on the nonlinearity and the (SD) hypothesis.

The next Lemma establishes existence of an increasing sequence $\{t_n\}$ tending to infinity on which the solution converges to a zero of $V(u)$ at the origin, thus allowing only two possibilities for the value of $\lim_{n\to\infty} u(0,t_n)$.

Lemma 4.11. *If* $E[u(\cdot,t)]$ *is bounded from below, there exists an increasing sequence* $\{t_n\}$ *with* $\lim_{n\to\infty} t_n = \infty$ *such that either* $\lim_{n\to\infty} u(0,t_n) = 0$, *or* $\lim_{n\to\infty} u(0,t_n) = \theta^*$.

Proof. Multiplying u_x on both sides of equation (1.22), and integrating the products over $(-\infty, 0)$, then one has the following equality:

$$\begin{aligned}\int_{-\infty}^{0} u_x(x,t)u_t(x,t)dx &= \int_{-\infty}^{0} (u_{xx}(x,t) + f(u(x,t)))u_x(x,t)dx \\ &= \frac{1}{2}u_x^2(x,t)\Big|_{x=-\infty}^{0} - (V(u(0,t)) - V(u(-\infty,t))) \\ &= -V(u(0,t)). \end{aligned} \tag{4.33}$$

From monotonicity of u on $(-\infty, 0)$ and standard parabolic regularity, for $t \geq 1$ the left-hand side of (4.33) can be controlled by

$$\begin{aligned}\left|\int_{-\infty}^{0} u_x(x,t)u_t(x,t)dx\right| &\leq \|u_t(\cdot,t)\|_{L^2(-\infty,0)}\|u_x(\cdot,t)\|_{L^2(-\infty,0)} \\ &\leq \|u_t(\cdot,t)\|_{L^2(\mathbb{R})}\|u_x(\cdot,t)\|^{1/2}_{L^\infty(\mathbb{R})}|u(0,t)|^{1/2} \\ &\leq \|u_t\|_{L^2(\mathbb{R})}\|u_x\|^{1/2}_{L^\infty(\mathbb{R}\times(1,\infty))}\max\{1, \|\phi\|^{1/2}_{L^\infty(\mathbb{R})}\}. \end{aligned} \tag{4.34}$$

where the Cauchy-Schwarz inequality is applied in the first line. Since $E[u(\cdot,t)]$ is bounded from below, one has

$$\int_1^\infty \int_{\mathbb{R}} u_t^2(x,t)dxdt < \infty \tag{4.35}$$

by (3.4). Therefore, there exists an unbounded increasing sequence $\{t_n\}$ such that $\lim_{n\to\infty} \|u_t(\cdot,t_n)\|_{L^2(\mathbb{R})} = 0$. Since also $\|u_x\|_{L^\infty(\mathbb{R}\times(1,\infty))} < \infty$ by standard parabolic regularity, this implies that $\lim_{n\to\infty} V(u(0,t_n)) = 0$ by (4.33). Furthermore, since $\limsup_{n\to\infty} \|u(\cdot,t_n)\|_{L^\infty(\mathbb{R})} \leq 1$, by the assumptions on the nonlinearity either $\lim_{t\to\infty} u(0,t_n) = 0$ or $\lim_{t\to\infty} u(0,t_n) = \theta^*$. □

Remark 4.12. *The sequence $\{t_n\}$ in Lemma 4.11 satisfies $\|u_t(\cdot,t_n)\|_{L^2(\mathbb{R})} \to 0$ and can be chosen so as $t_{n+1} - t_n \leq 1$ for every n.*

The next result treats the first alternative in Lemma 4.11.

Lemma 4.13. *Suppose that there exists an increasing sequence* $\{t_n\}$ *such that* $\lim_{n\to\infty} t_n = \infty$*, and* $\lim_{n\to\infty} u(0, t_n) = 0$*, then* $\lim_{t\to\infty} u(x, t) = 0$ *uniformly in* $\mathbb{R}$*.*

Proof. Recall that the maximum of solution u is always at the origin. By the structure of the nonlinearity f, it is known that once $\max_{x\in\mathbb{R}} u(x, T) < \theta_0$ for some $T \geq 0$, then $\lim_{t\to\infty} u(x, t) = 0$ uniformly in $\mathbb{R}$. □

Combining the results of Lemma 4.11 and Lemma 4.13, the following result can be proved.

Lemma 4.14. *Suppose that* $E[u(\cdot, t)]$ *is bounded from below, then either* $\lim_{t\to\infty} u(x, t) = 0$*, or* $\lim_{t\to\infty} u(x, t) = v(x)$*, uniformly in* $\mathbb{R}$*.*

Proof. From Lemma 4.11 and Lemma 4.13, it is only necessary to prove that if the increasing sequence $\{t_n\}$ in Lemma 4.11 satisfies $\lim_{n\to\infty} u(0, t_n) = \theta^*$, then $\lim_{t\to\infty} u(x, t) = v(x)$ uniformly in $\mathbb{R}$. To prove this, it is necessary to prove the locally uniform convergence on the sequence $\{t_n\}$ first. Let $w(x, t) := u(x, t) - v(x)$, then in view of $v(0) = \theta^*$ by Proposition 4.4 the following equation holds:

$$w_t = w_{xx} + f'(\tilde{u})w, \quad w_x(0, t) = 0, \quad w(0, t) = u(0, t) - \theta^*, \tag{4.36}$$

where $\tilde{u}$ is between u and v. Then it is necessary to prove that

$$\lim_{n\to\infty} w(x, t_n) = 0, \tag{4.37}$$

locally uniformly in $\mathbb{R}$. The proof follows from the continuous dependence on the data for solutions of the initial value problem in x obtained from (4.36) for each $t = t_n$ fixed. Indeed, at $t = t_n \geq 1$, denoting $w_n(x) := w(x, t_n)$, $g_n(x) := u_t(x, t_n)$, $K_n(x) := f'(\tilde{u}(x, t_n))$, $\alpha_n := u(0, t_n) - \theta^*$, and considering (4.36) as an ordinary differential equation in $x > 0$, one has the following equation:

$$w_n'' = g_n - K_n w_n, \quad w_n'(0) = 0, \quad w_n(0) = \alpha_n. \tag{4.38}$$

For any $L > 0$,

$$\begin{aligned}\max_{0\le x\le L}|w_n'(x)| &\le \sqrt{L}\|g_n\|_{L^2(\mathbb{R})} + L\|K_n\|_{L^\infty(\mathbb{R})}\max_{0\le x\le L}|w_n(x)| \\ &\le \sqrt{L}\|g_n\|_{L^2(\mathbb{R})} + L\mathcal{K}\max_{0\le x\le L}|w_n(x)|, \end{aligned} \tag{4.39}$$

holds by integration over $(0, L)$ and an application of Cauchy-Schwarz inequality, where the constant $\mathcal{K}$ satisfies

$$|f'(s)| \le \mathcal{K}, \ \ 0 \le s \le \max\{1, \|\phi(x)\|_{L^\infty(\mathbb{R})}\}. \tag{4.40}$$

For fixed $L > 0$, there is a sufficiently large integer l such that $2\delta L\mathcal{K} \le 1$ for $\delta := L/l$. By taking

$$W_{n,k} := \max_{(k-1)\delta\le x\le k\delta}|w_n(x)|, \quad k \in \mathbb{N}, \tag{4.41}$$

$$m_{n,0} := \alpha_n, \qquad m_{n,k} := \max_{1\le k'\le k} W_{n,k'}, \tag{4.42}$$

$m_{n,k}$ is non-decreasing in k, and $m_{n,k} = \max_{0\le x\le k\delta}|w_n(x)|$. By (4.39) and the choice of δ,

$$\begin{aligned} m_{n,k} - m_{n,k-1} &\le \delta \max_{0\le x\le L}|w_n'(x)| \\ &\le \delta(\sqrt{L}\|g_n\|_{L^2(\mathbb{R})} + L\mathcal{K}m_{n,k}) \\ &\le \delta\sqrt{L}\|g_n\|_{L^2(\mathbb{R})} + \frac{m_{n,k}}{2} \end{aligned} \tag{4.43}$$

holds for any $1 \le k \le l$. This implies that

$$m_{n,k} \le 2m_{n,k-1} + G_n, \tag{4.44}$$

for any $1 \le k \le l$, where $G_n := 2\delta\sqrt{L}\|g_n\|_{L^2(\mathbb{R})}$. Since by definition $m_{n,0} = \alpha_n$,

$$\max_{-L\le x\le L}|w_n(x)| = m_{n,l} \le 2^l\alpha_n + (2^l - 1)G_n \tag{4.45}$$

holds by iteration and symmetry of $w_n(x)$.

