Perturbed spherical objects in acoustic and fluid flow fields

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ABSTRACT

PERTURBED SPHERICAL OBJECTS IN ACOUSTIC AND FLUID FLOW FIELDS

by

Manmeet Kaur

In this study, the time averaged acoustic radiation force and drag on a small, nearly spherical object suspended freely in a stationary sound wave field in a compressible, low viscosity fluid is to be calculated. This problem has been solved for a spherical object, and it has many important engineering applications related to segregation and separation processes for particles in fluids such as water. Small but significant errors have occurred in the predicted behavior of the particles using the existing approximate solutions based on perfect spheres. The classical approach has been extended in this research to objects that deviate slightly from spherical shape whose boundary may be expressed in the general form, in spherical coordinates $r_p = a \{1 + \epsilon \delta(\theta, \phi)\}$ where $a$ is the radius of an unperturbed sphere, $|\epsilon| \ll 1$ is a small radius variation parameter and $\delta(\theta, \phi)$ is a smooth, perturbation function, $\pi$ -periodic in $\theta$ and $2\pi$ -periodic in $\phi$, chosen such that

$$|\delta| = O(1) \text{ and } |\epsilon \delta(\theta, \phi)| < 1 : 0 \leq \theta \leq \pi, \ 0 \leq \phi \leq 2\pi$$

A simple and straightforward method for treating the scattering problem of an irregular shaped nearly spherical object is presented. Detailed calculations are carried out to estimate the contributions to the acoustic and drag forces generated by the perturbations in shape to the first order in the small parameter $\epsilon$. A mathematical model is developed to calculate trajectories of perturbed particles due to application of acoustic standing waves. The resulting system of second order ordinary differential equations does not have a closed form solution, and it is so stiff that it is even difficult to solve numerically. A combination of phase space and asymptotic analysis turns out
to be far more useful in obtaining approximate solutions. Analysis of the solutions shows that all particles move towards the pressure node to first order in $\epsilon$. Moreover, it is found that the first order correction term tends to be several orders of magnitude larger than the zeroth order terms, which is an indication of their importance in the analysis of the particle dynamics.
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by

Manmeet Kaur

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This work is dedicated to our Preceptor Satguru Sewa Ram ji and Sant Mata Gulbeer Kaur ji and family without whose blessings this could not be possible.
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Background</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Design of Experimental Chamber based on the Technology</td>
<td>5</td>
</tr>
<tr>
<td>2 ACOUSTIC RADIATION FORCE ON A PERTURBED SPHERE</td>
<td>12</td>
</tr>
<tr>
<td>2.1 Basic Definitions:</td>
<td>12</td>
</tr>
<tr>
<td>2.2 Method</td>
<td>14</td>
</tr>
<tr>
<td>2.2.1 Formulation</td>
<td>15</td>
</tr>
<tr>
<td>2.2.2 Acoustic Radiation Stress Tensor</td>
<td>17</td>
</tr>
<tr>
<td>2.2.3 Formula for Mean Excess Pressure/Pressure Variation in the Medium</td>
<td>18</td>
</tr>
<tr>
<td>2.3 Acoustic Radiation Force on a Perturbed Compressible Sphere</td>
<td>20</td>
</tr>
<tr>
<td>2.3.1 Formulation for Perturbed Spherical Object</td>
<td>23</td>
</tr>
<tr>
<td>2.4 Scattering of the Stationary Waves by a Compressible Perturbed Sphere</td>
<td>26</td>
</tr>
<tr>
<td>2.4.1 Boundary Conditions</td>
<td>30</td>
</tr>
<tr>
<td>2.5 The Radiation Pressure on a Compressible Sphere</td>
<td>39</td>
</tr>
<tr>
<td>3 LAMB'S GENERAL SOLUTION OF STOKES EQUATION</td>
<td>57</td>
</tr>
<tr>
<td>3.1 Drag on a Perturbed sphere</td>
<td>59</td>
</tr>
<tr>
<td>4 FORCES AND DYNAMIC EQUATION FOR PARTICLE</td>
<td>64</td>
</tr>
<tr>
<td>4.1 Approximate Analysis of Equation of Motion</td>
<td>67</td>
</tr>
<tr>
<td>4.2 An Example</td>
<td>72</td>
</tr>
<tr>
<td>4.3 Approximate Analysis of an Example</td>
<td>79</td>
</tr>
<tr>
<td>5 CONCLUSIONS AND FUTURE RESEARCH</td>
<td>83</td>
</tr>
<tr>
<td>APPENDIX A INCIDENT VELOCITY POTENTIAL</td>
<td>86</td>
</tr>
<tr>
<td>APPENDIX B SPHERICAL COMPONENTS OF TERMS IN ACOUSTIC FORCE</td>
<td>89</td>
</tr>
<tr>
<td>APPENDIX C ACOUSTIC FORCE ON A PERTURBED SPHERE</td>
<td>107</td>
</tr>
<tr>
<td>Chapter</td>
<td>Page</td>
</tr>
<tr>
<td>-----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>APPENDIX D ACOUSTIC FORCE FOR PARTICULAR PERTURBATION</td>
<td>127</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>142</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>4</td>
</tr>
<tr>
<td>1.2</td>
<td>7</td>
</tr>
<tr>
<td>2.1</td>
<td>13</td>
</tr>
<tr>
<td>2.2</td>
<td>24</td>
</tr>
<tr>
<td>2.3</td>
<td>56</td>
</tr>
<tr>
<td>4.1</td>
<td>73</td>
</tr>
<tr>
<td>4.2</td>
<td>73</td>
</tr>
<tr>
<td>4.3</td>
<td>81</td>
</tr>
</tbody>
</table>

1.1 Acoustic Forces
1.2 Forces on a Particle in an Acoustic and Fluid Field
2.1 Planes of Nodes and Antinodes
2.2 A Spherical Coordinate System: \( r \) is the distance from the origin (O), \( \phi \) is the horizontal azimuthal angle, and \( \theta \) is the polar angle measured from the Z axis.
2.3 2D-slice of obstacle placed in a plane standing acoustic-wave field.
4.1 Perturbed sphere for \( \epsilon = 0.2 \).
4.2 Perturbed sphere for \( \epsilon = 0.9 \).
4.3 Particle displacement from pressure antinode vs time.
CHAPTER 1
INTRODUCTION

The trapping, manipulation and separation of particles suspended in fluids often involve the use of external fields. Acoustic fields have been used in situations where there is an impedance difference between the particles and the surrounding fluid. As the acoustic force depends on the particle's size, density and compressibility, particles can be separated acoustically and this has been used in many bio-separation processes [20]. Separation of particles from the host media using acoustic standing wave fields has been studied and used in many practical applications [6, 19, 35, 38]. In medical science it has been used for separation of blood composition for treatment and diagnosis. Standing wave fields have been used to determine the acoustic properties of red blood cells, to accelerate immuno-precipitation and to enhance the sedimentation of mammalian cells [36]. Recently, exploratory research on applying an acoustic standing wave to a sediment flow stream to fractionate and segregate particles was investigated by Meegoda et al. [27]. Using fundamental physics of particles in an acoustic field, a mathematical model was developed to calculate trajectories of deflected particles due to the fluid flow and application of acoustic standing waves [2].

1.1 Background

The fractionation of particles by size or other physical properties such as density has many applications in a variety of technologies including materials processing, polymer recycling, food processing, and the fuel industry. The use of monodisperse particles has advantages for diagnostic and treatment purposes in various fields of the life sciences. More effective separation of cells, bacteria, or yeast is desired in biotechnology. Fractionation of particles can be used in engineering applications such as decontamination of dredged sediments. The dredged sediments of harbors
are composed of fine and coarse particles. Decontamination and separation of fine sediments is useful in treating sediments, leaving contaminated water to be treated separately, and cleaned sediments to be reused as construction soil or returned to the sea. Thus, separating the sediments from the contaminants and recycling the clean sediments back to the harbor may reduce the volume of the contaminated solid waste to be treated or disposed of [1].

Ultrasound energy can remove organic compounds and metals from dredged sediments. Application of ultrasound energy to a soil slurry such as dredged sediments causes acoustic cavitation, and the formation, growth, and implosive collapse of bubbles in a liquid. Shock waves from cavitation in liquid/solid slurries produce high-velocity inter-particle collisions, the impact of which is sufficient to desorb contaminants from soils. The separation of coarse particles from fines using ultrasound with vacuum pressure was studied and modeled by Meegoda et al. [27], and showed that silt sizes could also be separated from waste water using ultrasound with vacuum pressure utilizing specially designed microporous stones. However, there is difficulty in decontaminating or separating ultrafine particles such as clays.

Ultrafine particles, and particles with neutral buoyancy, or a uniform electromagnetically charged surface pose difficulties with fractionation. Existing methods require excessive time, prohibitive pressure drops, or extremely high electric or magnetic fields. Hence the exploratory research using an acoustic field was conducted to evaluate the feasibility of fractionating ultrafine suspended particles and at the same time segregating them so that the current technology can be coupled to ultrasound technology with vacuum pressure to decontaminate dredged sediments.

A complete literature search was done by Meegoda et al. [1, 27] which included a detailed description of particles suspended in acoustic wave fields, the parameters that contribute to the phenomenon and results from selected research projects. The aggregation of micron sized particles when exposed to an acoustic standing wave field
was first observed more than a century ago by Kundt and Lehmann (1874) [22]. King and Macdonald (1934) [20] presented a detailed theoretical formulation of acoustic forces for a rigid sphere in a plane standing or progressive wave field in an ideal fluid. Yosioka and Kawasima (1955) [40] extended the above formulation to include the effects on compressible spheres.

In an acoustic field, forces exerted on particles in the medium are proportional to the local velocities of the fluid and when averaged they become zero. In an acoustic standing wave field, the averaged forces acting on a particle are not zero and play an important part as they arise from second order effects (see Gorkov(1962) [15]). These forces are:

1. primary axial acoustic radiation force ($F_{PARF}$)
2. primary transverse acoustic radiation force ($F_{PTRF}$), and
3. secondary acoustic radiation force ($F_S$).

The primary acoustic radiation force is the time-averaged force generated due to the interaction between particles and the wave field. The primary axial acoustic component dominates over the transverse component. Secondary acoustic radiation forces may arise between particles due to the multiple scattering of acoustic waves. However these are several orders of magnitude smaller than the primary forces [36]. The primary axial radiation force drives dispersed particles toward the velocity antinodes of the resonance field, so that the average distance between the particles will decrease. The magnitude of the force depends on the compressibility and density difference between the particle and the medium. An excellent approximation of the primary acoustic force $F_{ac}$ on a spherical object was derived by Yosioka and Kawasima (1955) [40] and expressed as:

$$F_{ac} = V_0 E_{ac} k G \sin(2kx) \hat{k}$$
The acoustic energy density is \((J/m^3)\), and \(E_{ac}\) is a measure of the energy residing in a wave field, and is equal to the sum of the time-averaged potential and kinetic energy densities:

\[ V_0 = \text{volume of one particle (m}^3) \]

\[ k = 2\pi/\lambda_k \text{ is the wave number of the acoustic radiation m}^{-1} \]

\(\lambda_k = \text{sound wavelength} \]

\(x = \text{axial distance from a pressure node (m) and} \]

\(\vec{k} = \text{unit vector in the axial direction.} \]

The axial primary radiation force has sinusoidal variation and is proportional to the acoustic energy density. This force acts in the direction parallel to the propagation of waves. The response of any solid suspended in a fluid to a resonant acoustic field depends on the acoustic contrast factor \(G\). For any solid compressible particle suspended in fluid with size \(r \ll \lambda\), the acoustic contrast factor is given by Yosioka and Kawasima (1955) [40]

\[ G = \frac{\beta_f - \beta_p}{\beta_f} + \frac{3(\rho_p - \rho_f)}{\rho_f + 2\rho_p} \]

**Figure 1.1 Acoustic Forces**
where $\beta_f$ and $\beta_p$ are compressibilities of the fluid and particle, respectively, and $\rho_f$ and $\rho_p$ are densities of fluid and particle, respectively.

If $G$ is positive, the particles move toward the pressure nodes and if $G$ is negative, the particles move toward the pressure antinodes.

### 1.2 Design of Experimental Chamber based on the Technology

Segregation and fractionation of suspended particles using acoustic standing wave fields require a high degree of control of different parameters governing the technology. Based on the above discussion and a study of experimental observations of several researchers the following can be concluded and will be employed or considered during the implementation of the proposed technology:

1. The diameters of the particles to be separated have to be much smaller than the wavelength of sound. The original equation of acoustic force on the particle derived by King, Yasioka and others was based on this assumption.

2. Only a sinusoidal wave of a single frequency will be used to produce sonic oscillation in the system.

3. To produce a stationary field in the system a reflector is required which reflects the waves with the same amplitude in the opposite direction. This can be achieved by requiring that the distance between the transducer and reflector be an integral multiple of a half wavelength.

4. The secondary interaction force between particles is influenced by concentration. Taking a dilute suspension will reduce secondary radiation forces.

5. The sound field should not be disturbed by ultrasonic cavitation or suspended particle interaction so that the energy density remains constant in the entire field.
6. There should be no change in temperature due to the radiation of sound. A large acoustic chamber will be built to have a large volume of liquid where the temperature effect will be negligible due to the dissipation of heat through a larger area.