Then, as $n \to \infty$, by Lemma 4.11 and Remark 4.12, one knows that $u(0,t_n) - \theta^* \to 0$ and $\|u_t(x,t_n)\|_{L^2(\mathbb{R})} \to 0$, so that $\alpha_n \to 0$, $G_n \to 0$, and $\max_{-L \le x \le L} |w_n(x)| \to 0$, i.e., $u(x,t_n)$ converges to $v(x)$ locally uniformly. Then by Proposition 3.8 and the fact that by Remark 4.12 the sequence $\{t_n\}$ can be chosen so as $t_{n+1} - t_n \le 1$, the full limit convergence is obtained. Indeed, since the Hölder constant in t of $u(x,t)$ converges to 0 as $n \to \infty$ uniformly for all $|x| \le L$ and all $t_n < t < t_{n+1}$,

$$|u(x,t) - v(x)| \le |u(x,t_n) - v(x)| + |u(x,t) - u(x,t_n)| \to 0 \text{ as } n \to \infty \tag{4.46}$$

is obtained.

Finally, it can be proved that the convergence of $u(x,t)$ to $v(x)$ is, in fact, uniform. Indeed, since $u(x,t)$ is symmetric decreasing in x and $v(x) \to 0$ as $|x| \to \infty$,

$$\begin{aligned}
\sup_{|x| \ge L} |w(x,t)| &= \sup_{|x| \ge L} |u(x,t) - v(x)| \\
&\le \max_{|x| \ge L} \{u(x,t), v(x)\} \\
&\le \max\{u(L,t), v(L)\} \\
&\le v(L) + \max_{|x| \le L} |w(x,t)|
\end{aligned} \tag{4.47}$$

holds for any $L > 0$, $t > 0$. This implies that

$$\sup_{x \in \mathbb{R}} |w(x,t)| \le v(L) + \max_{|x| \le L} |w(x,t)|. \tag{4.48}$$

Then, for any $\varepsilon > 0$, there exists a sufficiently large constant $L > 0$ such that $v(L) < \varepsilon/2$. In addition, there exists a constant $T > 0$ such that $|w(x,t)| < \varepsilon/2$ for any $x \in [-L, L]$ and any $t > T$. So that

$$\lim_{t \to \infty} |w(x,t)| = 0, \tag{4.49}$$

uniformly in $x \in \mathbb{R}$, which proves the lemma. $\square$

Note that in view of the results in the preceding lemmas, by proving Lemma 4.14, Theorem 4.2 is proved.

Remark 4.15. *By standard parabolic regularity, under the assumptions of Lemma 4.14,*

$$\lim_{t\to\infty} u(x,t) = v(x) \text{ in } C^1(\mathbb{R}) \tag{4.50}$$

also holds.

Then the limit value of energy will be studied. At first, it will be proved that the energy of the solution goes to zero, provided extinction occurs.

Lemma 4.16. *If* $\lim_{t\to\infty} u(x,t) = 0$ *uniformly in* $\mathbb{R}$*, then* $\lim_{t\to\infty} E[u(\cdot,t)] = 0$.

Proof. From the (SD) hypothesis, the following inequality holds:

$$\begin{aligned} \int_{\mathbb{R}} \frac{1}{2} u_x^2(x,t)dx &= \int_0^\infty u_x^2(x,t)dx \\ &\leq \|u_x(x,t)\|_{L^\infty(\mathbb{R})}\, u(0,t). \end{aligned} \tag{4.51}$$

By standard parabolic regularity, if $\lim_{t\to\infty} u(x,t) = 0$ uniformly in $\mathbb{R}$, then

$$\lim_{t\to\infty} \int_{\mathbb{R}} \frac{1}{2} u_x^2(x,t)dx \to 0. \tag{4.52}$$

So it is only necessary to show that $\lim_{t\to\infty} \int_{\mathbb{R}} V(u(x,t))dx = 0$.

If $f'(0) < 0$, there exists $C > 0$ such that

$$0 \leq V(u) \leq Cu^2, \tag{4.53}$$

for sufficiently small u. Then from the usual energy estimate, $\lim_{t\to\infty} \|u(\cdot,t)\|^2_{L^2(\mathbb{R})} = 0$ exponentially, so that $\lim_{t\to\infty} \int_{\mathbb{R}} V(u(x,t))dx = 0$ as well. Alternatively, if $f'(0) = 0$, then by (4.5) the estimate

$$0 \leq V(u) \leq Cu^{p+1} \tag{4.54}$$

can be obtained for some $C > 0$ and sufficiently small u. So it is sufficient to show that $\lim_{t\to\infty} \|u(\cdot,t)\|^{p+1}_{L^{p+1}(\mathbb{R})} = 0$. In view of (4.5) the solution $\bar{u}(x,t)$ of the heat equation:

$$\bar{u}_t = \bar{u}_{xx},\ x \in \mathbb{R},\ t > T, \qquad \bar{u}(x,T) = u(x,T),\ x \in \mathbb{R}, \tag{4.55}$$

can be used as a supersolution to obtain (see, e.g., [43, Proposition 48.4])

$$\|u(\cdot,t)\|_{L^{p+1}(\mathbb{R})} \le \|\bar{u}(\cdot,t)\|_{L^{p+1}(\mathbb{R})} \le (4\pi t)^{-\frac{p-1}{4(p+1)}} \|u(\cdot,T)\|_{L^2(\mathbb{R})} \to 0 \text{ as } t \to \infty, \tag{4.56}$$

and the statement follows. □

If, on the other hand, $\lim_{t\to\infty} u(x,t) = v(x)$ uniformly in $\mathbb{R}$, then it can be proved that $E[u(\cdot,t)]$ has a limit as $t \to \infty$, and the value of the limit is equal to E_0 defined in Proposition 4.4. The proof begins with the analysis of the non-degenerate case.

Lemma 4.17. *Suppose that $f'(0) < 0$, then $\lim_{t\to\infty} u(x,t) = v(x)$ uniformly in $\mathbb{R}$ implies $\lim_{t\to\infty} E[u(\cdot,t)] = E_0$.*

Proof. At first, it will be shown that for any fixed $L > 0$, the energy $E[u(\cdot,t);L]$ of $u(x,t)$ restricted to $[-L,L]$, namely $E[u(\cdot,t);L] := \int_{-L}^{L} \left(\frac{1}{2}u_x^2 + V(u)\right) dx$, converges to the energy $E[v;L]$ of $v(x)$ restricted to $[-L,L]$. Then it will be proved that $E[u(\cdot,t)] - E[u(\cdot,t);L]$ converges to $E_0 - E[v;L]$ for sufficiently large L.

Since $|v'| = \sqrt{2V(u)}$ on the interval $[-L,L]$, which follows upon integration of (1.24),

$$E[v;L] = \int_{-L}^{L} \left(\frac{1}{2}|v'|^2 + V(v)\right) dx = 2\sqrt{2} \int_{v(L)}^{\theta^*} \sqrt{V(u)} du \tag{4.57}$$

is obtained. It is also obtained that $u(x,t) \to v(x)$, $u_x(x,t) \to v'(x)$ uniformly in $x \in [-L,L]$, as $t \to \infty$, by Lemma 4.15. This implies that

$$\lim_{t\to\infty} E[u(\cdot,t);L] = E[v;L]. \tag{4.58}$$

By symmetry of the solution, the remaining part of energy can be estimated as follows:

$$E[u(\cdot,t)] - E[u(\cdot,t);L] = \int_L^\infty (u_x^2(x,t) + 2V(u(x,t)))dx. \tag{4.59}$$

And by decrease of the solution for $x > 0$,

$$\int_L^\infty u_x^2(x,t)dx \le u(L,t)\|u_x(x,t)\|_{L^\infty(\mathbb{R})}. \tag{4.60}$$

is obtained. By standard parabolic regularity, for $t \ge 1$, the above expression converges to 0 as $L \to \infty$. In addition, $E[v] - E[v;L] \to 0$ holds as $L \to \infty$. So it is only necessary to show that for any $\delta > 0$ there exist a sufficiently large $L_\delta > 0$, $T_\delta > 0$ such that for any $t > T_\delta$,

$$\left| \int_{L_\delta}^\infty V(u(x,t))dx \right| < \delta. \tag{4.61}$$

If $f'(0) < 0$, then there exists $K > 0$ such that $f(u) \le -Ku$ for all $u \in [0, \theta_0/2]$. The proof of the lemma can be completed by an L^2 decay estimate similar to the one in the proof of Lemma 4.16. Taking $L > 0$ satisfying $v(L) < \theta_0/4$, there exists $T > 0$ such that $u(x,t) < \theta_0/2$ for any $x \in (L,\infty)$ and any $t > T$. Then

$$\begin{aligned} \frac{d}{dt}\int_L^\infty u^2 dx &= 2\int_L^\infty u(x,t)(u_{xx}(x,t) + f(u))dx \\ &\le 2u(L,t)|u_x(L,t)| - 2K\int_L^\infty u^2(x,t)dx \end{aligned} \tag{4.62}$$

holds for $t > T$. Since $\lim_{t\to\infty} u(L,t)|u_x(L,t)| = v(L)|v'(L)|$, from the above inequality and the relation $0 \le V(u(x,t)) \le Cu^2$ on $u \in [0,\theta_0]$ for some $C > 0$, there exists $\hat{T} > T$ such that for any $t > \hat{T}$

$$0 \le \int_L^\infty V(u(x,t))dx < \frac{2Cv(L)|v'(L)|}{K}. \tag{4.63}$$

Since $v(L)v'(L) \to 0$ as $L \to \infty$, the desired conclusion is proved. □

Now to the degenerate case.