7. Viscosity of the suspending fluid can have a considerable influence on the primary axial acoustic radiation force ($F_{P\ ARF}$). The ideal fluid approximation provides a good estimate of the $F_{P\ ARF}$ in cases where the particle radius is much smaller than the wavelength and much greater than the viscous boundary layer thickness. The radiation force in the viscous medium can be several orders of magnitude higher than in the nonviscous case. On the other hand, this effect of viscosity decreases rapidly with increase in frequency. When dealing with low viscosity fluids like water, the impact of viscosity on the technology is minimal. When dealing with suspended particles in a low viscosity liquid, appropriate resonance frequency and solution concentration can control the effect of viscosity on the process. The design of a nearly ideal acoustic separation chamber was achieved by considering the above points simultaneously.
The proposed technology was implemented by building a rectangular narrow channel of upward fluid flow. Two walls of the channel consisted of a piezoelectric transducer and rigid reflecting surface, respectively. When the transducer was energized at the proper frequency to maintain a stationary acoustic field, it formed a pressure node located on the mid-plane of the chamber, and pressure antinodes located on the chamber walls. The acoustic force on a suspended particle results from the particle - fluid interaction that arises when the particle and suspending fluid have different acoustic properties. When the particle is at the pressure node at a quarter wavelength, mid-plane of the chamber width, the magnitude of this force is the maximum. For a dilute suspension, the secondary radiation forces, the body forces and the hydrodynamic interactions were neglected. The rate of change of particle momentum is equal to:

\[
(\rho_p + 0.5\rho_f V_0) \frac{d\vec{v}}{dt} = F_{PARF} + F_{PTRF} + V_0(\rho_p - \rho_f)\vec{g} + \vec{F_D};
\]

\[
m_0 \ddot{\vec{a}} = F_{PARF} + F_{PTRF} + V_0(\rho_p - \rho_f)\vec{g} + \vec{F_D};
\]
where $\bar{v}$ is the particle velocity and $\bar{g}$ is the gravitational acceleration. The mass in the momentum term is the virtual mass of the particle, $m_v$, that is the sum of true particle mass and a fluid mass that behaves effectively as if it was entrained with the particle. The particle’s velocity $\bar{v}$ in the fluid due to the primary radiation force is very small, which yields a Reynolds number much less than 1; hence the drag force is given by Stokes’ law as $F_D = -6\pi \mu r \bar{v}$, where $\mu$ is the viscosity of the fluid and $r$ is the radius of the particle in suspension. The summation of the forces in the direction of the acoustic wave propagation gives:

$$F_{ac} = F_{PARF} = m_o a + 6\pi \mu r v = V_0 E_{ac} k G \sin(2kx). \tag{1.3}$$

Thus the acoustic force, $F_{ac}$, on a spherical particle in an acoustic field is mainly due to the primary axial radiation force and can be used to calculate the particle trajectories.

Using the particle physics in an acoustic field, a mathematical model was developed to calculate trajectories of deflected particles due to application of acoustic standing waves [2]. This model was used to interpret/validate the experimental data. Rearranging Equations (1.2) and (1.3) yields the following equation:

$$(\rho_p V_0 + 0.5 \rho_f V_0) \frac{dv}{dt} + 6\pi \mu r \frac{dx}{dt} - 4\pi r^3 k E_{ac} G \sin(2kx) = 0. \tag{1.4}$$

Simplifying Equation (1.4) using properties of particles and the fluid that were used in the research yields:

$$x'' + cx' - k \sin \beta x = 0 \tag{1.5}$$

where, the parameters $c$, $k$, and $\beta$ are all positive constants having the following orders ($O$) of magnitude: $c = O(10^6)$, $k = O(10^8)$, and $b = O(10^{-3})$. This equation is used throughout the mathematical derivation. The notation ($'$) indicates the derivative, $\frac{d}{dt}$, and $x$ is the position of the migrating particle in the $x$-direction.
between a transducer and a reflector separated by one half wavelength at the given frequency.

The resulting second order ordinary differential equation is very stiff and hence difficult to solve numerically. Moreover, no closed form solution could be found. The analysis of the above equation showed that the basic problems with numerical solutions could not even be effectively ameliorated through the use of standard ODE solvers employing rescaling techniques. A combination of phase space and asymptotic analysis turns out to be far more useful in obtaining approximate solutions. An approximate solution was derived which enabled the calculation of the particle trajectories and concentration at collection planes in the acoustic field. Analysis of the solution showed that all the particles move toward the pressure node to which the particles are supposed to migrate.

The technology was evaluated using uniform size silicon dioxide ($SiO_2$) and silicon carbide particle ($SiC$) suspensions in de-ionized water. A plexiglas acoustic resonator chamber was built with two transducers fixed to opposite ends. One transducer was used as a load to transfer energy into the fluid, and the other transducer was used as a reflector or receiver to reflect the waves in order to produce standing waves. The load transducer was excited using a sinusoidal signal amplified by a power amplifier. When the acoustic energy was supplied, $SiO_2$ particles migrated toward the pressure nodes at half wavelength intervals at an optimum frequency of 333 kHz and 40 W power. Dark lines representing particle columns were formed after the application of the acoustic field, which was recorded in videotape. However, due to the small particle size of $SiO_2$, particle trajectories could not be recorded, hence the slightly larger sized $SiC$ was used to track particle trajectories. The displacements of $SiC$ particles due to an acoustic force were compared with the mathematical model predictions. After using statistical analysis and a data optimization procedure, for input power level between 3.0 and 5.0 W, it was demonstrated that the experimental
data were comparable to mathematical model predictions. (with the maximum error not exceeding 10%)

Hence based on preliminary results it can be concluded that the acoustic field can be effectively used either to segregate or fractionate fine particles. Using the mathematical model developed to calculate the trajectories and concentration of the micron size particles suspended in a fluid with acoustic standing waves resulted in excellent qualitative but only fair to good quantitative agreement with experiments. Because no exact solution appears to exist and due to extremely large parameters in the governing equations it was found that asymptotic methods produce the best approximate solutions. The reason for the discrepancy between the theory and experiments might be due to the assumption of spherical shaped object particles in the theory. For example, SiC particles are cubical crystals. So it was proposed to reanalyze the theory for different shaped particles which motivated us to take into consideration irregular shaped geometries, and in particular, those having a perturbed spherical shape. Our hope is that by considering objects of slightly perturbed spherical shape we will obtain better quantitative agreement with experiments.

The main contribution is our derivation of the approximate acoustic force on an immersed (in a fluid) particle whose shape differs slightly from that of a sphere. We combine this with the first-order approximation for the drag of such a particle, obtain the resulting equation of motion, and analyze its solutions. Thus, we add an important new piece to the research on the dynamics of particles in combined acoustic and fluid flow fields. Since the work on the acoustic radiation force on a rigid sphere in an axisymmetric wave field by King [20], the radiation force on a sphere with different mechanical properties (compressible, fluid, elastic) has been investigated [7, 15, 17, 18, 31, 40]. Fairly recent examples of this research include: Mitri [28, 29, 30] who calculated the acoustic radiation force on elastic, and viscoelastic spheres, cylinders and shells immersed in non-viscous fluids in plane stationary and quasistationary
waves. Donikov [9, 10, 11] analyzed the effects of viscosity and heat conduction on spheres, gas bubbles and liquid drops placed in progressive and standing waves. These dissipative effects were found to highly influence the radiation force for progressive waves at low $ka$ values, while for the case of standing waves, these effects are negligible. Despite the extensive theoretical and experimental studies performed taking into account material properties, irregular geometric shapes are still to be explored and need to be studied beyond what is done here.
CHAPTER 2

ACOUSTIC RADIATION FORCE ON A PERTURBED SPHERE

The main purpose of the present work is to find the trajectories of slightly perturbed spherical particles suspended in stationary sound wave fields in a low viscosity fluid. It is necessary to understand the behavior of micron-sized particles suspended in a fluid and subjected to a standing wave field under various forces such as added mass force, acoustic field forces and hydrodynamic forces [2].

2.1 Basic Definitions:

1. Acoustics: The science that deals with the production, control, transmission, reception, and effects of sound is called acoustics.

2. Acoustic waves: An oscillatory motion with small amplitude in a compressible fluid is called a sound wave. Acoustic waves are one of a variety of pressure disturbances that produce the sensation of sound and propagate through a compressible fluid. Such waves are generated by pressure variation in a medium. Sound waves can be produced in a medium by a mechanical excitation generated by a sound source.

3. Standing wave field: When traveling waves meet a perpendicular boundary, they are reflected back to the source. These reflected waves carry some amount of energy. If waves are continuously sent out and reflected back, the two waves can reach a state of equilibrium, when the distance, between the source and the reflector is a multiple of a half-wavelength, $n\lambda/2$, where $n$ is an integer and $\lambda$ is the sound wavelength. This type of wavefield is called an acoustic standing wave field. Thus, the necessary and sufficient condition for the formation of a standing wave is that the distance between the source and the reflection source
Figure 2.1 Planes of Nodes and Antinodes

\( L = n\lambda k/2 \). A standing wave with \( n = 1 \) is called the fundamental or first harmonic. The wave for \( n = 2 \) is the second harmonic, the wave for \( n = 3 \) is the third harmonic and so forth. A plane standing-wave field arises from the superposition of two waves of equal wavelength and amplitude traveling in opposite directions. Equal wavelengths are necessary for reflection of the waves, and equal amplitudes are required to have constant pressure values at any time along the wave-guide. Under a stationary wave field, it should be noted that pressure oscillation and velocity fluctuation are out of phase; that is, a pressure node is a velocity anti-node and a pressure anti-node is a velocity node in the medium.

4. Acoustic radiation: The total energy in a sound wave is the summation of the kinetic and potential energy. The waves of energy, which travel at the speed of sound in an acoustic field, are the acoustic radiation. Under a stable situation in a standing wave, the amount of energy at various points along a path for the standing wave may vary and points of maximum and minimum energy occurs.
If the frequency is changed, the wavelength change causes a shift in the positions of maximum and minimum energy.

5. Acoustic radiation force: Acoustic radiation force is a phenomenon associated with the nature of acoustic wave propagation in a medium. The force is caused by the transfer of momentum from the wave to an obstacle in the wave path. The magnitude of the radiation force exerted on the object by a wave depends upon both the medium's mechanical characteristics and the object scattering properties. With respect to time dependence, radiation force is either static or dynamic. Static radiation force is a time averaged quantity produced on a target by a monochromatic wave (an incident sound wave field whose intensity does not change with time). The dynamic counterpart is an oscillatory force exerted on a target by a bichromatic acoustic wave (an incident field whose intensity varies with time).

2.2 Method

The acoustic force is calculated by integrating the time-averaged radiation stress tensor over the target surface. The radiation stress tensor is expressed in terms of the total (incident and scattered) linear acoustic velocity potential or pressure fields. Therefore, the calculation consists essentially of two parts: the scattering problem for the wave and the evaluation of the radiation force on the perturbed sphere.

The equations describing the motion of acoustic waves are nonlinear. In the interest of simplifying the analysis and since nearly ideal fluids are considered here, it is sufficient to linearize these equations and to retain only first order terms of the velocity or particle displacements, regarding them as small quantities. Radiation force, however, is connected with energy densities, which are quadratic terms containing squares of velocities or displacements. Therefore, any theory dealing
with acoustic radiation force must retain at least all second order terms to be valid even for small amplitude acoustic waves [25, 26].

2.2.1 Formulation

Consideration is given to an isentropic motion of a lossless, non-dispersive, compressible and sound propagating fluid. The wavelength of the sound wave is assumed to be much greater than the diameter of the perturbed sphere. We shall study the radiation force without the complications caused by gravity, which is consistent with ignoring higher order effects. To study the system with the sound on we start with the Euler equation (as viscosity can be ignored as far as the sound field and radiation force is concerned [25]). The fundamental Eulerian field equations are:

The equation of motion:

\[ \rho \left( \frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \text{grad}) \bar{u} \right) = -\text{grad} p \]

\[ (2.1) \]

The equation of continuity (mass conservation law):

\[ \frac{\partial \rho}{\partial t} + \text{div}(\rho \bar{u}) = 0 \]

\[ (2.2) \]

The equation of state:

\[ p = f(\rho) \]

\[ (2.3) \]

Since oscillations are small the velocity \( \bar{u} \) is also small. So the term \( (\bar{u} \cdot \text{grad}) \bar{u} \) in Euler’s equation may be neglected. For the same reasons, the relative changes in the fluid density and pressure are small. We can write the variables \( p \) and \( \rho \) for small fluctuations in the form:

\[ p = p_0 + p' \quad \text{is the total pressure, } p' \ll p \]

\[ (2.4) \]
The pressure $p$ in Equation (2.1) is a constant equilibrium pressure $p_0$ in the absence of sound and $p'$ is the acoustic pressure. Similarly,

$$
\rho = \rho_0 + \rho' \quad \text{is the total density, } \rho' \ll \rho
$$

(2.5)

where $\rho_0$ is a constant equilibrium density and $\rho'$ takes care of acoustic pressure and rarefaction of the medium due to the propagation of sound. As there is no mean flow, there is no constant flow $\vec{u}$ identified as acoustic velocity. The quantities $\vec{u}$, $\rho'$ and $p'$ are first order quantities oscillating with the frequency $\omega$ of the incident wave; $\rho_0$ and $p_0$ are assumed to be negligible time-independent quantities. By putting Equations (2.4) and (2.5) into (2.1) and (2.2) and equating orders we obtain the first order momentum and continuity equation as:

$$
\rho_0 \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla} p'
$$

(2.6)

$$
\frac{\partial \rho'}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{u} = 0
$$

(2.7)

where $\nabla$ denotes the usual "del" operator. The equation of state is

$$
p = f(p)
$$

Since changes in pressure and density are assumed to be small, this equation can be developed into a Taylor series.

$$
p - p_0 = \left( \frac{\partial p}{\partial \rho} \right)_{\rho = \rho_0} (\rho - \rho_0) + \frac{1}{2} \left( \frac{\partial^2 p}{\partial \rho^2} \right)_{\rho = \rho_0} (\rho - \rho_0)^2 + \cdots
$$

(2.8)

So, the small change $p'$ in the pressure is related to the small change $\rho'$ in the density by

$$
p' = \left( \frac{\partial p}{\partial \rho} \right) \rho'
$$

(2.9)
Thus for an ideal fluid, the linearized equation of state is

\[ p' = c_0^2 \rho' \]  

(2.10)

where,

\[ c_0^2 = \left( \frac{\partial p}{\partial \rho} \right)_{\rho=\rho_0} \]

is the square of the speed of sound in the fluid under normal, room temperature atmospheric conditions.

Since a sound field in an inviscid fluid is irrotational, we can write \( \vec{u} = -\nabla \Phi \), where \( \Phi \) is the scalar velocity potential. Equations (2.6), (2.7) and (2.10) can be combined to yield a single differential equation with one independent variable; namely

\[ \frac{\partial \Phi^2}{\partial t^2} = c_0^2 \nabla^2 \Phi \]

(2.11)

Assuming the solution to be time harmonic i.e \( \Phi = \Phi(x, y, z)e^{i\omega t} \), one obtains the Helmoltz equation:

\[ (\nabla^2 + k^2)\Phi = 0 \]

where the compressional wave number in the fluid is \( k = \frac{\omega}{c_0} \).