Lemma 4.18. *If $f'(0) = 0$, then $\lim_{t\to\infty} u(x,t) = v(x)$ uniformly in $\mathbb{R}$ implies $\lim_{t\to\infty} E[u(\cdot,t)] = E_0$, when (4.4) and (4.5) hold.*

Proof. In the spirit of Lemma 4.17, it is only necessary to show that

$$\limsup_{t\to\infty} \int_L^\infty V(u(x,t))dx \to 0 \ \text{ as } L \to \infty. \tag{4.64}$$

By (4.5),

$$0 \le V(u) \le \frac{2ku^{p+1}}{p+1} \qquad \forall u \in [0,\delta], \tag{4.65}$$

for any sufficiently small $\delta > 0$. Furthermore, by Proposition 4.4, a constant $L \sim \delta^{-\frac{p-1}{2}} \gg 1$ can be chosen such that $v(L) = \delta/2$. Because $u(L,t)$ converges to $v(L)$ as $t \to \infty$, for sufficiently large t, $u(x,t) \le \delta$ holds for all $x \ge L$ and

$$0 \le \int_L^\infty V(u(x,t))dx \le \frac{2k}{p+1} \|u(\cdot,t)\|^{p+1}_{L^{p+1}(L,\infty)}. \tag{4.66}$$

Then it is only necessary to control $\|u(\cdot,t)\|_{L^{p+1}(L,\infty)}$ by δ for large enough t.

Denote by $\bar{v}(x)$ a shift of the bump solution $v(x)$ from Proposition 4.4 which satisfies $0 < \bar{v} \le \delta$ for all $x > L$ and

$$\begin{cases} 0 &= \bar{v}'' + f(\bar{v}), \quad x > L, \\ \bar{v}(L) &= \delta, \\ \bar{v}(\infty) &= 0. \end{cases} \tag{4.67}$$

Then a supersolution $\bar{u}$ can be constructed, which solves the half-line problem:

$$\begin{cases} \bar{u}_t &= \bar{u}_{xx} + f(\bar{u}), \quad x > L, \ t > T, \\ \bar{u}(L,t) &= \delta, \\ \bar{u}(x,T) &= \max\{u(x,T), \bar{v}(x)\}. \end{cases} \tag{4.68}$$

Note that since $\hat{u}(x,t) \equiv \delta$ is a supersolution for $\bar{u}(x,t)$, $\bar{u}(x,t) \le \delta$ for all $x \ge L$ and $t \ge T$. And by comparison principle, $u(x,t) \le \bar{u}(x,t)$ for all $x \ge L$ and $t \ge T$.

By definition

$$w(x,t) := \bar{u}(x,t) - \bar{v}(x) \geq 0, \ \ x > L, \ t > T, \tag{4.69}$$

$w(x,t)$ satisfies the linear equation:

$$w_t = w_{xx} + f'(\tilde{w})w, \ \ x > L, \ t > T, \tag{4.70}$$

for some $\bar{v} \leq \tilde{w} \leq \bar{u}$, with homogeneous Dirichlet boundary condition

$$w(L,t) = 0, \ \ t > T. \tag{4.71}$$

Since $0 \leq w(x,T) \leq u(x,T)$, $w(\cdot,T) \in L^2(\mathbb{R})$ holds by Proposition 3.1. Furthermore, in view of (4.4) the solution $\bar{w}$ of the heat equation with the same initial and boundary conditions:

$$\bar{w}_t = \bar{w}_{xx}, \ x > L, \ t > T, \qquad \bar{w}(L,t) = 0, \ t > T, \qquad \bar{w}(x,T) = w(x,T), \ x > L, \tag{4.72}$$

is a supersolution for w. Then, by the estimate similar to the one in (4.56) and comparison principle, there exists $C > 0$ such that

$$\|w(\cdot,t)\|_{L^{p+1}(\mathbb{R})} \leq \|\bar{w}(\cdot,t)\|_{L^{p+1}(\mathbb{R})} \leq Ct^{-\frac{p-1}{4(p+1)}} \|w(\cdot,T)\|_{L^2(\mathbb{R})} \to 0 \text{ as } t \to \infty. \tag{4.73}$$

Estimating $\|\bar{u}(\cdot,t)\|_{L^{p+1}(\mathbb{R})}$ in terms of $\|w(\cdot,t)\|_{L^{p+1}(\mathbb{R})}$, the following inequality is obtained:

$$\|\bar{u}(\cdot,t)\|_{L^{p+1}(L,\infty)} \leq \|w(\cdot,t)\|_{L^{p+1}(L,\infty)} + \|\bar{v}\|_{L^{p+1}(L,\infty)} \qquad \forall t \geq T. \tag{4.74}$$

On the other hand, it is clear that the estimates in Proposition 4.4 apply to $\bar{v}$ as well. Therefore

$$\|\bar{v}\|^{p+1}_{L^{p+1}(L,\infty)} \leq C\delta^{\frac{p+3}{2}}, \tag{4.75}$$

for some $C > 0$ and all $\delta > 0$ sufficiently small. Finally, combining (4.73) and (4.75) in (4.74), and by comparison principle, it is known that $\|u(\cdot,t)\|_{L^{p+1}(L,\infty)}$ can be made arbitrarily small for all $t \geq T$ by choosing a sufficiently small δ in the limit $t \to \infty$. $\square$

Note that the Theorem 4.3 is proved.

Finally, the question of threshold phenomena is considered. A similar notations as in [10] is used. Let $X := \{\phi(x) : \phi(x)$ satisfies (1.23) and the (SD) hypothesis$\}$. A one-parameter family of initial conditions ϕ_λ, $\lambda > 0$ is considered, which satisfy the following conditions:

(P1) For any $\lambda > 0$, $\phi_\lambda \in X$, the map $\lambda \mapsto \phi_\lambda$ is continuous from $\mathbb{R}_+$ to $L^2(\mathbb{R})$;

(P2) If $0 < \lambda_1 < \lambda_2$, then $\phi_{\lambda_1} \leq \phi_{\lambda_2}$ and $\phi_{\lambda_1} \neq \phi_{\lambda_2}$ in $L^2(\mathbb{R})$.

(P3) $\lim\limits_{\lambda \to 0} \phi_\lambda(x) = 0$ in $L^2(\mathbb{R})$.

$u_\lambda(x,t)$ is denoted by the solution of (1.22) with the initial datum ϕ_λ.

Here is the main result concerning sharp transition and threshold phenomena for bistable nonlinearities.

Theorem 4.4. *Under the same conditions as in Theorem 4.2, suppose that (P1) through (P3) hold. Then one of the following two conclusions is true:*

1. $\lim\limits_{t\to\infty} u_\lambda(x,t) = 0$ *uniformly in* $\mathbb{R}$ *for every* $\lambda > 0$*;*

2. There exists $\lambda^* > 0$ *such that*

$$\lim_{t\to\infty} u_\lambda(x,t) = \begin{cases} 0, & \text{uniformly in } \mathbb{R}, \quad \text{for } 0 \leq \lambda < \lambda^*, \\ v(x), & \text{uniformly in } \mathbb{R}, \quad \text{for } \lambda = \lambda^*, \\ 1, & \text{locally uniformly in } \mathbb{R}, \quad \text{for } \lambda > \lambda^*. \end{cases}$$

Proof. The following definitions are used:

$$\Sigma_0 := \{\lambda > 0 : \; u_\lambda(x,t) \to 0 \; as \; t \to \infty \text{ uniformly in } x \in \mathbb{R}\},$$

$$\Sigma_1 := \{\lambda > 0 : \ u_\lambda(x,t) \to 1 \ as \ t \to \infty \text{ locally uniformly in } x \in \mathbb{R}\}.$$

It is known that $\lambda \in \Sigma_0$ if and only if there exists $T \geq 0$ such that $u(0,T) < \theta_0$. Clearly the set Σ_0 is open. Furthermore, by comparison principle, if $\hat{\lambda} \in \Sigma_0$, then for any $\lambda < \hat{\lambda}$, $\lambda \in \Sigma_0$. So Σ_0 is an open interval.