### 2.2.2 Acoustic Radiation Stress Tensor

The momentum equation and the continuity equation of an ideal fluid can be combined to yield

\[ \frac{\partial (\rho u_i)}{\partial t} + \frac{\partial \Pi_{ij}}{\partial x_j} = 0 \]

(2.12)

where \( \Pi_{ij} = p\delta_{ij} + \rho u_i u_j \), is the stress tensor in an ideal fluid.
For acoustic oscillations, we can define the time average over a wave cycle, denoted as $< >$. On averaging Equation (2.12), $\frac{\partial (\rho u_i)}{\partial t}$ vanishes in a steady state (or we can say that the system is so slowly changing compared with the oscillation that it can be considered as stationary; therefore after time averaging the term vanishes). Also, $< \frac{\partial (\rho u_i)}{\partial t} >$ is the mean value of the momentum change per unit time. An ideal fluid by definition cannot absorb momentum, therefore $< \frac{\partial (\rho u_i)}{\partial t} >= 0$ and Equation (2.12) reduces to:

$$\frac{\partial S_{ij}}{\partial x_j} = 0$$ (2.13)

where,

$$S_{ij} = -< p > \delta_{ij} - < \rho u_i u_j >$$ (2.14)

$S_{ij}$: is called the acoustic radiation stress tensor, which was derived by Brillouin [25] and $\delta_{ij}$: is the Kronecker delta.

Since, $p_0$ is a constant, it will not make any difference in Equation (2.13) if we replace $p$ by $p - p_0$ in Equation (2.14). Also, $u_i$ is of first order, with $\rho = \rho + \rho'$, so Equation (2.14) becomes, at 2nd order,

$$S_{ij} = -< p - p_0 > \delta_{ij} - < \rho_0 u_i u_j >$$ (2.15)

### 2.2.3 Formula for Mean Excess Pressure/Pressure Variation in the Medium

The quantity $< p - p_0 >$ in Equation (2.15) is the mean excess pressure and it is non-zero at finite amplitudes. As, $\vec{u} = -\nabla \Phi$, where $\Phi$ is the velocity potential and $(\vec{u} \cdot \nabla)\vec{u} = \nabla (\frac{1}{2} \vec{u} \cdot \vec{u}) - \vec{u} \times (\nabla \times \vec{u})$, Equation (2.1) then becomes

$$\nabla \left[ \frac{\partial \Phi}{\partial t} + \frac{1}{2} |\nabla \Phi|^2 \right] = -\frac{\nabla p}{\rho}$$ (2.16)
If $T$ is the temperature, and $s$ and $w$ are the entropy per unit mass and the enthalpy per unit mass of the fluid, respectively, then

$$dw = Tds + \frac{dp}{\rho}$$

For an adiabatic process,

$$\nabla w = \frac{\nabla p}{\rho}$$

and Equation (2.16) becomes

$$w = -\frac{\partial \Phi}{\partial t} - \frac{1}{2} \vert \nabla \Phi \vert^2 + f(t)$$

after being integrated in space, where $f(t)$ is constant in space but can depend on time.

Absorbing the arbitrary function $f(t)$ into $\frac{\partial \Phi}{\partial t}$ or setting $f(t) = 0$, we obtain

$$\omega = \frac{\partial \Phi}{\partial t} - \frac{1}{2} \vert \nabla \Phi \vert^2$$

The pressure can be expressed in a Taylor series in $\omega$ as follows:

$$p = p_0 + \left( \frac{\partial p}{\partial \omega} \right)_{s, 0} \omega + \frac{1}{2} \left( \frac{\partial^2 p}{\partial \omega^2} \right)_{s, 0} \omega^2 + \cdots \quad (2.17)$$

where the subscripts “$s$, 0” means evaluated at constant entropy and at equilibrium.

Since

$$\left( \frac{\partial \omega}{\partial p} \right)_{s} = \frac{1}{\rho}$$

$$\Rightarrow \left( \frac{\partial p}{\partial \omega} \right)_{s} = \rho$$

$$\Rightarrow \left( \frac{\partial^2 p}{\partial \omega^2} \right)_{s} = \left( \frac{\partial p}{\partial \omega} \right)_{s} \left( \frac{\partial p}{\partial \omega} \right)_{s} = \frac{\rho}{c^2}$$
where the fundamental relation \( \left( \frac{\partial p}{\partial t} \right)_s = c^2 \) was used.

Now let each of these quantities take its equilibrium value. Then (2.17) becomes,

\[
p = p_0 + \rho_0 \left( \frac{\partial \phi}{\partial t} - \frac{1}{2} |\nabla \phi|^2 \right) + \frac{1}{2} \rho_0 \left( \frac{\partial \phi}{\partial t} - \frac{1}{2} |\nabla \phi|^2 \right)^2 + \cdots
\]

On time averaging the above equation and retaining terms only up to the 2\textsuperscript{nd} order, we obtain

\[
(p - p_0) = -\frac{1}{2} \rho_0 \left( |\nabla \Psi|^2 \right) + \frac{1}{2} \frac{\rho_0}{c_0^2} \left( \frac{\partial \Psi}{\partial t} \right)^2
\]

which is finite on the nonlinear level or in the second order.

### 2.3 Acoustic Radiation Force on a Perturbed Compressible Sphere

Let an object with a closed boundary \( S \) in the medium be in motion with small velocity \( \vec{u} \) and \( S(t) \) be its position at time \( t \). The time-averaged radiation force, \( F_i \), is obtained by integrating the normal component of the radiation stress tensor in Equation (2.15) over the target surface \( S(t) \).

\[
F_i = \iint_{\partial S(t)} S_{ij} n_j ds
\]

Integrating Equation (2.13) over the space between \( S(t) \) and a much larger surface \( S_F \) concentric with the object, and using Gauss's theorem, we obtain a surface integral of \( S_{ij} n_j \) over \( S' = S(t) + S_F \), equated to zero, where \( n_j \) is a component of the normal unit vector on the surface \( S' \) pointing away from the enclosed space. Since the integral \(- S_{ij} n_j \) over \( S \) is the force \( F_i \) acting on the sphere, we have

\[
F_i = \iint_{S_F} S_{ij} n_j ds
\]
Equation (2.19) represents the time-averaged radiation force in terms of the far field, which can be rewritten as:

\[ F_i = \iint_{S_F} S_{ij} n_j ds - \iint_S S_{ij} n_j ds + \iint_S S_{ij} n_j ds \]
\[ = \iint_{S_F} S_{ij} n_j ds + \iint_S S_{ij} \tilde{n}_j ds + \iint_S S_{ij} n_j ds \]
\[ = \iiint \frac{\partial S_{ij}}{\partial x_j} dV + \iint_S S_{ij} n_j ds \]
\[ = 0 + \iint_S S_{ij} n_j ds \quad \text{as} \quad \frac{\partial S_{ij}}{\partial x_j} = 0 \]

where, \( S_F \): encloses the equilibrium surface \( S \),
and \( \tilde{n} = -\bar{n} \) is the inward normal to the surface \( S \).

Therefore,

\[ F_i = \iint_{S^{(i)}} S_{ij} n_j ds \]
\[ = \iint_{S_F} S_{ij} n_j ds \]
\[ = \iint_S S_{ij} n_j ds \]
\[ = -\iint_S \langle p - p_0 > \delta_{ij} + \rho_0 u_i u_j \rangle n_j ds \]
\[ = -\iint_S \langle p - p_0 > n_i + \rho_0 (u_i u_j n_j) \rangle ds \]
In vector form this is

\[
\vec{F} = - \iint_S (< p - p_0 > \vec{n} + \rho_0 < \vec{u}(\vec{u} \cdot \vec{n}) >) ds \\
= - \iint_S \left[ \left( -\frac{1}{2} \rho_0 < |\vec{\nabla} \Phi|^2 > + \frac{1}{2} c_0^2 \frac{\partial^2 \Phi}{\partial t^2} \right) \vec{n} + \rho_0 < \vec{\nabla} \Phi \left( \vec{\nabla} \Phi \cdot \vec{n} \right) > \right] ds
\]

(2.20)

where \( S \) is the target surface at its equilibrium position.

According to Equation (2.20), the acoustic radiation force is determined by solving the linear wave equation along with its boundary conditions.

\[
\frac{\partial^2 \Phi}{\partial t^2} - c_0^2 \nabla^2 \Phi = 0 \quad \text{(from Equation (2.11))}
\]

If the incident wave field is assumed to be composed of monochromatic stationary waves, the solution of Equation (2.11) is of the form \( \Phi(r, \theta, \phi, t) = Re[\Phi(\theta, \phi, \omega)e^{i\omega t}] = \Phi_{\text{tot}}(\theta, \phi, \omega)e^{i\omega t} \), where "Re" indicates the real part of a complex number \( \Phi_{\text{tot}} \). The total scalar velocity potential field \( \Phi_{\text{tot}} \) outside the obstacle, which can be complex, is the sum of the incidence wave field \( \Phi_i \) and scattered wave field \( \Phi_s \).

\[
\Phi_{\text{tot}} = \Phi_i + \Phi_s,
\]

where,

\( \Phi_i \) : is determined by the transducer

\( \Phi_s \) : is the wave scattered from the obstacle

The boundary conditions come from the continuity in the normal component of the velocity across the boundary \( S \) of the object, as well as continuity of pressure across
\(S\), which are, respectively

\[
(\nabla \Phi_{\text{tot}}) \cdot \hat{n} |_{S} = (\nabla \Phi^*) \cdot \hat{n} |_{S} \tag{2.21}
\]

\[
\rho_0 \frac{\partial \Phi_{\text{tot}}}{\partial t} |_{S} = \rho^* \frac{\partial \Phi^*}{\partial t} |_{S} \tag{2.22}
\]

where,

\(\Phi^* \) : is the velocity potential inside the object.

\(\nabla :=\) is the usual gradient operator

\(\hat{n} :=\) is the outward unit vector normal to \(S\)

\(\rho_0, \rho^* :=\) are the densities exterior and interior to object, both of which are assumed to be constant

whence,

\[
\vec{F} = -\iint_{S} \left[ \left( -\frac{1}{2} \rho_0 \left\langle |\vec{n} \Phi_{\text{tot}}| \right\rangle + \frac{1}{2} \rho_0 \frac{c_0}{2} \left\langle \left( \frac{\partial \Phi_{\text{tot}}}{\partial t} \right)^2 \right\rangle \right) \hat{n} + \rho_0 \left\langle \vec{n} \Phi_{\text{tot}} \cdot (\nabla \Phi_{\text{tot}} \cdot \hat{n}) \right\rangle \right] dS \tag{2.23}
\]

### 2.3.1 Formulation for Perturbed Spherical Object

The problem under consideration is the scattering of a given acoustic wave field by a perturbed spherical shaped object. The surface of the perturbed sphere is assumed to have radius \(a(1 + \epsilon \delta(\theta, \phi)) > 0\), an a perturbed 2-sphere.

\[S_p := \{ \vec{r} = \vec{r}_p(\theta, \phi) \}
\]

\[:= \{ a(1 + \epsilon \delta(\theta, \phi)) \cos \phi \sin \theta i + \sin \phi \sin \theta j + \cos \theta k : (\theta, \phi) \in [0, \pi] \times [0, 2\pi] \}, \tag{2.24}\]

where \(|\epsilon| \ll 1\) and \(\delta(\theta, \phi)\) is an arbitrary, smooth, perturbation function, \(\pi\)-periodic in \(\theta\) and \(2\pi\)-periodic in \(\phi\), chosen such that

\[|\delta| = O(1) \text{ and } |\epsilon \delta(\theta, \phi)| < 1 : 0 \leq \theta \leq \pi, \ 0 \leq \phi \leq 2\pi\]
and $a > 0$ is a constant, representing the radius of the unperturbed spherical object with boundary the unperturbed 2-sphere

$$
\tilde{S}_0 := \{ r = r_0(\theta, \phi) \} := \{ a[\cos \phi \sin \theta \hat{i} + \sin \phi \sin \theta \hat{j} + \cos \theta \hat{k} : (\theta, \phi) \in [0, \pi] \times [0, 2\pi] \}, \quad (2.25)
$$

First we compute the usual outer normal to $\tilde{S}_p$ (denoted as $\tilde{N}_p$) and compare it with

$\tilde{N}_p = \frac{\partial \tilde{r}}{\partial \theta} \times \frac{\partial \tilde{r}}{\partial \phi}$

$$
= \hat{i}[a^2 \cos(\phi) \sin^2(\theta) + a^2 \cos(\phi) \sin^2(\theta) \epsilon \delta(\theta, \phi)^2 + 2a^2 \cos(\phi) \sin^2(\theta) \epsilon \delta(\theta, \phi)] \\
+ a^2 \sin(\phi) \epsilon \delta(\theta, \phi) + a^2 \sin(\phi) \epsilon \delta(\theta, \phi) \epsilon \delta(\theta, \phi) \\
- a^2 \cos(\theta) \cos(\phi) \sin(\theta) \epsilon \delta(\theta, \phi) + a^2 \cos(\theta) \cos(\phi) \sin(\theta) \epsilon \delta(\theta, \phi) \epsilon \delta(\theta, \phi)] \\
+ \hat{j}[a^2 \sin^2(\theta) \sin(\phi) \epsilon \delta(\theta, \phi)^2 + a^2 \sin^2(\theta) \sin(\phi) + 2a^2 \sin^2(\theta) \sin(\phi) \epsilon \delta(\theta, \phi)] \\
- a^2 \cos(\phi) \epsilon \delta(\theta, \phi) - a^2 \cos(\phi) \epsilon \delta(\theta, \phi) \epsilon \delta(\theta, \phi) \\
- a^2 \cos(\theta) \sin(\phi) \epsilon \delta(\theta, \phi) - a^2 \cos(\theta) \sin(\phi) \epsilon \delta(\theta, \phi) \epsilon \delta(\theta, \phi)] \\
+ \hat{k}[a^2 \cos(\theta) \sin(\theta) \epsilon \delta(\theta, \phi)^2 + a^2 \cos(\theta) \sin(\theta) + 2a^2 \cos(\theta) \sin(\theta) \epsilon \delta(\theta, \phi)] \\
+ a^2 \sin^2(\theta) \epsilon \delta(\theta, \phi) + a^2 \sin^2(\theta) \epsilon \delta(\theta, \phi) \epsilon \delta(\theta, \phi)]
$$

**Figure 2.2** A Spherical Coordinate System: $r$ is the distance from the origin(O), $\phi$ is the horizontal azimuthal angle, and $\theta$ is the polar angle measured from the Z axis.
When $\epsilon \to 0$, $\vec{N}_p$ in cartesian coordinates reduces to

$$
\vec{N}_p = a \sin \theta \vec{r}_0
$$

$$
= \frac{\partial \vec{r}_0}{\partial \theta} \times \frac{\partial \vec{r}_0}{\partial \phi}
$$

$$
= \vec{N}_0
$$

where $\vec{r}_0$ is the position vector of any point on the unperturbed sphere. Using the Jacobian transformation matrix, $\vec{N}_p$ in cartesian coordinates can be transformed into spherical coordinates. $\vec{N}_p$ in spherical polar coordinates is given by:

$$
\{a^2 \sin(\theta)(\epsilon \delta(\theta, \phi) + 1)^2, -a^2 \sin(\theta)(\epsilon \delta(\theta, \phi) + 1)\epsilon \delta(\theta, \phi), -a^2(\epsilon \delta(\theta, \phi) + 1)\epsilon \delta(\theta, \phi)\}
$$

which reduces to

$$
\vec{N}_p = a^2 \sin \theta \hat{e}_r = \vec{N}_0 \text{ as } \epsilon \to 0
$$

The differential of surface area on $S_0$ is

$$
d\sigma_0 = ||\vec{N}_0||d\theta d\phi
$$

$$
= a^2 \sin \theta d\theta d\phi
$$

The differential of surface area on $S$ is

$$
d\sigma = ||\vec{N}_p||d\theta d\phi \quad (2.26)
$$

Thus acoustic radiation force reduces to:

$$
\vec{F} = - \left\langle \int_0^{2\pi} \int_0^\pi \left\{ \left( \frac{-1}{2} \rho_0 |\vec{\nabla}\Phi_{tot}|^2 + \frac{1}{2} \rho_0 \left( \frac{\partial \Phi_{tot}}{\partial t} \right)^2 \right) \vec{n} + \rho_0 (\vec{\nabla}\Phi_{tot} (\vec{\nabla}\Phi_{tot} \cdot \vec{n})) \right\} |_{r=r_p} \right\rangle
$$

$$
(2.27)
$$
2.4 Scattering of the Stationary Waves by a Compressible Perturbed Sphere

We consider the obstacle to be a compressible perturbed sphere of mean radius \( r_p \) whose center is at distance \( h \) in the z-direction from the nearest pressure antinode of a plane stationary wave field. Assuming that the incident wave is axisymmetric and its wave fronts are normal to the z-axis, the incident velocity potential referred to the equilibrium position of the center as origin of a rectangular coordinate system \((x, y, z)\) can be written as

\[
-\vec{u} = \vec{\nabla}\Phi_{\text{tot}} = \left( \frac{\partial \Phi_{\text{tot}}}{\partial r}, \frac{1}{r} \frac{\partial \Phi_{\text{tot}}}{\partial \theta}, r \sin \theta \frac{\partial \Phi_{\text{tot}}}{\partial \phi} \right)
\]

\( \vec{n} \) = spherical polar coordinates of \( \frac{\vec{N}_p}{||\vec{N}_p||} \)

where

\[ d\sigma : \text{is given by} \ (2.26) \]

\[
-\vec{u} = \vec{\nabla}\Phi_{\text{tot}} = \left( \frac{\partial \Phi_{\text{tot}}}{\partial r}, \frac{1}{r} \frac{\partial \Phi_{\text{tot}}}{\partial \theta}, r \sin \theta \frac{\partial \Phi_{\text{tot}}}{\partial \phi} \right)
\]

\[ \vec{n} = \text{spherical polar coordinates of} \ \frac{\vec{N}_p}{||\vec{N}_p||} \]

\[
-\vec{u} = \vec{\nabla}\Phi_{\text{tot}} = \left( \frac{\partial \Phi_{\text{tot}}}{\partial r}, \frac{1}{r} \frac{\partial \Phi_{\text{tot}}}{\partial \theta}, r \sin \theta \frac{\partial \Phi_{\text{tot}}}{\partial \phi} \right)
\]

\[ \vec{n} = \text{spherical polar coordinates of} \ \frac{\vec{N}_p}{||\vec{N}_p||} \]

\[ d\sigma : \text{is given by} \ (2.26) \]

\[
-\vec{u} = \vec{\nabla}\Phi_{\text{tot}} = \left( \frac{\partial \Phi_{\text{tot}}}{\partial r}, \frac{1}{r} \frac{\partial \Phi_{\text{tot}}}{\partial \theta}, r \sin \theta \frac{\partial \Phi_{\text{tot}}}{\partial \phi} \right)
\]

\[ \vec{n} = \text{spherical polar coordinates of} \ \frac{\vec{N}_p}{||\vec{N}_p||} \]

\[
-\vec{u} = \vec{\nabla}\Phi_{\text{tot}} = \left( \frac{\partial \Phi_{\text{tot}}}{\partial r}, \frac{1}{r} \frac{\partial \Phi_{\text{tot}}}{\partial \theta}, r \sin \theta \frac{\partial \Phi_{\text{tot}}}{\partial \phi} \right)
\]

\[ \vec{n} = \text{spherical polar coordinates of} \ \frac{\vec{N}_p}{||\vec{N}_p||} \]

\[
\Phi_i = e^{i\omega t} \left[ e^{ik(z+h)} + e^{-ik(z+h)} \right]
\]

(2.28)

where the first term and second term represent the plane progressive waves propagating in the direction of the \(-z\) and \(+z\) axes respectively.

In a system of spherical coordinates \((r, \theta, \phi)\), \( z = r \cos \theta \), the origin being center of mass of perturbed sphere, it can be rewritten as (see Appendix A):

\[
\Phi_i = \sum_{n=0}^{\infty} (2n+1)(-i)^n E_n j_n (kr) P_n (\cos \theta) e^{i\omega t}
\]

(2.29)
where,

\[ E_n = E_n(k, h) = e^{-ikh} + (-1)^n e^{ikh}, \]

\[ j_n(\cdot) : \text{is the spherical Bessel function of order } n, \]

\[ k := \frac{\omega}{c_0} \text{ is the acoustic wave number in the surrounding fluid} \]

\[ P_n(\cdot) : \text{is the Legendre polynomial of order } n \]

Of course, (2.29) represents but one of the solution forms to the Helmholtz equation, \((\nabla^2 + k^2)\Phi = 0\). That arises in consideration with \(\Phi\) having harmonic time dependence and the Spherical Laplacian being given by:

\[
\nabla^2 = \frac{1}{r^2} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}
\]

The scattered wave due to the presence of the perturbed sphere can also be expressed as a series of spherical waves from the origin:

\[
\Phi_{sc} = e^{i\omega t} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{nm} h_n^{(2)}(kr) P_n^m(\cos \theta) e^{im\phi}
\]

where,

\[ A_{nm} : \text{are constants (uniquely determined by } \Phi_{sc}) \]

\[ h_n^{(2)}(\cdot) : \text{is the spherical Hankel function of the second kind.} \]

\[ P_n^m(\cos \theta) : \text{is an associated Legendre function of order } n, m. \]

\[ P_n^m(\cos \theta) e^{im\phi} : \text{is called a spherical surface harmonic.} \]

Therefore, the total velocity potential outside the perturbed sphere \(\Phi_{tot}\) is the superposition of the incident field \(\Phi_i\), which would be the field if the sphere were absent, and the scattered field \(\Phi_{sc}\).
There also is a velocity potential inside the perturbed sphere induced by the impinging exterior wave that can be expressed as

$$\Phi^* = e^{i\omega t} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} B_{nm} j_n(k^*r) P_n^m(\cos \theta)e^{im\phi}$$  \hspace{1cm} (2.32)$$

where

$$\Phi^*: \text{is the velocity potential inside the object.}$$
$$k^* := \frac{\omega}{c^*} \text{is the acoustic wave number inside the perturbed sphere.}$$
$$B_{nm}: \text{are constants (uniquely determined by } \Phi^* \text{)}$$

The symbol * is used hereafter to indicate quantities in the perturbed sphere. Observe that it is assumed that the origin of the coordinate system is at the center of mass of the object, which leads to the choice of the spherical Bessel function rather than the Hankel function which has been selected in $\Phi_{sc}$ to represents waves diverging from the object. Note that the quantity on the right hand side of (2.28) is non-dimensional. This is permissible because we can always assume that it is divided by a unit base potential. In what follows, we are going to use perturbation theory in the small (amplitude) parameter $\epsilon$. It is assumed that the acoustic and velocity fields can be expanded in powers of $\epsilon$. The zeroth order field will be designated by a superscript 0,
We shall determine the dimensionless scattering coefficients $\Phi_{(0)}^{(0)}, \Phi_{(1)}^{(1)}, \Phi_{(2)}^{(2)}, B_{(0)}^{(0)}, B_{(1)}^{(1)}, B_{(2)}^{(2)}$ by applying the natural boundary conditions at the surface of the object in a fluid in which an acoustic field has been imposed (apart from the auxiliary condition at infinity already taken care of by $\Phi_i$). Which will lead directly to the solution of the wave equation.

Similar technique have been used by Erma [13] in his treatment of the electrostatic problem for irregular shaped conductors, which he successfully applied in many other problems [14]. Yeh [39] also applied the boundary perturbation technique in solving a problem of the diffraction of electromagnetic waves by a dielectric body with a perturbed spherical boundary.
2.4.1 Boundary Conditions

The boundary conditions come from the continuity in the normal component of the velocity across the boundary $S_p$ of the object, as well as continuity of pressure across $S_p$, which are, respectively

\[
(\nabla \Phi_{tot}) \cdot \hat{n}\big|_{r=r_p} = (\nabla \Phi^*) \cdot \hat{n}\big|_{r=r_p}
\]

(2.35)

\[
\rho_0 \frac{\partial \Phi_{tot}}{\partial t}\big|_{r=r_p} = \rho^* \frac{\partial \Phi^*}{\partial t}\big|_{r=r_p}
\]

(2.36)

where,

\[\nabla := \text{is the usual gradient operator}\]

\[\hat{n} := \text{is the outward unit vector normal to } S_p\]

\[\rho_0, \rho^* := \text{are the densities exterior and interior to object, both of which are assumed to be constant}\]

\[r_p := a(1 + \varepsilon \delta(\theta, \phi)) > 0, \text{ is the radius of the perturbed sphere}\]

Matching the boundary conditions at the perturbed sphere boundary to any order of $\varepsilon$ in order to get scattering coefficients involves the use of series expansions in terms of Taylor series and spherical harmonics on the boundary of the object. This, in effect, transforms the boundary conditions at the perturbed spherical boundary into a succession of boundary conditions on the surface of an unperturbed sphere. For the sake of clarity and simplicity, only the first-order solution will be carried out in detail. The higher order solutions can be obtained in a similar fashion. We use this approach to approximate the acoustic radiation force. First, we deal with the second
boundary condition, which is due to the continuity of pressure:

\[
\left\{ \begin{align*}
\rho_0 \frac{\partial \Phi_{\text{tot}}}{\partial t} - \rho^* \frac{\partial \Phi^*}{\partial t} \bigg|_{r=r_p} &= 0 \\
(\rho_0 i\omega \Phi_{\text{tot}} - \rho^* i\omega \Phi^*) \bigg|_{r=r_p} &= 0 \\
(\rho_0 \Phi_{\text{tot}} - \rho^* \Phi^*) \bigg|_{r=r_p} &= 0
\end{align*} \right.
\]

Substituting of Equations (2.33) and (2.34) into the above boundary condition and equating like powers in \( \epsilon \), where we use Taylor expansions such as

\[ j_n(\epsilon \delta) = j_n(ak) + a \epsilon j'_n(ak) \delta(\theta, \phi) + \frac{1}{2} a^2 \epsilon^2 j''_n(ak) \delta(\theta, \phi)^2 + \cdots, \]

leads to the following recursion relations owing to the orthonormality of the spherical harmonics. We shall use the notation:

\[
e_n = \frac{E_n}{\sqrt{\frac{2n+1}{4\pi}}} \quad a_{nm} = \frac{A_{nm}}{\sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}}} \quad b_{nm} = \frac{B_{nm}}{\sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}}} \tag{2.37}
\]

\[
Y^m_n(\theta, \phi) = N^n_m P^m_n(\cos \theta) e^{in\phi} \quad \forall \ n \geq 0 \text{ and } |m| \leq n
\]

where, the functions \( Y^m_n(\theta, \phi) \) are the spherical harmonics and \( N^n_m = \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} \) is the normalizing factor. In what follows, we only retain first-order perturbations:

\[
e^0_0 : \quad \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (\rho_0 a_{nm}^{(0)} h_n^{(2)}(ak) - \rho^* b_{nm}^{(0)} j_n(ak^*)) Y^m_n(\theta, \phi) =
\]

\[
- \sum_{n=0}^{\infty} (-i)^n \rho_0 (1 + 2n) e_n j_n(ak) Y^0_n(\theta, \phi)
\]
Therefore, for all \( n \), the \( \epsilon^0 \) zeroth order terms are:

\[
\rho_0 h_n^2 (ak) a_{nm}^{(0)} - \rho^* j_n (ak^*) b_{nm}^{(0)} = \begin{cases} 
-(-i)^n \rho_0 (1 + 2n) j_n (ka) e_n, & \text{if } m = 0, \\
0, & \text{if } m \neq 0.
\end{cases} \tag{2.38}
\]