If $\Sigma_0 \neq (0,\infty)$, then the set Σ_1 is an open interval (semi-infinite) as well. Indeed, by Corollary 4.10 for every $\lambda \in \Sigma_1$ there exists $T \geq 0$ such that $E[u_\lambda(\cdot,T)] < 0$. Then by continuity of the energy functional in $H^1(\mathbb{R})$ and continuous dependence in $H^1(\mathbb{R})$ of the solution at $t > 0$ on the initial data in $L^2(\mathbb{R})$ (see Proposition 3.1), there exists $\delta > 0$ such that $E[u_{\lambda'}(\cdot,T)] < 0$ holds for all $|\lambda' - \lambda| < \delta$. Hence $\lambda' \in \Sigma_1$ as well. In addition, by comparison principle, if $\tilde{\lambda} \in \Sigma_1$, then for any $\lambda > \tilde{\lambda}$, $\lambda \in \Sigma_1$. Then it is known that $\mathbb{R}_+ \setminus (\Sigma_0 \cup \Sigma_1)$ is a closed set, and, more precisely, a closed interval.

It is necessary to prove that the set $\mathbb{R}_+ \setminus (\Sigma_0 \cup \Sigma_1)$ contains only one point, if it is not empty. The Schrödinger-type operator

$$\mathfrak{L} = -\frac{d^2}{dx^2} + V(x), \qquad V(x) := -f'(v(x)), \tag{4.76}$$

and the associated Rayleigh quotient (for technical background, see, e.g., [26, Chapter 11]):

$$\mathfrak{R}(\phi) := \frac{\int_{\mathbb{R}} \left(|\phi'|^2 + V(x)\phi^2\right) dx}{\int_{\mathbb{R}} \phi^2 dx}. \tag{4.77}$$

are employed. Since $v' \in H^1(\mathbb{R})$ by Proposition 4.4, translational symmetry of the problem yields (weakly differentiate (1.24) and test with v'):

$$\mathfrak{R}(v') = 0. \tag{4.78}$$

Furthermore, since the function v' changes sign and $\lim_{|x|\to\infty} V(x) \geq 0$, v' is not a minimizer of $\mathfrak{R}$. Therefore, since $\inf \mathfrak{R} < 0$, and $V - \lim\limits_{|x|\to\infty} V(x) \in L^1(\mathbb{R})$ is obtained by Proposition 4.4, then there exists a positive function $\phi_0 \in H^1(\mathbb{R})$ that minimizes $\mathfrak{R}$, with $\min_{\phi\in H^1(\mathbb{R})}\mathfrak{R}(\phi) =: \nu_0 < 0$ [26, Theorem 11.5]. Approximating ϕ_0 by a function with compact support and using it as a test function, then $\min_{\phi\in H_0^1(-L,L)}\mathfrak{R}(\phi) =: \nu_0^L < 0$ holds for a sufficiently large $L > 0$, and in this case there exists a positive minimizer $\phi_0^L \in H_0^1(-L,L)\cap C^2(-L,L)\cap C^1([-L,L])$ such that

$$\mathfrak{L}(\phi_0^L) = \nu_0^L\phi_0^L. \tag{4.79}$$

If Σ_1 is not empty and the threshold set $\mathbb{R}_+\setminus(\Sigma_0\cup\Sigma_1)$ does not contain only one point, then there exist two distinct values $0<\lambda_1<\lambda_2$ in the threshold set. Since $f(u)\in C^1([0,\infty))$, $f'(u)$ is uniformly continuous on $[0,\max\{1,\|\phi\|_{L^\infty}\}]$. Thus, there exists $\delta>0$ such that

$$|f'(u_1)-f'(u_2)| < \frac{|\nu_0^L|}{2}, \tag{4.80}$$

for any $u_1,u_2\in[0,\max\{1,\|\phi\|_{L^\infty}\}]$ satisfying $|u_1-u_2|<\delta$. By the definition of the threshold set $\mathbb{R}_+\setminus(\Sigma_0\cup\Sigma_1)$, $\lim\limits_{t\to\infty} u_{\lambda_{1,2}}(x,t) = v(x)$ uniformly in $x\in\mathbb{R}$. Then, there exists T sufficiently large, such that $|u_{\lambda_{1,2}}(x,t)-v(x)|<\delta$ for any $t\geq T$ and all $x\in\mathbb{R}$. So that

$$\max_{x\in[-L,L]}|f'(v(x))-f'(\tilde{u}(x,t))| < \frac{|\nu_0^L|}{2}, \tag{4.81}$$

for every $u_{\lambda_1}(x,t)\leq\tilde{u}(x,t)\leq u_{\lambda_2}(x,t)$ and all $t\geq T$. However, taking $w(x,t) = u_{\lambda_2}(x,t)-u_{\lambda_1}(x,t)$, then $w(x,t)$ satisfies the following equation,

$$w_t = w_{xx} + f'(\tilde{u})w, \quad x\in\mathbb{R},\ t>0, \tag{4.82}$$

for some $u_{\lambda_1}(x,t)\leq\tilde{u}(x,t)\leq u_{\lambda_2}(x,t)$. By the strong maximum principle $w(x,t)>0$ for any $x\in\mathbb{R}$ and $t>0$. Hence there exists $\varepsilon>0$ such that $w(x,T)>\varepsilon\phi_0^L(x)$.

Taking $\varepsilon\phi_0^L(x) =: \underline{w}(x,t)$, then

$$\begin{aligned}
\underline{w}_t - \underline{w}_{xx} - f'(\tilde{u})\underline{w} &= -\underline{w}_{xx} - f'(v)\underline{w} + (f'(v) - f'(\tilde{u}))\underline{w} \\
&= \nu_0^L \underline{w} + (f'(v) - f'(\tilde{u}))\underline{w} \\
&\leq \frac{\nu_0^L}{2}\underline{w} \\
&\leq 0, \qquad (4.83)
\end{aligned}$$

which implies that $\underline{w}(x,t)$ is a subsolution for $t \geq T$. So by comparison principle

$$u_{\lambda_2} - u_{\lambda_1} \geq \varepsilon\phi_0^L(x), \quad \forall t \geq T, \qquad (4.84)$$

i.e., there exists a barrier between u_{λ_1} and u_{λ_2}, which contradicts the assumption that both $u_{\lambda_1}(x,t)$ and $u_{\lambda_2}(x,t)$ converge to $v(x)$ uniformly in $\mathbb{R}$, as $t \to \infty$. It means that if $\mathbb{R}_+ \setminus (\Sigma_0 \cup \Sigma_1)$ is not empty, then it only contains one point. $\square$

Remark 4.19. *By Corollary 4.10 and comparison principle, to ensure that $\lambda^* < \infty$ in Theorem 4.4 it is enough if there exists $\lambda > 0$ and $\tilde{\phi}_\lambda \in L^2(\mathbb{R})$ such that $0 \leq \tilde{\phi}_\lambda \leq \phi_\lambda$ and $E[\tilde{\phi}_\lambda] < 0$. This condition is easily seen to be verified for the family of characteristic functions of growing symmetric intervals studied by Kanel' [23]. Also, by Theorem 4.3 and the monotone decrease of the energy evaluated on solutions the condition $E[\phi_\lambda] < E_0$ for some $\lambda > 0$ implies that $u_\lambda(x,t) \not\to v(x)$. In particular, if $\sup_{0<\lambda<\bar{\lambda}} E[\phi_\lambda] < E_0$, then $\lambda^* > \bar{\lambda}$.*

CHAPTER 5

MONOSTABLE NONLINEARITY

In this chapter, the monostable nonlinearity is studied, i.e., $f(u) \in C^1([0,\infty),\mathbb{R})$ satisfies condition (1.15). Moreover, $f(u)$ also satisfies

$$f'(0) = 0. \tag{5.1}$$

Typical examples are the Arrhenius type combustion nonlinearity

$$f(u) = (1-u)e^{-\frac{a}{u}}, \quad a > 0, \tag{5.2}$$

and the generalized Fisher nonlinearity, i.e., the nonlinearity

$$f(u) = u^p(1-u), \tag{5.3}$$

with exponent $p > 1$.

Under conditions (1.15), there exists one root of $V(u)$: $u = 0$, and possibly a second root $u = \theta^\diamond > 1$. However, since $\lim\limits_{t\to\infty} \|u(x,t)\|_{L^\infty(\mathbb{R})} \leq 1$, without loss of generality, it can be supposed that $\|\phi\|_{L^\infty(\mathbb{R})} < \theta^\diamond$. So that $V(u) \leq 0$ always holds.

The following theorems are about convergence and one-to-one relations between the limit value of the energy and the long time behavior of solutions, similar to the bistable case.

Theorem 5.1. *Let f satisfy conditions (1.15) and (5.1), and let $\phi(x)$ satisfy condition (1.23) and hypothesis (SD). Then one of the following holds.*

1. $\lim\limits_{t\to\infty} u(x,t) = 1$ *locally uniformly in* $\mathbb{R}$*,*

2. $\lim\limits_{t\to\infty} u(x,t) = 0$ *uniformly in* $\mathbb{R}$*.*

Theorem 5.2. *Under the same conditions as in Theorem 5.1, the following one-to-one relation holds.*

1. $\lim_{t\to\infty} u(x,t) = 1$ *locally uniformly in* $\mathbb{R} \Leftrightarrow \lim_{t\to\infty} E[u(\cdot,t)] = -\infty$.

2. $\lim_{t\to\infty} u(x,t) = 0$ *uniformly in* $\mathbb{R} \Leftrightarrow \lim_{t\to\infty} E[u(\cdot,t)] = 0$.

Throughout the rest of this chapter, the hypotheses of Theorem 5.1 are assumed to be satisfied. The following conclusion is straight from the structure of nonlinearity.

Lemma 5.1. *If* $\lim_{t\to\infty} u(x,t) = 1$ *locally uniformly in* $\mathbb{R}$*, then* $\lim_{t\to\infty} E[u(\cdot,t)] = -\infty$*. And if* $\lim_{t\to\infty} u(x,t) = 0$ *uniformly in* $\mathbb{R}$*, then* $\lim_{t\to\infty} E[u(\cdot,t)] \leq 0$*.*

Proof. Under condition (1.15), $\int_{\mathbb{R}} V(u)dx \leq 0$ is always true. In addition, if $u \to 1$ locally uniformly in $\mathbb{R}$, then

$$\lim_{t\to\infty} \int_{\mathbb{R}} V(u(x,t))dx = -\infty. \tag{5.4}$$

By hypothesis (SD), one has the following estimate:

$$\begin{aligned} \int_{\mathbb{R}} \frac{1}{2} u_x^2(x,t)dx &= \int_0^\infty u_x^2(x,t)dx \\ &\leq \|u_x(x,t)\|_{L^\infty(\mathbb{R})} u(0,t). \end{aligned} \tag{5.5}$$

Then by standard parabolic regularity the left-hand side of (5.5) is bounded uniformly in time. So the first conclusion is proved. On the other hand, if $u \to 0$ uniformly in $\mathbb{R}$, then the right-hand side of (5.5) converges to 0, so as its left-hand side. In view of $V(u(x,t)) \leq 0$, the second conclusion is proved. $\square$

Similarly to the Lemma 4.5 for the bistable case, the following lemma holds for the monostable case.

Lemma 5.2. *Assume that there exists* $c_0 > 0$ *such that* $\phi(x) \in H^1_{c_0}(\mathbb{R})$*. If there exists* $T \geq 0$ *such that* $E[u(\cdot,T)] < 0$*, then* $u(x,t)$ *is wave-like.*

Proof. Since $\phi(x) \in H^1_{c_0}(\mathbb{R})$, $u(x,T) \in H^1(\mathbb{R}) \cap H^1_{c_0}(\mathbb{R})$. For any small $\varepsilon > 0$, when $E[u(x,T)] = -\varepsilon < 0$, there exists $L > 0$ such that

$$0 \leq \frac{1}{2}\int_L^\infty e^{c_0 x} u_x^2(x,T)dx < \frac{\varepsilon}{8}, \tag{5.6}$$

$$-\frac{\varepsilon}{8} < \int_L^\infty e^{c_0 x} V(u(x,T))dx \leq 0. \tag{5.7}$$

Note that if smaller $c \geq 0$ is chosen to instead of c_0 in the above inequalities, then they still hold. Furthermore, by the definition of L the following inequality holds:

$$\int_{-L}^{L} \left(\frac{1}{2}u_x^2(x,T) + V(u(x,T))\right) dx < -\frac{3\varepsilon}{4}. \tag{5.8}$$

So that there exists a sufficiently small $c > 0$ such that $c < c_0$ and

$$\int_{-L}^{L} e^{cx} \left(\frac{1}{2}u_x^2(x,T) + V(u(x,T))\right) dx < -\frac{\varepsilon}{2}. \tag{5.9}$$

Then the following inequality holds:

$$\Phi_c[u(\cdot,T)] = \int_{\mathbb{R}} e^{cx} \left(\frac{1}{2}u_x^2(x,T) + V(u(x,T))\right) dx < 0. \tag{5.10}$$

So u is wave-like. □

In contrast to the bistable case, for monostable case boundedness of energy always implies extinction.

Lemma 5.3. *Suppose that $E[u(\cdot,t)]$ is bounded from below for all $t \geq 1$, then $\lim_{t\to\infty} u(x,t) = 0$ uniformly in $\mathbb{R}$.*

Proof. Since the unique root of $V(u)$ is 0, by the similar argument in Lemma 4.11, $u(0,t) \to 0$ as $t \to \infty$. Then this lemma is proved by using Proposition 3.8. □

Lemma 5.4. *Suppose that there exists $T \geq 0$ such that $E[u(\cdot,T)] < 0$. Then $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly in $\mathbb{R}$.*

Proof. The proof is similar to the proof of Lemma 4.7. If $E[u(\cdot,T)] < 0$ for some $T \geq 0$, then there exists a sufficiently small $c > 0$ such that $\Phi_c[\varphi_L u(\cdot,T)] < 0$ for large enough $L > 0$, where the cutoff function φ_L is as in Lemma 4.7. In addition, by the conditions (1.15) and (5.1), there exists $\delta_0 > 0$, such that condition (3.8) holds. Then from Proposition 3.7, $R_{\delta_0}(t) > ct + R_0$ is obtained for some $R_0 \in \mathbb{R}$. Similarly to Lemma 4.6, since the unique solution of equation (1.24) larger than δ_0 is 1 in the whole of $\mathbb{R}$, $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly in $\mathbb{R}$ is proved. □

An immediate consequence of Lemma 5.1 and Lemma 5.4 is the following.

Corollary 5.5. *Suppose that* $\lim_{t\to\infty} u(x,t) = 0$ *uniformly in* $\mathbb{R}$*, then* $\lim_{t\to\infty} E[u(\cdot,t)] = 0.$

Theorems 5.1 and 5.2 have been established.

The last theorem in this chapter concerns with the threshold phenomena for monostable nonlinearities.

Theorem 5.3. *Under the same conditions as in Theorem 5.1, suppose that (P1) through (P3) hold. Then one of the following holds:*

1. $\lim_{t\to\infty} u_\lambda(x,t) = 0$ *uniformly in* $x \in \mathbb{R}$ *for every* $\lambda > 0$*;*
2. $\lim_{t\to\infty} u_\lambda(x,t) = 1$ *locally uniformly in* $x \in \mathbb{R}$ *for every* $\lambda > 0$*;*
3. *There exists* $\lambda^* > 0$ *such that*

$$\lim_{t\to\infty} u_\lambda(x,t) = \begin{cases} 0, & \textit{uniformly in } x \in \mathbb{R}, \quad \textit{for } 0 < \lambda \leq \lambda^*, \\ 1, & \textit{locally uniformly in } x \in \mathbb{R}, \quad \textit{for } \lambda > \lambda^*. \end{cases}$$

Proof. Similarly to the proof of Theorem 4.4, if neither $\Sigma_0 = \varnothing$ nor $\Sigma_1 = \varnothing$, then Σ_1 is an open interval. The conclusion then follows. □

Note that the sharp transition result above is nontrivial, e.g., for the generalized Fisher nonlinearity in (5.3) with $p > p_c$, where $p_c = 3$ is the Fujita exponent (see, e.g., [4, Theorem 3.2]).