Similarly, for all \( n \) the first order \( \epsilon^1 \) terms lead to:

\[
\rho_0 h_n^2 (ak) a_{nm}^{(1)} - \rho^* j_n (ak^*) b_{nm}^{(1)} = \int_0^{2\pi} \int_0^\pi F_n (\theta, \phi) Y_n^m (\theta, \phi) \sin \theta \, d\theta \, d\phi \tag{2.39}
\]

where,

\[
F_n (\theta, \phi) = -(-i)^n \rho_0 ak (1 + 2n) j'_n (ka) e_n \delta(\theta, \phi) Y_n^0 (\theta, \phi) \\
- \sum_{m=-n}^n (\rho_0 a h_n^{(2)} (ak) a_{nm}^{(0)} - \rho^* a k j'_n (ak^*) b_{nm}^{(0)} \delta(\theta, \phi) Y_n^m (\theta, \phi) \tag{2.40}
\]

Now we turn to first boundary condition, which is bit more complicated; namely

\[ \nabla \left( \Phi_{tot} - \Phi^* \right) \cdot \hat{n}_{|_{r=r_p}} = 0 \]

where,

\[ \nabla := \text{the usual gradient operator} \]

\[ \hat{n} := \text{the unit outward normal vector to } S_p. \]
The latter can be written as:

\[ \vec{n} = \nabla h; \quad h \equiv r - r_p(\theta, \phi) \]

\[ \vec{n} = e_r - \frac{ae\delta_{\phi}(\theta, \phi)}{r} e_\theta - \frac{a \csc(\theta) \epsilon\delta_{\phi}(\theta, \phi)}{r} e_\phi \]

We note that normal vector \( \vec{n} \) defined in this manner is not a unit vector; however, inasmuch as the corresponding normal component of the velocity field vanishes, only the direction of the normal vector is of significance. The boundary condition expressing continuity of the normal component of velocity vanishes at each point of the surface may be represented by the equation:

\[
\left. \frac{\partial}{\partial r} (\Phi_{tot} - \Phi^*) - \frac{ae\delta_{\phi}(\theta, \phi)}{r^2} \frac{\partial}{\partial \theta} (\Phi_{tot} - \Phi^*) - \frac{a \csc(\theta) \epsilon\delta_{\phi}(\theta, \phi)}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} (\Phi_{tot} - \Phi^*) \right|_{r=r_p} = 0
\]

(2.41)

Substituting Equations (2.33) and (2.33) in the above equation and incorporating Taylor series expansion, we obtain:

\[
a \sum_{n=0}^{\infty} (-i)^n k(1 + 2n) e_n Y_n(\theta, \phi) \left( \frac{1}{2} a^2 \epsilon^2 \delta(\theta, \phi)^2 j_n^2(ak) + j_n^0(ak) \right) \\
+ a \sum_{n=0}^{\infty} (-i)^n k(1 + 2n) e_n Y_n(\theta, \phi) (ak \epsilon \delta(\theta, \phi) j_n^0(ak)) \\
+ a \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (ka_{nm}^0 Y_n^m(\theta, \phi) h_n^2(ak) + ke Y_n(\theta, \phi) (a_{nm}^1 h_n^2(ak))) \\
+ a \sum_{n=0}^{\infty} \sum_{m=-n}^{n} ke Y_n(\theta, \phi) (aka_{nm}^0 \delta(\theta, \phi) h_n^2(ak)) \\
- a \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (b_{nm}^0 k Y_n(\theta, \phi) j_n^0(ak^*) + \epsilon k Y_n(\theta, \phi) (b_{nm}^1 j_n^0(ak^*))) \\
- a \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \epsilon k Y_n(\theta, \phi) (b_{nm}^0 k \delta(\theta, \phi) j_n^0(ak^*)) \\
- \csc^2(\theta) \left( \sum_{n=0}^{\infty} (-i)^n (1 + 2n) e_n (j_n(ak) + ak \epsilon \delta(\theta, \phi) j_n^0(ak)) Y_n^{(0,1)}(\theta, \phi) \right) \epsilon \delta(\theta, \phi) \\
- \csc^2(\theta) \left( \sum_{n=0}^{\infty} (-i)^n (1 + 2n) e_n \left( \frac{1}{2} a^2 \epsilon^2 \delta(\theta, \phi)^2 j_n^2(ak) \right) Y_n^{(0,1)}(\theta, \phi) \right) \epsilon \delta(\theta, \phi) \\
- \csc^2(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( a_{nm}^0 \delta(\theta, \phi) h_n^2(ak) Y_n^m(\theta, \phi) \right) \right) \epsilon \delta(\theta, \phi) \\
+ \csc^2(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( \beta_{nm}^0 j_n^0(ak^*) Y_n^m(\theta, \phi) \right) \right) \epsilon \delta(\theta, \phi) \\
+ \csc^2(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( \epsilon a_{nm}^0 k Y_n^m(\theta, \phi) \right) \right) \epsilon \delta(\theta, \phi) \\
+ 2 \csc^2(\theta) \left( \sum_{n=0}^{\infty} (-i)^n (1 + 2n) e_n (j_n(ak)) Y_n^{(0,1)}(\theta, \phi) \right) \epsilon \delta(\theta, \phi) \epsilon \delta(\theta, \phi) \]
Equating terms involving like orders of $\epsilon$, we find that the contribution from the $\epsilon^0$, (zeroth) order term is

$$\sum_{n=0}^{\infty} (-i)^n k (1 + 2n) e_n (a k e \delta (\theta, \phi) j_n' (ak)) Y_n (\theta, \phi) + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} k a_{nm}^0 Y_m (\theta, \phi) h_n^2 (ak)$$

$$- \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_{nm}^0 k^* Y_m (\theta, \phi) j_n (ak^*) = 0$$
The contribution from the $\epsilon^1$, (first) order term is

$$a \sum_{n=0}^{\infty} (-i)^n k(1 + 2n)e_n ak\delta(\theta, \phi) j_n''(ak) Y_n(\theta, \phi)$$

$$+ a \sum_{n=0}^{\infty} \sum_{m=-n}^{n} k Y_n^m(\theta, \phi) \left( a_{nm}^{(1)} h_n^2(ak) + ak a_{nm}^{(0)} \delta(\theta, \phi) h_n''(ak) \right)$$

$$- a \sum_{n=0}^{\infty} \sum_{m=-n}^{n} k^* Y_n^m(\theta, \phi) \left( b_{nm}^{(1)} j_n'(ak^*) + ak b_{nm}^{(0)} k^* \delta(\theta, \phi) j_n''(ak^*) \right)$$

$$- \csc^2 \theta \sum_{n=0}^{\infty} (-i)^n (1 + 2n)e_n j_n(ak) Y_{n\phi}(\theta, \phi) \delta_{\phi}(\theta, \phi)$$

$$- \csc^2 \theta \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm}^{(0)} h_n^2(ak) Y_{n\phi}^m(\theta, \phi) \delta_{\phi}(\theta, \phi)$$

$$+ \csc^2 \theta \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_{nm}^{(0)} j_n'(ak^*) Y_{n\phi}^m(\theta, \phi) \delta_{\phi}(\theta, \phi)$$

$$- \sum_{n=0}^{\infty} (-i)^n (1 + 2n)e_n j_n(ak) Y_{n\theta}(\theta, \phi) \delta_{\theta}(\theta, \phi)$$

$$- \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm}^{(0)} h_n^2(ak) Y_{n\theta}^m(\theta, \phi) \delta_{\theta}(\theta, \phi)$$

$$+ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_{nm}^{(0)} j_n'(ak^*) Y_{n\theta}^m(\theta, \phi) \delta_{\theta}(\theta, \phi) = 0$$

Therefore, for all $n$, the $\epsilon^0$ zeroth order and $\epsilon^1$ first order terms lead to:

$$k h_n''(ak) a_{nm}^{(0)} - k^* j_n'(ak^*) b_{nm}^{(0)} = \begin{cases} (-i)^n k(1 + 2n) j_n'(ka) e_n, & \text{if } m = 0, \\ 0, & \text{if } m \neq 0. \end{cases} \quad (2.42)$$

$$ak h_n''(ak) a_{nm}^{(1)} - ak^* j_n'(ak^*) b_{nm}^{(1)} = \int_0^{2\pi} \int_0^{\pi} G_n(\theta, \phi) \overline{Y_n^m(\theta, \phi)} \sin(\theta) d\theta d\phi \quad (2.43)$$
where

\[
G_n(\theta, \phi) = \left( -i \right)^n (ak)^2 (1 + 2n) \epsilon_n \delta_{m,0} [k^* j_n(ak) j_n'(ak) - k^* \rho_0 j_n(ak) j_n'(ak)] \frac{\delta(\theta, \phi) Y_n^m(\theta, \phi)}{D_n}
\]

\[
+ \sum_{m=-n}^{n} \left[ (ak^2) j_n^\prime(ak) b_{nm}^{(0)} - (ak)^2 h_n^\prime(ak) a_{nm}^{(0)} \right] \delta(\theta, \phi) Y_n^m(\theta, \phi)
\]

\[
+ \csc^2 \theta \left( -i \right)^n (1 + 2n) \epsilon_n j_n(ak) Y_n^0(\theta, \phi) \delta_\phi(\theta, \phi)
\]

\[
- \csc^2 \theta \sum_{m=-n}^{n} \left[ h_n^2(ak) a_{nm}^{(0)} - j_n(ak^2) b_{nm}^{(0)} \right] Y_n^m(\theta, \phi) \delta_\phi(\theta, \phi)
\]

\[
+ \left( -i \right)^n (1 + 2n) \epsilon_n j_n(ak) Y_n^0(\theta, \phi) \delta_\phi(\theta, \phi)
\]

\[
- \sum_{m=-n}^{n} \left[ h_n^2(ak) a_{nm}^{(0)} - j_n(ak^2) b_{nm}^{(0)} \right] Y_n^m(\theta, \phi) \delta_\theta(\theta, \phi)
\]

(2.44)

Solving together Equations (2.38) and (2.42) uniquely determines the \(a_{nm}^{(0)}\) and \(b_{nm}^{(0)}\) in accord with the classical solutions

\[
a_{nm}^{(0)} = \frac{(-i)^n (1 + 2n) \epsilon_n \delta_{m,0}}{D_n} \left[ k^* \rho_0 j_n(ak) j_n'(ak) - k^* \rho_0 j_n(ak) j_n'(ak^*) \right]
\]

(2.45)

\[
b_{nm}^{(0)} = \frac{(-i)^n \rho_0 k (1 + 2n) \epsilon_n \delta_{m,0}}{D_n} \left[ j_n(ak) h_n^{(2)'}(ak) - j_n'(ak) h_n^{(2)}(ak) \right]
\]

(2.46)

Using the Wronskian relation, \( (j_n h_n' - j_n' h_n)(x) = \frac{i}{x^2} \), we get

\[
b_{nm}^{(0)} = \frac{(-1)^{n+1} (i)^{n+1} \rho_0 k (1 + 2n) \epsilon_n \delta_{m,0}}{(ak)^2 (D_n)}
\]

(2.46)

where,

\[
D_n = \left[ k^* \rho_0 j_n(ak^*) h_n^{(2)}(ak) - k^* \rho_0 j_n(ak^*) h_n^{(2)'}(ak) \right]
\]

(2.47)

Similarly, solving Equations (2.39) and (2.43) uniquely determines the coefficients \(a_{nm}^{(1)}\) and \(b_{nm}^{(1)}\) as
\[ a_{nm}^{(1)} = \frac{ak^* j_n'(ak^*)}{aD_n} \int_0^{2\pi} \int_0^\pi F_n(\theta, \phi) Y_n^m(\theta, \phi) \sin(\theta) d\theta d\phi \]
\[ - \rho j_n(ak^*) \int_0^{2\pi} \int_0^\pi G_n(\theta, \phi) Y_n^m(\theta, \phi) \sin(\theta) d\theta d\phi \]  
\[ b_{nm}^{(1)} = \frac{ak h_n^{(2)'}(ak)}{aD_n} \int_0^{2\pi} \int_0^\pi F_n(\theta, \phi) Y_n^m(\theta, \phi) \sin(\theta) d\theta d\phi \]
\[ - \rho_0 h_n^{(2)}(ak) \int_0^{2\pi} \int_0^\pi G_n(\theta, \phi) Y_n^m(\theta, \phi) \sin(\theta) d\theta d\phi \]  
\[ (2.48) \]
\[ (2.49) \]

using (2.37), (2.45), (2.46), (2.48), and (2.49), we get

\[ A_{nm}^{(0)} = E_n \delta_{m,0} \sqrt{\frac{(n-m)!}{(n+m)!}} \frac{(-i)^n(1+2n) \left[ k^* j_n(ak^*) j_n'(ak) - k^* \rho_0 j_n(ak) j_n'(ak^*) \right]}{D_n} \]  
\[ (2.50) \]

\[ A_{n,m}^{(0)} = \delta_{m,0} \sqrt{\frac{(n-m)!}{(n+m)!}} \frac{(-i)^n(1+2n) \left[ k^* j_n(ak^*) j_n'(ak) - k^* \rho_0 j_n(ak) j_n'(ak^*) \right]}{D_n} \]  
\[ (2.51) \]

\[ B_{nm}^{(0)} = E_n \delta_{m,0} \sqrt{\frac{(n-m)!}{(n+m)!}} \frac{(-1)^{n+1}(i)^{n+1} \rho_0 k(1+2n)}{(ak)^2 D_n} \]  
\[ (2.52) \]

\[ B_{n,m}^{(0)} = \delta_{m,0} \sqrt{\frac{(n-m)!}{(n+m)!}} \frac{(-1)^{n+1}(i)^{n+1} \rho_0 k(1+2n)}{(ak)^2 D_n} \]  
\[ (2.53) \]

\[ B_{nm}^{(1)} = \frac{ak h_n^{(2)'}(ak)}{aD^0} \int_0^{2\pi} \int_0^\pi \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} F_n(\theta, \phi) Y_n^m(\theta, \phi) \sin(\theta) d\theta d\phi \]
\[ - \rho_0 h_n^{(2)}(ak) \int_0^{2\pi} \int_0^\pi \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} G_n(\theta, \phi) Y_n^m(\theta, \phi) \sin(\theta) d\theta d\phi \]  
\[ (2.54) \]
Beginning with the zeroth-order field, each higher-order field can be successively determined by satisfying appropriate boundary conditions at the surface of the unperturbed sphere. The problem can in principle be solved to any order of approximation in $\epsilon$. In practice, the algebraic manipulations quickly become unwieldy. We limit ourselves therefore to calculating only the first order correction to Yosioka and Kawasima’s [40] results. Furthermore, we make no attempt to justify the proposed perturbat-
ion scheme as such discussions are available in the literature [13, 14, 39]. Questions of convergence are far too complex to be investigated here. Thus, the scattered velocity potential, $\Phi_{sc}$ and the velocity potential inside the sphere, $\Phi^*$, may be written as (2.33) and (2.34) where coefficients are determined by using (2.50), (2.52), (2.48) (2.54), and (2.37).

### 2.5 The Radiation Pressure on a Compressible Sphere

It is convenient to express $\vec{F}$ in terms of $\Phi^*$, and by using the boundary conditions, we have the following expressions evaluated at $r = r_p$:

\[
\vec{F} = - \left\langle \int_0^{2\pi} \int_0^\pi \left\{ \left( -\frac{1}{2} \rho_0 |\vec{\nabla} \Phi^*|^2 + \frac{\lambda^2}{2} \frac{\rho_0}{c_0^2} \left( \frac{\partial \Phi^*}{\partial t} \right)^2 \right) \vec{n} \cdot \vec{\epsilon} + \rho_0 \vec{\nabla} \Phi^* \cdot (\vec{\nabla} \Phi^* \cdot \vec{n}) \right\} \right|_{r=r_p} d\theta d\phi \right\rangle = -\vec{F}_u - \vec{F}_t - \vec{F}_{un}
\]

Therefore, the radiation forces experienced by a perturbed sphere in the direction of the $x-$, and $y-$ axis are expressed respectively as:

\[
F_x = - \left\langle \int_0^{2\pi} \int_0^\pi \left\{ \left( -\frac{1}{2} \rho_0 |\vec{\nabla} \Phi^*|^2 + \frac{\lambda^2}{2} \frac{\rho_0}{c_0^2} \left( \frac{\partial \Phi^*}{\partial t} \right)^2 \right) \vec{n} \cdot \vec{\epsilon}_x + \rho_0 \vec{\nabla} \Phi^* \cdot (\vec{\nabla} \Phi^* \cdot \vec{n}) \right\} \right|_{r=r_p} d\theta d\phi \right\rangle
\]

\[
F_y = - \left\langle \int_0^{2\pi} \int_0^\pi \left\{ \left( -\frac{1}{2} \rho_0 |\vec{\nabla} \Phi^*|^2 + \frac{\lambda^2}{2} \frac{\rho_0}{c_0^2} \left( \frac{\partial \Phi^*}{\partial t} \right)^2 \right) \vec{n} \cdot \vec{\epsilon}_y + \rho_0 \vec{\nabla} \Phi^* \cdot (\vec{\nabla} \Phi^* \cdot \vec{n}) \right\} \right|_{r=r_p} d\theta d\phi \right\rangle
\]

The radiation force experienced by the perturbed sphere in the direction of $z-$axis or along $\theta = 0$, which is also the direction of propagation for the incident stationary wave field, is:

\[
F_z = - \left\langle \int_0^{2\pi} \int_0^\pi \left\{ \left( -\frac{1}{2} \rho_0 |\vec{\nabla} \Phi^*|^2 + \frac{\lambda^2}{2} \frac{\rho_0}{c_0^2} \left( \frac{\partial \Phi^*}{\partial t} \right)^2 \right) \vec{n} \cdot \vec{\epsilon}_z + \rho_0 \vec{\nabla} \Phi^* \cdot (\vec{\nabla} \Phi^* \cdot \vec{n}) \right\} \right|_{r=r_p} d\theta d\phi \right\rangle
\]
\( \vec{n} \) is the normal to the object surface in spherical coordinates
\[
\vec{n} = \{ a^2 \sin(\theta)(1 + \epsilon \delta(\theta, \phi))^2, -a^2 \sin(\theta)(1 + \epsilon \delta(\theta, \phi))\epsilon \delta_\theta(\theta, \phi), -a^2(1 + \epsilon \delta(\theta, \phi))\epsilon \delta_\phi(\theta, \phi) \} \\
\lambda = \frac{\rho^*}{\rho_0}
\]
\( \hat{e}_x, \hat{e}_y, \) and \( \hat{e}_z \) are the unit vectors in cartesian coordinates resolved in terms of the unit vectors in spherical coordinates.
\[
\hat{e}_x = \cos(\phi) \sin(\theta) \hat{e}_r + \cos(\phi) \cos(\theta) \hat{e}_\theta - \sin(\phi) \hat{e}_\phi \\
\hat{e}_y = \sin(\theta) \sin(\phi) \hat{e}_r + \cos(\theta) \sin(\phi) \hat{e}_\theta + \cos(\phi) \hat{e}_\phi \\
\hat{e}_z = \cos(\theta) \hat{e}_r - \sin(\theta) \hat{e}_\theta + 0 \hat{e}_\phi
\]
Let,

\[ \bar{F}_u = \left( \frac{-\rho_0}{2} \int_0^{2\pi} \int_0^{\pi} \left| \nabla \Phi^* \right|^2 \hat{n} \bigg|_{r=r_p} \, d\theta d\phi \right) \]

\[ = \left( \int_0^{2\pi} \int_0^{\pi} \left\{ F_{ur} \hat{e}_r + F_{u\theta} \hat{e}_\theta + F_{u\phi} \hat{e}_\phi \right\} d\theta d\phi \right) \]

\[ \bar{F}_t = \left( \frac{\lambda^2 \rho_0}{2c_0^2} \int_0^{2\pi} \int_0^{\pi} \left( \frac{\partial \Phi^*}{\partial t} \right)^2 \hat{n} \bigg|_{r=r_p} \, d\theta d\phi \right) \]

\[ = \left( \int_0^{2\pi} \int_0^{\pi} \left\{ F_{tr} \hat{e}_r + F_{t\theta} \hat{e}_\theta + F_{t\phi} \hat{e}_\phi \right\} d\theta d\phi \right) \]

\[ \bar{F}_{un} = \left( \rho_0 \int_0^{2\pi} \int_0^{\pi} \nabla \Phi^* (\nabla \Phi^* \cdot \hat{n}) \bigg|_{r=r_p} \, d\theta d\phi \right) \]

\[ = \left( \int_0^{2\pi} \int_0^{\pi} \left\{ F_{unr} \hat{e}_r + F_{un\theta} \hat{e}_\theta + F_{un\phi} \hat{e}_\phi \right\} d\theta d\phi \right) \]

\[ F_z = - \left( \int_0^{2\pi} \int_0^{\pi} \left\{ F_{ur} \hat{e}_r + F_{u\theta} \hat{e}_\theta + F_{u\phi} \hat{e}_\phi \right\} \cdot \hat{e}_z d\theta d\phi \right) \]

\[ - \left( \int_0^{2\pi} \int_0^{\pi} \left\{ F_{tr} \hat{e}_r + F_{t\theta} \hat{e}_\theta + F_{t\phi} \hat{e}_\phi \right\} \cdot \hat{e}_z d\theta d\phi \right) \]

\[ - \left( \int_0^{2\pi} \int_0^{\pi} \left\{ F_{unr} \hat{e}_r + F_{un\theta} \hat{e}_\theta + F_{un\phi} \hat{e}_\phi \right\} \cdot \hat{e}_z d\theta d\phi \right) \]

\[ F_y = - \left( \int_0^{2\pi} \int_0^{\pi} \left\{ F_{ur} \hat{e}_r + F_{u\theta} \hat{e}_\theta + F_{u\phi} \hat{e}_\phi \right\} \cdot \hat{e}_y d\theta d\phi \right) \]

\[ - \left( \int_0^{2\pi} \int_0^{\pi} \left\{ F_{tr} \hat{e}_r + F_{t\theta} \hat{e}_\theta + F_{t\phi} \hat{e}_\phi \right\} \cdot \hat{e}_y d\theta d\phi \right) \]

\[ - \left( \int_0^{2\pi} \int_0^{\pi} \left\{ F_{unr} \hat{e}_r + F_{un\theta} \hat{e}_\theta + F_{un\phi} \hat{e}_\phi \right\} \cdot \hat{e}_y d\theta d\phi \right) \]

\[ F_z = - \left( \int_0^{2\pi} \int_0^{\pi} \left\{ F_{ur} \hat{e}_r + F_{u\theta} \hat{e}_\theta + F_{u\phi} \hat{e}_\phi \right\} \cdot \hat{e}_z d\theta d\phi \right) \]

\[ - \left( \int_0^{2\pi} \int_0^{\pi} \left\{ F_{tr} \hat{e}_r + F_{t\theta} \hat{e}_\theta + F_{t\phi} \hat{e}_\phi \right\} \cdot \hat{e}_z d\theta d\phi \right) \]

\[ - \left( \int_0^{2\pi} \int_0^{\pi} \left\{ F_{unr} \hat{e}_r + F_{un\theta} \hat{e}_\theta + F_{un\phi} \hat{e}_\phi \right\} \cdot \hat{e}_z d\theta d\phi \right) \]
Details involved in calculating the spherical components of \( \frac{\lambda \rho_0}{2c_0^2} \left( \frac{\partial \Phi^*}{\partial t} \right)^2 \mathbf{n} \), and \( \rho_0 \nabla \Phi^* (\nabla \Phi^* \cdot \mathbf{n}) \) can be found in Appendix 2.

\[
F_{ur} = \frac{1}{2} \rho_0 \sin(\theta) J_1 J_1 + \frac{\lambda^2}{2} \rho_0 \csc(\theta) J_3 J_3 + \frac{\lambda^2}{2} \rho_0 \sin(\theta) J_5 J_5 \\
+ \epsilon \rho_0 \sin(\theta) J_1 J_2 + \epsilon \lambda^2 \rho_0 \csc(\theta) J_3 J_4 \\
+ \epsilon \lambda^2 \rho_0 \sin(\theta) J_5 J_6 + \epsilon \rho_0 \sin(\theta) J_1 J_1 \delta(\theta, \phi)
\]

\[
F_{u\theta} = -\frac{1}{2} \rho_0 \sin(\theta) J_1 J_1 \delta_\theta(\theta, \phi) - \frac{\lambda^2}{2} \rho_0 \csc(\theta) J_3 J_3 \delta_\theta(\theta, \phi) \\
- \epsilon \lambda^2 \rho_0 \sin(\theta) J_5 J_5 \delta_\theta(\theta, \phi) + O(\epsilon^2)
\]

\[
F_{u\phi} = -\frac{1}{2} \rho_0 J_1 J_1 \delta_\phi(\theta, \phi) - \frac{\lambda^2}{2} \rho_0 \csc^2(\theta) J_3 J_3 \delta_\phi(\theta, \phi) \\
- \epsilon \lambda^2 \rho_0 J_5 J_5 \delta_\phi(\theta, \phi) + O(\epsilon^2)
\]

\[
F_{tr} = -\lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} J_7 J_7 \sin(\theta) - \epsilon \lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} J_7 J_8 \sin(\theta) \\
- \epsilon \lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} J_7 J_7 \sin(\theta) \delta(\theta, \phi) + O(\epsilon^2)
\]

\[
F_{t\theta} = \epsilon \lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} J_7 J_7 \sin(\theta) \delta(\theta, \phi) + O(\epsilon^2)
\]

\[
F_{t\phi} = \epsilon \lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} J_7 J_7 \delta_\phi(\theta, \phi) + O(\epsilon^2)
\]

\[
F_{unr} = \rho_0 \sin(\theta) J_1 J_1 + \epsilon 2a^2 \rho_0 \sin(\theta) J_1 J_2 - \epsilon \lambda \rho_0 \csc(\theta) J_3 J_3 \delta_\phi(\theta, \phi) \\
- \epsilon \lambda \rho_0 \sin(\theta) J_1 J_5 \delta_\theta(\theta, \phi) + \epsilon 2 \rho_0 \sin(\theta) J_1 J_1 \delta(\theta, \phi) + O(\epsilon^2)
\]

\[
F_{un\theta} = \lambda \rho_0 \sin(\theta) J_1 J_5 + \epsilon \lambda \rho_0 \sin(\theta) J_1 J_6 - \epsilon \lambda a \rho_0 \sin(\theta) J_1 J_5 \delta(\theta, \phi) \\
- \epsilon \lambda^2 \rho_0 \csc(\theta) J_3 J_3 \delta_\phi(\theta, \phi) - \epsilon \lambda^2 \rho_0 \sin(\theta) J_5 J_5 \delta_\theta(\theta, \phi) \\
+ \epsilon \lambda 2 \rho_0 \sin(\theta) J_1 J_5 \delta(\theta, \phi) + O(\epsilon^2)
\]

\[
F_{un\phi} = \lambda \rho_0 J_1 J_3 + \epsilon \lambda \rho_0 J_1 J_4 + \epsilon \lambda \rho_0 J_3 J_3 - \epsilon \lambda \rho_0 J_1 J_3 \delta(\theta, \phi) \\
- \epsilon \lambda^2 \rho_0 \csc^2(\theta) J_3 J_3 \delta_\phi(\theta, \phi) - \epsilon \lambda^2 \rho_0 J_3 J_5 \delta_\theta(\theta, \phi) \\
+ \epsilon \lambda 2 \rho_0 J_1 J_3 \delta(\theta, \phi) + O(\epsilon^2)
\]
where

\[
\Re(J_i) = \left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \Re(I_{i,n}^m) \right\} 
\]

(2.55)

\[
\Re(I_{1,n}^m) = P_n^m (\cos \theta) M_{1,n}^m 
\]

\[
\Re(I_{2,n}^m) = P_n^m (\cos \theta) M_{2,n}^m 
\]

\[
\Re(I_{3,n}^m) = P_n^m (\cos \theta) M_{3,n}^m 
\]

\[
\Re(I_{4,n}^m) = P_n^m (\cos \theta) M_{4,n}^m 
\]

\[
\Re(I_{5,n}^m) = P_n^m (\cos \theta) M_{5,n}^m 
\]

\[
\Re(I_{6,n}^m) = P_n^m (\cos \theta) M_{6,n}^m 
\]

\[
\Re(I_{7,n}^m) = P_n^m (\cos \theta) M_{7,n}^m 
\]

\[
\Re(I_{8,n}^m) = P_n^m (\cos \theta) M_{8,n}^m 
\]