CHAPTER 6

IGNITION NONLINEARITY

In this chapter, the ignition nonlinearity is studied, i.e., $f(u)$ satisfies condition (1.21). In addition, there exists $\delta > 0$ such that

$$f(u) \text{ is convex on } [\theta_0, \theta_0 + \delta]. \tag{6.1}$$

Under (1.13) and (1.21), except on the interval $[0, \theta_0]$, there exists at most one root $u = \theta^\diamond > 1$ of $V(u)$. However, since $\limsup_{t\to\infty} \|u(x,t)\|_{L^\infty(\mathbb{R})} \leq 1$, without loss of generality, it can be supposed that $\|\phi\|_{L^\infty(\mathbb{R})} < \theta^\diamond$. So that $V(u) \leq 0$ is always true.

The following theorems are main results concerning the long time behavior of solutions and their energy.

Theorem 6.1. *Let f satisfy conditions (1.21) and (6.1). Let $\phi(x)$ satisfy condition (1.23) and hypothesis (SD). Then one of the following holds.*

1. $\lim_{t\to\infty} u(x,t) = 1$ *locally uniformly in* $x \in \mathbb{R}$,
2. $\lim_{t\to\infty} u(x,t) = \theta_0$ *locally uniformly in* $x \in \mathbb{R}$,
3. $\lim_{t\to\infty} u(x,t) = 0$ *uniformly in* $x \in \mathbb{R}$.

Theorem 6.2. *Under the same assumptions as in Theorem 6.1, the following one-to-one relation holds.*

1. $\lim_{t\to\infty} u(x,t) = 1$ *locally uniformly in* $x \in \mathbb{R} \Leftrightarrow \lim_{t\to\infty} E[u(\cdot,t)] = -\infty$.
2. $\lim_{t\to\infty} u(x,t) = \theta_0$ *locally uniformly in* $x \in \mathbb{R}$ *or* $\lim_{t\to\infty} u(x,t) = 0$ *uniformly in* $x \in \mathbb{R} \Leftrightarrow \lim_{t\to\infty} E[u(\cdot,t)] = 0$.

The above theorems can be proved via a sequence of lemmas.

Lemma 6.1. *Suppose that there exists* $T \geq 0$ *such that* $E[u(\cdot, T)] < 0$*, then* $\lim_{t\to\infty} u(x,t) = 1$ *locally uniformly in* $\mathbb{R}$.

Proof. The arguments follow those in the proof of Lemma 4.7. If $E[u(\cdot, T)] < 0$ for some $T \geq 0$, then there exists a sufficiently small $c > 0$ such that $\Phi_c[\varphi_L u(\cdot, T)] < 0$ for large enough $L > 0$, where the cutoff function φ_L is as in Lemma 4.7. And by the condition (1.21), there exists $\delta_0 > \theta_0$, such that condition (3.8) holds. Then from Proposition 3.7, $R_{\delta_0}(t) > ct + R_0$ is obtained for some $R_0 \in \mathbb{R}$. Similarly to Lemma 4.6, since the unique solution of equation (1.24) larger than δ_0 is 1 in the whole of $\mathbb{R}$, $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly in $\mathbb{R}$ is proved. □

Lemma 6.2. *Suppose that* $E[u(\cdot, t)]$ *is bounded from below. Then either* $\lim_{t\to\infty} u(x,t) = 0$ *uniformly in* $\mathbb{R}$*, or* $\lim_{t\to\infty} u(x,t) = \theta_0$ *locally uniformly in* $\mathbb{R}$.

Proof. Same as in Lemma 4.11, there exists an unbounded increasing sequence $\{t_n\}$ such that

$$\lim_{n\to\infty} u(0, t_n) = \alpha, \tag{6.2}$$

for some $\alpha \in [0, \theta_0]$. In addition, in the spirit of Lemma 4.14,

$$\lim_{n\to\infty} u(x, t_n) \equiv \alpha, \tag{6.3}$$

locally uniformly in $x \in \mathbb{R}$. It is necessary to prove that α is either 0 or θ_0. This conclusion can be proved by contradiction. Assume that $0 < \alpha < \theta_0$, then there exists $T \geq 0$ sufficiently large such that $u(0, T) < \theta_0$. In addition, for any $t > T$, $u(x,t) \equiv \theta_0$ is a supersolution of (1.22), so that $0 \leq u(x,t) \leq \theta_0$ uniformly in $\mathbb{R}$. From the definition of $f(u)$, it then implies that equation (1.22) becomes

$$u_t(x,t) = u_{xx}(x,t), \tag{6.4}$$

for any $t > T$. However, the L^2 norm of the solution of the heat equation is non-increasing, contradicting the assumption that $u(x,t)$ converges to α locally uniformly. So either $\alpha = 0$ or $\alpha = \theta_0$, which proves the lemma. $\square$

Corollary 6.3. *Suppose that* $\lim_{t\to\infty} u(x,t) = 1$ *locally uniformly in* $\mathbb{R}$*, then* $\lim_{t\to\infty} E[u(\cdot,t)] = -\infty$.

Lemma 6.4. *Both* $\lim_{t\to\infty} u(x,t) = 0$ *uniformly in* $\mathbb{R}$ *and* $\lim_{t\to\infty} u(x,t) = \theta_0$ *locally uniformly in* $\mathbb{R}$ *imply* $\lim_{t\to\infty} E[u(\cdot,t)] = 0$.

Proof. By Lemma 6.1, $E[u(\cdot,t)] \geq 0$ for these behaviors. And since $V(u) \leq 0$ for any u, the following inequality holds:

$$E[u(\cdot,t)] \leq \int_{\mathbb{R}} \frac{1}{2} u_x^2(x,t)dx. \tag{6.5}$$

So it is only necessary to prove that the right-hand side of (6.5) converges to 0 as $t \to \infty$. By hypothesis (SD), one has the following inequality:

$$\int_{\mathbb{R}} \frac{1}{2} u_x^2(x,t)dx = \int_0^\infty u_x^2(x,t)dx \leq \|u_x(\cdot,t)\|_{L^\infty(\mathbb{R})} u(0,t). \tag{6.6}$$

It is done if $\lim_{t\to\infty} u(x,t) = 0$ uniformly in $\mathbb{R}$, because $\|u_x(\cdot,t)\|_{L^\infty(\mathbb{R})}$ is bounded by standard parabolic regularity. So it is only necessary to prove that $\|u_x(\cdot,t)\|_{L^\infty(\mathbb{R})} \to 0$ as $t \to \infty$ for the case $\lim_{t\to\infty} u(x,t) = \theta_0$ locally uniformly in $\mathbb{R}$. Note that $|u_{xx}(x,t)|$ is uniformly bounded for all $x \in \mathbb{R}$ and all $t \geq 1$. Using the convergence result $\lim_{t\to\infty} u(0,t) = \theta_0$, by standard parabolic regularity, the following estimate holds:

$$\lim_{t\to\infty} \sup_{|x| \leq R_{\theta_0}(t)} |u_x(x,t)| = 0. \tag{6.7}$$

Multiplying (1.22) by u_x and integrating from the leading edge $R_\delta(t)$ to ∞, which is justified by Proposition 3.1, for any $\theta \in (0, \theta_0]$,

$$\int_{R_\theta(t)}^\infty u_x(x,t)u_t(x,t)dx = \int_{R_\theta(t)}^\infty u_x(x,t)u_{xx}(x,t)dx, \tag{6.8}$$

is obtained, since $f(u) = 0$ for any $u \in [0, \theta_0]$. Integrating by part and applying Cauchy-Schwarz inequality, the inequality

$$\begin{aligned}
\frac{1}{2}u_x^2(R_\theta(t), t) &\leq \left(\int_{R_\theta(t)}^\infty u_x^2(x,t)dx \int_{R_\theta(t)}^\infty u_t^2(x,t)dx\right)^{\frac{1}{2}} \\
&\leq \left(\theta \max_{x\in\mathbb{R}} |u_x(x,t)| \int_{R_\theta(t)}^\infty u_t^2(x,t)dx\right)^{\frac{1}{2}} \\
&\leq \left(\theta_0 \max_{x\in\mathbb{R}} |u_x(x,t)| \int_{\mathbb{R}} u_t^2(x,t)dx\right)^{\frac{1}{2}}. \qquad (6.9)
\end{aligned}$$

is obtained. Since $E[u(\cdot, t)]$ is bounded from below in t, there exists an increasing sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}} u_t^2(x, t_n)dx = 0. \qquad (6.10)$$

In turn, since θ is arbitrary in $(0, \theta_0]$, the following estimate holds:

$$\lim_{n\to\infty} \sup_{x>R_{\theta_0}(t_n)} |u_x(x, t_n)| = 0. \qquad (6.11)$$

This means that the right-hand side of (6.5) converges to 0 on sequence $\{t_n\}$. The statement of the lemma then follows, since $E[u(\cdot, t)]$ is non-increasing. □

Theorems 6.1 and 6.2 have been established.

Studying the threshold phenomena for ignition nonlinearity is a little different from the situation with bistable nonlinearity. The main difficulty is to show that the threshold set contains only a single point, since constructing the type of barrier used in the proof of Theorem 4.4 do not seem to be available. Instead a modified proof by Zlatoš in [48] is given. The original proof by Zlatoš is based on a rescaling technique, and only valid when the initial condition is in the form of a characteristic function.

Lemma 6.5. *Let $f : [0, \infty) \to \mathbb{R}$ be a Lipschitz function with $f(0) = 0$. Let $U(x, t)$: $\mathbb{R} \times [0, \infty) \to [0, \infty)$ be a classical solution of*

$$U_t = U_{xx} + f(U), \qquad (6.12)$$

which is uniformly continuous up to $t = 0$. *Denote by* $U_1(x,t)$ *and* $U_2(x,t)$ *the solutions of equation (6.12) with initial conditions* $U_1(x,0)$ *and* $U_2(x,0)$*, respectively, and assume* $0 \le U_1(x,0) \le U_2(x,0)$ *for any* $x \in \mathbb{R}$*, and* $U_1(x_0,0) < U_2(x_0,0)$ *for some* $x_0 \in \mathbb{R}$. *Assume also that for any* $\rho > 0$ *the set* $\Omega_{0,\rho} = \{x \in \mathbb{R} : U_2(x,0) \ge \rho\}$ *is compact. Finally, assume that there are* $0 < \theta_1 < \theta_2$ *and* $\varepsilon_1 > 0$ *such that for any* $\theta \in [\theta_1, \theta_2]$ *and* $\varepsilon \in [0, \varepsilon_1]$ *the inequality*

$$f(\theta + \varepsilon(\theta - \theta_1)) \ge (1+\varepsilon) f(\theta), \tag{6.13}$$

is always true, and assume that $\|U_1\|_{L^\infty(\mathbb{R}\times(0,\infty))} < \theta_2$ *for any* $t \in [0,\infty)$. *Then one has*

$$\liminf_{t\to\infty} \inf_{U_1(x,t)>\theta_1} \frac{U_2(x,t)-\theta_1}{U_1(x,t)-\theta_1} > 1, \tag{6.14}$$

with the convention that the infimum over an empty set is ∞.

Proof. It is essentially [48, Lemma 4]. □

Theorem 6.3. *Under the same conditions as in Theorem 6.1, suppose that (P1) through (P3) hold. Then one of the following holds:*

1. $\lim_{t\to\infty} u_\lambda(x,t) = 0$ *uniformly in* $x \in \mathbb{R}$ *for every* $\lambda > 0$*;*
2. *There exists* $\lambda^* > 0$ *such that*

$$\lim_{t\to\infty} u_\lambda(x,t) = \begin{cases} 0, & \text{uniformly in } x \in \mathbb{R}, \quad \text{for } 0 < \lambda < \lambda^*, \\ \theta_0, & \text{locally uniformly in } x \in \mathbb{R}, \quad \text{for } \lambda = \lambda^*, \\ 1, & \text{locally uniformly in } x \in \mathbb{R}, \quad \text{for } \lambda > \lambda^*. \end{cases}$$

Proof. Similarly to the proof of Theorem 4.4, it can be proved that if $\Sigma_0 \ne (0,\infty)$, then both Σ_0 and Σ_1 are open intervals, and hence $\mathbb{R}_+ \setminus (\Sigma_0 \cup \Sigma_1)$ is a closed interval. Then it is only necessary to prove that $\mathbb{R}_+ \setminus (\Sigma_0 \cup \Sigma_1)$ contains only a single point. It necessary to verify that if $f(u)$ satisfies (1.21) and (6.1), then there exists $\varepsilon_1 > 0$, and

$0 < \theta_1 < \theta_0 < \theta_2 < 1$ such that condition (6.13) holds. Note that convexity of $f(u)$ on $[\theta_0, \theta_0+\delta]$ implies that $f(u)$ is nondecreasing on $[\theta_0, \theta_0+\delta]$, and $\theta_0+\delta < 1$. Taking $\varepsilon_1 = \delta/2$, $\theta_1 = \theta_0/2$, $\theta_2 = (3\theta_0 + \delta)/3$, it is only necessary to prove that (6.13) holds for any $\varepsilon \in [0, \varepsilon_1]$ and $\theta \in [\theta_0, \theta_2]$. Denote by $\alpha := \theta - \theta_0 \in [0, \delta/3]$. The following estimate of the left-hand side of (6.13) holds:

$$f(\theta + \varepsilon(\theta - \theta_1)) = f\left(\theta_0 + (1+\varepsilon)\alpha + \frac{\varepsilon\theta_0}{2}\right) \geq f(\theta_0 + (1+\varepsilon)\alpha), \tag{6.15}$$

since $\theta + \varepsilon(\theta - \theta_1) < \theta_0 + \delta$. By convexity the following inequality also holds:

$$f(\theta_0 + \alpha) \leq \frac{f(\theta_0 + (1+\varepsilon)\alpha)}{1+\varepsilon} + \frac{\varepsilon f(\theta_0)}{1+\varepsilon}, \tag{6.16}$$

which proves

$$f(\theta_0 + (1+\varepsilon)\alpha) \geq (1+\varepsilon) f(\theta_0 + \alpha), \tag{6.17}$$

for any $\varepsilon \in [0, \varepsilon_1]$ and $\alpha \in [0, \theta_2 - \theta_0]$. Hence (6.13) holds for any $\theta \in [\theta_1, \theta_2]$ and $\varepsilon \in [0, \varepsilon_1]$.

Assume that there exist two distinct values $0 < \lambda_1 < \lambda_2$ in the threshold set $\mathbb{R}_+ \setminus (\Sigma_0 \cup \Sigma_1)$. Denote by $u_{\lambda_1}(x,t)$ and $u_{\lambda_2}(x,t)$ these solutions with initial conditions ϕ_{λ_1} and ϕ_{λ_2}, respectively. Taking θ_1, θ_2 as above, there exists $T > 0$ such that $\|u_{\lambda_1}\|_{L^\infty(\mathbb{R}\times(0,\infty))} < \theta_2$, for any $t \geq T$. Moreover, let $U_1(x,t) := u_{\lambda_1}(x, t-T)$ and $U_2(x,t) := u_{\lambda_2}(x, t-T)$ for any $t \geq T$. Obviously all the assumptions of Lemma 6.5 hold. So there exists $r > 1$ such that

$$\liminf_{t\to\infty} \frac{U_1(0,t) - \theta_1}{U_2(0,t) - \theta_1} \geq r. \tag{6.18}$$

But both $U_1(0,t)$ and $U_2(0,t)$ converge to θ_0 as $t \to \infty$. So that the left-hand side of (6.18) must be 1, which is a contradiction. $\square$

Remark 6.6. *The C^1 property of $f(u)$ and condition (1.21) imply that $f'(\theta_0) = 0$. If $f(u) \in C[0,\infty) \cap C^1(\theta_0, \infty)$ is supposed, together with (1.21), and $\lim_{u\to\theta_0^+} f'(u) > 0$,* *then without local convexity condition (6.1) all the conclusions about convergence, equivalence, and sharp transition in this chapter still hold.*

CHAPTER 7

ONGOING AND FUTURE WORK

7.1 Summary

In this dissertation, a comprehensive study of the long time behavior of one-dimensional reaction-diffusion equation with bistable, monostable, or ignition nonlinearity and localized initial data is presented. The common feature of all these nonlinearities is the presence of a sharp transition from extinction to propagation.

The meaning of the concept of propagation as the long time behavior of the solution is clarified in this dissertation. Since the leading edges of a symmetric decreasing wave-like solution propagate faster than some constant asymptotic speed, in this case $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly implies propagation in the strict sense. Note that in general there is no constant limit of the asymptotic propagation speed of the leading edges for monostable nonlinearities [21]. As a general question, the convergence to a pair of traveling fronts for degenerate bistable nonlinearities will be considered in the future [37].

In this dissertation, one-to-one relation between the long time behavior of solution and the limit value of energy is proved. In addition, a sufficient condition of propagation is that $E[u(\cdot,T)] < 0$ at some $T \geq 0$, which is easy to verify. Moreover, if the energy is bounded from below, then convergence of the solution to a stationary solution is not only (locally) uniform, but also in energy and in the stronger $L^{p+1}(\mathbb{R}) \cap D^1(\mathbb{R})$ norm, where p is the order in the condition (4.5).