(2.56)

and,

\[
M_{1,n}^m = \Re \left\{ a k^* E_n \tilde{B}_{nm}^{(0)} j_n^* (ak^*) e^{im\phi} e^{i\omega t} \right\} 
\]

\[
M_{2,n}^m = \Re \left\{ a k^* \left( E_n \tilde{B}_{nm}^{(1)} j_n^* (ak^*) + ak^* E_n \tilde{B}_{nm}^{(0)} \delta(\theta, \phi) j_n'' (ak^*) \right) e^{im\phi} e^{i\omega t} \right\} 
\]

\[
M_{3,n}^m = \Re \left\{ im E_n \tilde{B}_{nm}^{(0)} j_n (ak^*) e^{im\phi} e^{i\omega t} \right\} 
\]

\[
M_{4,n}^m = \Re \left\{ im \left( E_n \tilde{B}_{nm}^{(1)} j_n (ak^*) + ak^* E_n \tilde{B}_{nm}^{(0)} \delta(\theta, \phi) j_n' (ak^*) \right) e^{im\phi} e^{i\omega t} \right\} 
\]

\[
M_{5,n}^m = \Re \left\{ E_n \tilde{B}_{nm}^{(0)} j_n (ak^*) e^{im\phi} e^{i\omega t} \right\} 
\]

\[
M_{6,n}^m = \Re \left\{ \left( E_n \tilde{B}_{nm}^{(1)} j_n (ak^*) + ak^* E_n \tilde{B}_{nm}^{(0)} \delta(\theta, \phi) j_n' (ak^*) \right) e^{im\phi} e^{i\omega t} \right\} 
\]

\[
M_{7,n}^m = \Re \left\{ E_n \tilde{B}_{nm}^{(0)} j_n (ak^*) e^{im\phi} e^{i\omega t} \right\} 
\]

\[
M_{8,n}^m = \Re \left\{ \left( E_n \tilde{B}_{nm}^{(1)} j_n (ak^*) + ak^* E_n \tilde{B}_{nm}^{(0)} \delta(\theta, \phi) j_n' (ak^*) \right) e^{im\phi} e^{i\omega t} \right\} , 
\]

with \( \Re \) the real part of a complex number.
Therefore, we compute that

\[
F_x = - \left\langle \int_0^{2\pi} \int_0^\pi \mathcal{R} \{ F_{ur} \cos(\phi) \sin(\theta) + F_{u\theta} \cos(\theta) \cos(\phi) - F_{u\phi} \sin(\phi) \} \, d\theta d\phi \right\rangle \\
- \left\langle \int_0^{2\pi} \int_0^\pi \mathcal{R} \{ F_{tr} \cos(\phi) \sin(\theta) + F_{t\theta} \cos(\theta) \cos(\phi) - F_{t\phi} \sin(\phi) \} \, d\theta d\phi \right\rangle \\
- \left\langle \int_0^{2\pi} \int_0^\pi \mathcal{R} \{ F_{unr} \cos(\phi) \sin(\theta) + F_{un\theta} \cos(\theta) \cos(\phi) - F_{un\phi} \sin(\phi) \} \, d\theta d\phi \right\rangle 
\]

\[
F_y = - \left\langle \int_0^{2\pi} \int_0^\pi \mathcal{R} \{ F_{ur} \sin(\theta) \sin(\phi) + F_{u\theta} \cos(\theta) \sin(\phi) + F_{u\phi} \cos(\phi) \} \, d\theta d\phi \right\rangle \\
- \left\langle \int_0^{2\pi} \int_0^\pi \mathcal{R} \{ F_{tr} \sin(\theta) \sin(\phi) + F_{t\theta} \cos(\theta) \sin(\phi) + F_{t\phi} \cos(\phi) \} \, d\theta d\phi \right\rangle \\
- \left\langle \int_0^{2\pi} \int_0^\pi \mathcal{R} \{ F_{unr} \sin(\theta) \sin(\phi) + F_{un\theta} \cos(\theta) \sin(\phi) + F_{un\phi} \cos(\phi) \} \, d\theta d\phi \right\rangle 
\]

\[
F_z = - \left\langle \int_0^{2\pi} \int_0^\pi \mathcal{R} \{ F_{ur} \cos(\theta) - F_{u\theta} \sin(\theta) \} \, d\theta d\phi \right\rangle \\
- \left\langle \int_0^{2\pi} \int_0^\pi \mathcal{R} \{ F_{tr} \cos(\theta) - F_{t\theta} \sin(\theta) \} \, d\theta d\phi \right\rangle \\
- \left\langle \int_0^{2\pi} \int_0^\pi \mathcal{R} \{ F_{unr} \cos(\theta) - F_{un\theta} \sin(\theta) \} \, d\theta d\phi \right\rangle 
\]

After some straightforward but involved calculations, one obtains the acoustic radiation force to the first order in \( \epsilon \) (see Appendix C for details).

\[
F_x \approx F_{xcc} \cos^2(\kappa h) + F_{xss} \sin^2(\kappa h) + F_{xsc} \sin(2\kappa h) \\
F_y \approx F_{ycc} \cos^2(\kappa h) + F_{yss} \sin^2(\kappa h) + F_{ysc} \sin(2\kappa h) \\
F_z \approx F_{zcc} \cos^2(\kappa h) + F_{zss} \sin^2(\kappa h) + F_{zsc} \sin(2\kappa h) \\
F_{xcc} = F_{x0cc} + \epsilon F_{x1cc}, \quad F_{xss} = F_{x0ss} + \epsilon F_{x1ss}, \quad F_{xsc} = F_{x0sc} + \epsilon F_{x1sc} \\
F_{ycc} = F_{y0cc} + \epsilon F_{y1cc}, \quad F_{yss} = F_{y0ss} + \epsilon F_{y1ss}, \quad F_{ysc} = F_{y0sc} + \epsilon F_{y1sc} \\
F_{zcc} = F_{z0cc} + \epsilon F_{z1cc}, \quad F_{zss} = F_{z0ss} + \epsilon F_{z1ss}, \quad F_{zsc} = F_{z0sc} + \epsilon F_{z1sc} 
\]
\[ F_{x0cc} = \int_0^{2\pi} \int_0^{\pi} \{- F_{r0cc} \sin(\theta) \cos(\phi) - F_{\theta0cc} \cos(\theta) \cos(\phi) + F_{\phi0cc} \sin(\phi)\} \, d\theta \, d\phi \]

\[ F_{x1cc} = \int_0^{2\pi} \int_0^{\pi} \{- F_{r1cc} \sin(\theta) \cos(\phi) - F_{\theta1cc} \cos(\theta) \cos(\phi) + F_{\phi1cc} \sin(\phi)\} \, d\theta \, d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \{- F_{r2cc} \sin(\theta) \cos(\phi) - F_{\theta2cc} \cos(\theta) \cos(\phi) + F_{\phi2cc} \sin(\phi)\} \, d\theta \, d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \{- F_{r3cc} \sin(\theta) \cos(\phi) - F_{\theta3cc} \cos(\theta) \cos(\phi) + F_{\phi3cc} \sin(\phi)\} \, d\theta \, d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \{- F_{r4cc} \sin(\theta) \cos(\phi) - F_{\theta4cc} \cos(\theta) \cos(\phi) + F_{\phi4cc} \sin(\phi)\} \, d\theta \, d\phi \]

\[ F_{x0ss} = \int_0^{2\pi} \int_0^{\pi} \{- F_{r0ss} \sin(\theta) \cos(\phi) - F_{\theta0ss} \cos(\theta) \cos(\phi) + F_{\phi0ss} \sin(\phi)\} \, d\theta \, d\phi \]

\[ F_{x1ss} = \int_0^{2\pi} \int_0^{\pi} \{- F_{r1ss} \sin(\theta) \cos(\phi) - F_{\theta1ss} \cos(\theta) \cos(\phi) + F_{\phi1ss} \sin(\phi)\} \, d\theta \, d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \{- F_{r2ss} \sin(\theta) \cos(\phi) - F_{\theta2ss} \cos(\theta) \cos(\phi) + F_{\phi2ss} \sin(\phi)\} \, d\theta \, d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \{- F_{r3ss} \sin(\theta) \cos(\phi) - F_{\theta3ss} \cos(\theta) \cos(\phi) + F_{\phi3ss} \sin(\phi)\} \, d\theta \, d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \{- F_{r4ss} \sin(\theta) \cos(\phi) - F_{\theta4ss} \cos(\theta) \cos(\phi) + F_{\phi4ss} \sin(\phi)\} \, d\theta \, d\phi \]

\[ F_{x0sc} = \int_0^{2\pi} \int_0^{\pi} \{- F_{r0sc} \sin(\theta) \cos(\phi) - F_{\theta0sc} \cos(\theta) \cos(\phi) + F_{\phi0sc} \sin(\phi)\} \, d\theta \, d\phi \]

\[ F_{x1sc} = \int_0^{2\pi} \int_0^{\pi} \{- F_{r1sc} \sin(\theta) \cos(\phi) - F_{\theta1sc} \cos(\theta) \cos(\phi) + F_{\phi1sc} \sin(\phi)\} \, d\theta \, d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \{- F_{r2sc} \sin(\theta) \cos(\phi) - F_{\theta2sc} \cos(\theta) \cos(\phi) + F_{\phi2sc} \sin(\phi)\} \, d\theta \, d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \{- F_{r3sc} \sin(\theta) \cos(\phi) - F_{\theta3sc} \cos(\theta) \cos(\phi) + F_{\phi3sc} \sin(\phi)\} \, d\theta \, d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \{- F_{r4sc} \sin(\theta) \cos(\phi) - F_{\theta4sc} \cos(\theta) \cos(\phi) + F_{\phi4sc} \sin(\phi)\} \, d\theta \, d\phi \]
\[ F_{y0cc} = \int_0^{2\pi} \int_0^{\pi} \left\{ -F_{r0cc} \sin(\theta) \sin(\phi) - F_{\theta0cc} \cos(\theta) \sin(\phi) - F_{\phi0cc} \cos(\phi) \right\} d\theta d\phi \]

\[ F_{y1cc} = \int_0^{2\pi} \int_0^{\pi} \left\{ -F_{r1cc} \sin(\theta) \sin(\phi) - F_{\theta1cc} \cos(\theta) \sin(\phi) - F_{\phi1cc} \cos(\phi) \right\} d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \left\{ -F_{r2cc} \sin(\theta) \sin(\phi) - F_{\theta2cc} \cos(\theta) \sin(\phi) - F_{\phi2cc} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \left\{ -F_{r3cc} \sin(\theta) \sin(\phi) - F_{\theta3cc} \cos(\theta) \sin(\phi) - F_{\phi3cc} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \left\{ -F_{r4cc} \sin(\theta) \sin(\phi) - F_{\theta4cc} \cos(\theta) \sin(\phi) - F_{\phi4cc} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]

\[ F_{y0ss} = \int_0^{2\pi} \int_0^{\pi} \left\{ -F_{r0ss} \sin(\theta) \sin(\phi) - F_{\theta0ss} \cos(\theta) \sin(\phi) - F_{\phi0ss} \cos(\phi) \right\} d\theta d\phi \]

\[ F_{y1ss} = \int_0^{2\pi} \int_0^{\pi} \left\{ -F_{r1ss} \sin(\theta) \sin(\phi) - F_{\theta1ss} \cos(\theta) \sin(\phi) - F_{\phi1ss} \cos(\phi) \right\} d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \left\{ -F_{r2ss} \sin(\theta) \sin(\phi) - F_{\theta2ss} \cos(\theta) \sin(\phi) - F_{\phi2ss} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \left\{ -F_{r3ss} \sin(\theta) \sin(\phi) - F_{\theta3ss} \cos(\theta) \sin(\phi) - F_{\phi3ss} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \left\{ -F_{r4ss} \sin(\theta) \sin(\phi) - F_{\theta4ss} \cos(\theta) \sin(\phi) - F_{\phi4ss} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]

\[ F_{y0sc} = \int_0^{2\pi} \int_0^{\pi} \left\{ -F_{r0sc} \sin(\theta) \sin(\phi) - F_{\theta0sc} \cos(\theta) \sin(\phi) - F_{\phi0sc} \cos(\phi) \right\} d\theta d\phi \]

\[ F_{y1sc} = \int_0^{2\pi} \int_0^{\pi} \left\{ -F_{r1sc} \sin(\theta) \sin(\phi) - F_{\theta1sc} \cos(\theta) \sin(\phi) - F_{\phi1sc} \cos(\phi) \right\} d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \left\{ -F_{r2sc} \sin(\theta) \sin(\phi) - F_{\theta2sc} \cos(\theta) \sin(\phi) - F_{\phi2sc} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \left\{ -F_{r3sc} \sin(\theta) \sin(\phi) - F_{\theta3sc} \cos(\theta) \sin(\phi) - F_{\phi3sc} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^{\pi} \left\{ -F_{r4sc} \sin(\theta) \sin(\phi) - F_{\theta4sc} \cos(\theta) \sin(\phi) - F_{\phi4sc} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]
\[ F_{x0cc} = \int_0^{2\pi} \int_0^\pi \{-F_{r0cc} \cos(\theta) + F_{\phi0cc} \sin(\theta)\} d\theta d\phi \]
\[ F_{z1cc} = \int_0^{2\pi} \int_0^\pi \{-F_{r1cc} \cos(\theta) + F_{\phi1cc} \sin(\theta)\} d\theta d\phi \]
\[ {} \quad {} + \int_0^{2\pi} \int_0^\pi \{-F_{r2cc} \cos(\theta) + F_{\phi2cc} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi \]
\[ {} \quad {} + \int_0^{2\pi} \int_0^\pi \{-F_{r3cc} \cos(\theta) + F_{\phi3cc} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi \]
\[ {} \quad {} + \int_0^{2\pi} \int_0^\pi \{-F_{r4cc} \cos(\theta) + F_{\phi4cc} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi \]
\[ F_{z0ss} = \int_0^{2\pi} \int_0^\pi \{-F_{r0ss} \cos(\theta) + F_{\phi0ss} \sin(\theta)\} d\theta d\phi \]
\[ F_{z1ss} = \int_0^{2\pi} \int_0^\pi \{-F_{r1ss} \cos(\theta) + F_{\phi1ss} \sin(\theta)\} d\theta d\phi \]
\[ {} \quad {} + \int_0^{2\pi} \int_0^\pi \{-F_{r2ss} \cos(\theta) + F_{\phi2ss} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi \]
\[ {} \quad {} + \int_0^{2\pi} \int_0^\pi \{-F_{r3ss} \cos(\theta) + F_{\phi3ss} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi \]
\[ {} \quad {} + \int_0^{2\pi} \int_0^\pi \{-F_{r4ss} \cos(\theta) + F_{\phi4ss} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi \]
\[ F_{z0sc} = \int_0^{2\pi} \int_0^\pi \{-F_{r0sc} \cos(\theta) + F_{\phi0sc} \sin(\theta)\} d\theta d\phi \]
\[ F_{z1sc} = \int_0^{2\pi} \int_0^\pi \{-F_{r1sc} \cos(\theta) + F_{\phi1sc} \sin(\theta)\} d\theta d\phi \]
\[ {} \quad {} + \int_0^{2\pi} \int_0^\pi \{-F_{r2sc} \cos(\theta) + F_{\phi2sc} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi \]
\[ {} \quad {} + \int_0^{2\pi} \int_0^\pi \{-F_{r3sc} \cos(\theta) + F_{\phi3sc} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi \]
\[ {} \quad {} + \int_0^{2\pi} \int_0^\pi \{-F_{r4sc} \cos(\theta) + F_{\phi4sc} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi \]
\[ F_{r0cc} = \frac{3}{2} \rho_0 \sin(\theta) A_{1,1} + \frac{\lambda^2}{2} \rho_0 \csc(\theta) A_{3,3} + \frac{\lambda^2}{2} \rho_0 \sin(\theta) A_{5,5} - \lambda^2 2a^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) A_{7,7} \]