7.2 Compactly Supported Initial Data

Remark 7.1. *For both bistable nonlinearity case under the same assumptions as in Theorem 4.2 and ignition nonlinearity case under the same assumptions as in*

Theorem 6.1, if the symmetric decreasing hypothesis (SD) of initial data is replaced by an assumption of compactly supported initial data, then after a suitable translation all the conclusions in this dissertation still hold.

Since the convergence and sharp transition results for compactly supported initial data have been proved by Du and Matano [10], it is only necessary to prove that the one-to-one relation between the long time behavior of solution and the limit value of energy, and the behavior $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly in $\mathbb{R}$ implies that propagation (in the sense defined in the introduction) occurs indeed.

Proposition 7.2. *For both bistable nonlinearity case and ignition nonlinearity case under the same assumptions as in Remark 7.1, if* $E[u(\cdot,T)] < 0$ *for some* $T \geq 0$*, then* $\lim_{t\to\infty} u(x,t) = 1$ *locally uniformly in* $\mathbb{R}$*.*

Proof. It is sufficient to give an outline of proof in the bistable nonlinearity case. The ignition nonlinearity case is similar.

For compactly supported initial condition $\phi(x)$, there exists $L > 0$, such that the support of $\phi(x)$ is contained in the interval $[-L, L]$. By [10, Lemma 2.1],

$$u_x > 0, \text{ for } x < -L,\ t > 0; \quad u_x < 0, \text{ for } x > L,\ t > 0. \tag{7.1}$$

If $E[u(\cdot,t)] < 0$ for some $T \geq 0$, by the cutoff argument in the proof of Lemma 4.7 and in the spirit of Lemma 4.6, there exist $\delta_0 > \theta^*$ and $c > 0$ such that the leading edge position can be estimated as $R_{\delta_0}(t) > ct/2$ for sufficiently large t. Then, by monotonicity of $u(x,t)$ on (L,∞), for any $L^* > 0$ there exists a sufficiently large $T = T(L^*)$ such that $u(x,t) > \delta_0$ on the interval $(L, L+2L^*)$, whenever $t > T$.

Introduce the new variable $y := x - (L + L^*)$. By the same argument in the proof of Lemma 4.7, for any sufficiently large L^* a function $w(y,T)$ can be constructed with the following properties: $w(y,T)$ is symmetric decreasing with respect to $y = 0$ and supported on $-L^* < y < L^*$, $w(y,T) \leq u(y,T)$ and $E[w(\cdot,T)] < 0$. In addition,

$w(y,T) \in H^1_c(\mathbb{R})$ for any $c > 0$, since it is compactly supported. By the proof of Lemma 4.5, $\Phi_{c_0}[w(\cdot,T)] < 0$ for some $c_0 > 0$.

Let $w(y,t)$ be the solution of

$$w_t = w_{yy} + f(w), \quad y \in \mathbb{R},\ t > T, \tag{7.2}$$

with initial value $w(y,T)$. For any $t > T$, $w(y,t)$ is symmetric decreasing with respect to the space variable y. By Lemma 4.6, $\lim_{t\to\infty} u(y,t) = 1$ locally uniformly in $\mathbb{R}$. Then $u(x,t) > w(x,t)$ for any $t > T$ by comparison principle.

Moreover, by the definition of $w(y,T)$ and Proposition 3.4, for any $c' \in (0,c^\dagger)$, where $c^\dagger$ is the same as in Proposition 3.3, there exists $T_1 \geq 0$ such that $w(y,t) > \delta_0$ for all $y \in (-c't, c't)$. It is easy to see that for any $\delta < 1$ and $c'' \in (0,c^\dagger)$, there exists $T_2 \geq 0$ such that $u(x,t) > \delta$ on $(-c''t, c''t)$ for all $t > T_2$. Therefore, propagation does occur. $\square$

For compactly supported initial data, the proof of the one-to-one relation is almost the same as the proof for the symmetric decreasing case. The only difficulty is from the bistable nonlinearity case. Under the (SD) hypothesis, the unique maximum of $u(x,t)$ is at $x = 0$, for all $t > 0$. In addition, by Lemma 4.14, if there exists an increasing sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ and $t_{n+1} - t_n \leq 1$ such that $\lim_{n\to\infty} u(0,t_n) = \theta^*$, then $\lim_{n\to\infty} u(x,t) = v(x)$, uniformly in $\mathbb{R}$. However, in principle, the location of the maximum of $u(x,t)$ may oscillate without the (SD) hypothesis.

Du and Matano solved this difficulty by using a zero-counting argument [10, Lemma 2.8]. At the moment, a proof of the same result does not seem to be available by energy methods in this dissertation. (As a comparison result, under different assumptions Fife proved that if $E[u(\cdot,t)]$ is bounded from below, and $u(x,t)$ crosses θ_0 twice for all sufficiently large t, then $\lim_{t\to\infty} |u(x,t) - v(x-\xi(t))| = 0$ for some function $\xi(t)$ [15, Lemma 10]. However, the behavior of $\xi(t)$ as $t \to \infty$ is unknown.)

With the help of [10, Lemma 2.8], the one-to-one relation can be proved by the same energy arguments as in Chapters 4 and 6, with minor modifications. But [10, Lemma 2.8] itself appears to be too deep to be replaced by an energy argument. Combining energy methods with zero-counting argument is left for future work.

7.3 Future Work

There is an open problem about the long time behavior of solutions in the case of a balanced bistable nonlinearity, i.e., bistable nonlinearity with $V(1) = 0$. Though propagation is impossible in this case, at least for exponentially decaying initial data, the possible spreading behavior, i.e., $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly cannot be excluded by the current energy methods.

One possible extension of this dissertation is to the radial solutions of higher-dimensional reaction-diffusion equations. More precisely, the Cauchy problem

$$u_t(\vec{x},t) = \Delta u(\vec{x},t) + f(u(\vec{x},t)), \quad \vec{x} \in \mathbb{R}^n,\ t > 0, \tag{7.3}$$

$$u(\vec{x},0) = \phi(\vec{x}), \quad \vec{x} \in \mathbb{R}^n, \tag{7.4}$$

is considered. If the initial condition $\phi(\vec{x})$ is radial decreasing with respect to the origin, i.e., $\phi = \phi(r)$ is nonincreasing for $r > 0$, where r is the radial variable, then for any $t > 0$, the solution $u(\vec{x},t)$ of equation (7.3) is also radial decreasing with respect to the origin. Therefore, $u = u(r,t)$ satisfies

$$u_t(r,t) = \frac{n-1}{r} u_r(r,t) + f(u(r,t)), \quad r > 0,\ t > 0, \tag{7.5}$$

with the Neumann boundary condition

$$u_r(0,t) = 0, \quad \forall t > 0. \tag{7.6}$$

For bistable nonlinearity (1.19) with (4.1), under conditions $f'(0) < 0$ and $f'(1) < 0$, the long time behavior of solution has been studied by Jones [22]. For radial decreasing $\phi(r)$ with $\phi(\infty) < \theta_0$, Jones proved that one of the following holds.

1. $\lim_{t\to\infty} u(r,t) = 1$ locally uniformly in $\mathbb{R}$,
2. $\lim_{t\to\infty} u(x,t) = 0$ uniformly in $\mathbb{R}$,
3. $\lim_{t\to\infty} u(r,t) = v(r)$ uniformly in $\mathbb{R}$,

where $v(r)$ satisfies

$$v_{rr} + \frac{n-1}{r} v_r + f(v) = 0, \quad r > 0, \quad v_r(0) = 0, \quad v(\infty) = 0. \tag{7.7}$$

Moreover, if $\lim\limits_{t\to\infty} u(r,t) = 1$ locally uniformly in $\mathbb{R}$, then $u(r,t)$ converges to a traveling wave solution.

For the degenerate bistable nonlinearity, the problem is still open. It is expected that it may be studied by energy-based methods with a suitable energy functional. The exponentially weighted energy functional Φ_c in this dissertation may be a choice. But since there exists a u_r term when $n > 1$, the exponential weight e^{cx} may need to be modified.

Another possible extension is to the problems with general initial data, i.e., relaxing the symmetric decreasing hypothesis. Feireisl et al. have proved that "moving-bump"-type solutions may exist. By zero-counting argument, the equivalence between the boundedness of E and the boundedness of $H^1(\mathbb{R})$ norm (or $L^{p+1}(\mathbb{R}) \cap D^1(\mathbb{R})$ under different assumptions on f) can be proved. It is expected that the approach by Feireisl can be improved, i.e., the long time behavior can be classified by employing concentration compactness principle. But the estimate of the number of "moving-bumps" and the existence of a sharp transition from extinction to propagation are still left for future work.

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