\[ F_{r1cc} = 3 \rho_0 \sin(\theta) A_{1,2} + \lambda^2 \rho_0 \csc(\theta) A_{3,4} + \lambda^2 \rho_0 \sin(\theta) A_{5,6} - \lambda^2 2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) A_{7,8} \]

\[ F_{r2cc} = 3 \rho_0 \sin(\theta) A_{1,1} - \lambda^2 2a^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) A_{7,7} \]

\[ F_{r3cc} = -\lambda \rho_0 \sin(\theta) A_{1,5} \]

\[ F_{r4cc} = -\lambda \rho_0 \csc(\theta) A_{1,3} \]

\[ F_{r0ss} = \frac{3}{2} \rho_0 \sin(\theta) B_{1,1} + \frac{\lambda^2}{2} \rho_0 \csc(\theta) B_{3,3} + \frac{\lambda^2}{2} \rho_0 \sin(\theta) B_{5,5} - \lambda^2 2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) B_{7,7} \]

\[ F_{r1ss} = 3 \rho_0 \sin(\theta) B_{1,2} + \lambda^2 \rho_0 \csc(\theta) B_{3,4} + \lambda^2 \rho_0 \sin(\theta) B_{5,6} - \lambda^2 2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) B_{7,8} \]

\[ F_{r2ss} = 3 a^2 \rho_0 \sin(\theta) B_{1,1} - \lambda^2 2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) B_{7,7} \]

\[ F_{r3ss} = -\lambda \rho_0 \sin(\theta) B_{1,5} \]

\[ F_{r4ss} = -\lambda \rho_0 \csc(\theta) B_{1,3} \]

\[ F_{r0sc} = \frac{3}{2} \rho_0 \sin(\theta) C_{1,1} + \frac{\lambda^2}{2} \rho_0 \csc(\theta) C_{3,3} + \frac{\lambda^2}{2} \rho_0 \sin(\theta) C_{5,5} - \lambda^2 2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) C_{7,7} \]

\[ F_{r1sc} = 3 \rho_0 \sin(\theta) C_{1,2} + \lambda^2 \rho_0 \csc(\theta) C_{3,4} + \lambda^2 \rho_0 \sin(\theta) C_{5,6} - \lambda^2 2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) C_{7,8} \]

\[ F_{r2sc} = 3 \rho_0 \sin(\theta) C_{1,1} - \lambda^2 2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) C_{7,7} \]

\[ F_{r3sc} = -\lambda \rho_0 \sin(\theta) C_{1,5} \]

\[ F_{r4sc} = -\lambda \rho_0 \csc(\theta) C_{1,3} \]
\[ F_{\theta 0cc} = \lambda \rho_0 \sin(\theta) A_{1,5} \]
\[ F_{\theta 1cc} = \lambda \rho_0 \sin(\theta) A_{1,6} \]
\[ F_{\theta 2cc} = \lambda \rho_0 \sin(\theta) A_{1,5} \]
\[ F_{\theta 3cc} = -\frac{1}{2} \rho_0 \sin(\theta) A_{1,1} - \frac{\lambda^2}{2} \rho_0 \csc(\theta) A_{3,3} - \frac{3\lambda^2}{2} \rho_0 \sin(\theta) A_{5,5} + \lambda^2 \alpha^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) A_{7,7} \]
\[ F_{\theta 4cc} = -\lambda^2 \rho_0 \csc(\theta) A_{3,5} \]
\[ F_{\theta 0ss} = \lambda \rho_0 \sin(\theta) B_{1,5} \]
\[ F_{\theta 1ss} = \lambda \rho_0 \sin(\theta) B_{1,6} \]
\[ F_{\theta 2ss} = \lambda \rho_0 \sin(\theta) B_{1,5} \]
\[ F_{\theta 3ss} = -\frac{1}{2} \rho_0 \sin(\theta) B_{1,1} - \frac{\lambda^2}{2} \rho_0 \csc(\theta) B_{3,3} - \frac{3\lambda^2}{2} \rho_0 \sin(\theta) B_{5,5} + \lambda^2 \alpha^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) B_{7,7} \]
\[ F_{\theta 4ss} = -\lambda^2 \rho_0 \csc(\theta) B_{3,5} \]
\[ F_{\theta 0sc} = \lambda \rho_0 \sin(\theta) C_{1,5} \]
\[ F_{\theta 1sc} = \lambda \rho_0 \sin(\theta) C_{1,6} \]
\[ F_{\theta 2sc} = \lambda \rho_0 \sin(\theta) C_{1,5} \]
\[ F_{\theta 3sc} = -\frac{1}{2} \rho_0 \sin(\theta) C_{1,1} - \frac{\lambda^2}{2} \rho_0 \csc(\theta) C_{3,3} - \frac{3\lambda^2}{2} \rho_0 \sin(\theta) C_{5,5} + \lambda^2 \alpha^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) C_{7,7} \]
\[ F_{\theta 4sc} = -\lambda^2 \rho_0 \csc(\theta) C_{3,5} \]
\[ F_{\phi_{0cc}} = \lambda \rho_0 A_{1,3} \]
\[ F_{\phi_{1cc}} = \lambda \rho_0 A_{1,4} + \lambda \rho_0 A_{3,2} \]
\[ F_{\phi_{2cc}} = \lambda \rho_0 A_{1,3} \]
\[ F_{\phi_{3cc}} = -\lambda^2 \rho_0 A_{3,5} \]
\[ F_{\phi_{4cc}} = -\frac{1}{2} \rho_0 A_{1,1} - \frac{3 \lambda^2}{2} \rho_0 \csc^2(\theta) A_{3,3} - \frac{\lambda^2}{2} \rho_0 A_{5,5} + \lambda^2 \alpha^2 \rho_0 \omega^2 \frac{\omega^2}{2c_0^2} A_{7,7} \]
\[ F_{\phi_{0ss}} = \lambda \rho_0 B_{1,3} \]
\[ F_{\phi_{1ss}} = \lambda \rho_0 B_{1,4} + \lambda \rho_0 B_{3,2} \]
\[ F_{\phi_{2ss}} = \lambda \rho_0 B_{1,3} \]
\[ F_{\phi_{3ss}} = -\lambda^2 \rho_0 B_{3,5} \]
\[ F_{\phi_{4ss}} = -\frac{1}{2} \rho_0 B_{1,1} - \frac{3 \lambda^2}{2} \rho_0 \csc^2(\theta) B_{3,3} - \frac{\lambda^2}{2} \rho_0 B_{5,5} + \lambda^2 \alpha^2 \rho_0 \omega^2 \frac{\omega^2}{2c_0^2} B_{7,7} \]
\[ F_{\phi_{0sc}} = \lambda \rho_0 C_{1,3} \]
\[ F_{\phi_{1sc}} = \lambda \rho_0 C_{1,4} + \lambda \rho_0 C_{3,2} \]
\[ F_{\phi_{2sc}} = \lambda \rho_0 C_{1,3} \]
\[ F_{\phi_{3sc}} = -\lambda^2 \rho_0 C_{3,5} \]
\[ F_{\phi_{4sc}} = -\frac{1}{2} \alpha^2 \rho_0 C_{1,1} - \frac{3 \lambda^2}{2} \rho_0 \csc^2(\theta) C_{3,3} - \frac{\lambda^2}{2} \rho_0 C_{5,5} + \lambda^2 \alpha^2 \rho_0 \omega^2 \frac{\omega^2}{2c_0^2} C_{7,7} \]

Where, for \( l, p \in \{1, 3, 7\} \) we have,

\[ A_{l,p} = 4P_0^0(\cos \theta)P_0^0(\cos \theta) \langle \alpha_{l,0}^0 \alpha_{p,0}^0 \rangle \\
+ 4P_0^0(\cos \theta)P_2^0(\cos \theta) \{ \langle \alpha_{l,0}^0 \alpha_{p,2}^0 \rangle + \langle \alpha_{l,2}^0 \alpha_{p,0}^0 \rangle \} \\
+ 4P_2^0(\cos \theta)P_2^0(\cos \theta) \langle \alpha_{l,2}^0 \alpha_{p,2}^0 \rangle \]

\[ B_{l,p} = 4P_1^0(\cos \theta)P_1^0(\cos \theta) \langle \beta_{l,1}^0 \beta_{p,1}^0 \rangle \]

\[ C_{l,p} = 2P_0^0(\cos \theta)P_1^0(\cos \theta) \{ \langle \alpha_{l,0}^0 \beta_{p,1}^0 \rangle + \langle \alpha_{p,0}^0 \beta_{l,1}^0 \rangle \} \\
+ 2P_1^0(\cos \theta)P_2^0(\cos \theta) \{ \langle \alpha_{p,2}^0 \beta_{l,1}^0 \rangle + \langle \alpha_{l,2}^0 \beta_{p,1}^0 \rangle \} \]
For $l \in \{1, 3, 7\}$ $p = \{5\}$ we have,

\[
A_{l,5} = 4P_0^0(\cos \theta)P_{00}^0(\cos \theta) \langle \alpha_{l,0}^0 \alpha_{5,0}^0 \rangle \\
+ 4P_0^0(\cos \theta)P_{20}^0(\cos \theta) \langle \alpha_{l,0}^0 \alpha_{5,2}^0 \rangle \\
+ 4P_2^0(\cos \theta)P_{00}^0(\cos \theta) \langle \alpha_{l,2}^0 \alpha_{5,0}^0 \rangle \\
+ 4P_2^0(\cos \theta)P_{20}^0(\cos \theta) \langle \alpha_{l,2}^0 \alpha_{5,2}^0 \rangle \\
B_{l,5} = 4P_1^0(\cos \theta)P_{10}^0(\cos \theta) \langle \beta_{l,1}^0 \beta_{5,1}^0 \rangle \\
C_{l,p} = 2P_0^0(\cos \theta)P_{10}^0(\cos \theta) \langle \alpha_{l,0}^0 \beta_{5,1}^0 \rangle \\
\quad + 2P_1^0(\cos \theta)P_{00}^0(\cos \theta) \langle \beta_{l,1}^0 \alpha_{5,0}^0 \rangle \\
\quad + 2P_1^0(\cos \theta)P_{20}^0(\cos \theta) \langle \beta_{l,1}^0 \alpha_{5,2}^0 \rangle \\
\quad + 2P_2^0(\cos \theta)P_{10}^0(\cos \theta) \langle \alpha_{l,2}^0 \beta_{5,1}^0 \rangle
For \( l \in \{1, 3, 7\} \) and \( p \in \{2, 4, 8\} \) we have,

\[
A_{l,p} = 4P_0^0(\cos \theta)P_0^0(\cos \theta) \left\langle \alpha_{l,0}^0\alpha_{p,0}^0 \right\rangle \\
+ 4P_0^0(\cos \theta)P_2^{-2}(\cos \theta) \left\langle \alpha_{l,0}^0\alpha_{p,2}^{-2} \right\rangle \\
+ 4P_0^0(\cos \theta)P_2^{-1}(\cos \theta) \left\langle \alpha_{l,0}^0\alpha_{p,2}^{-1} \right\rangle \\
+ 4P_0^0(\cos \theta)P_2^0(\cos \theta) \left\{ \left\langle \alpha_{l,0}^0\alpha_{p,2}^0 \right\rangle + \left\langle \alpha_{l,2}^0\alpha_{p,0}^0 \right\rangle \right\} \\
+ 4P_0^0(\cos \theta)P_2^1(\cos \theta) \left\langle \alpha_{l,0}^0\alpha_{p,2}^1 \right\rangle \\
+ 4P_0^0(\cos \theta)P_2^2(\cos \theta) \left\langle \alpha_{l,0}^0\alpha_{p,2}^2 \right\rangle \\
+ 4P_2^0(\cos \theta)P_2^{-2}(\cos \theta) \left\langle \alpha_{l,2}^0\alpha_{p,2}^{-2} \right\rangle \\
+ 4P_2^0(\cos \theta)P_2^{-1}(\cos \theta) \left\langle \alpha_{l,2}^0\alpha_{p,2}^{-1} \right\rangle \\
+ 4P_2^0(\cos \theta)P_2^0(\cos \theta) \left\langle \alpha_{l,2}^0\alpha_{p,2}^0 \right\rangle \\
+ 4P_2^0(\cos \theta)P_2^1(\cos \theta) \left\langle \alpha_{l,2}^0\alpha_{p,2}^1 \right\rangle \\
+ 4P_2^0(\cos \theta)P_2^2(\cos \theta) \left\langle \alpha_{l,2}^0\alpha_{p,2}^2 \right\rangle \\
B_{l,p} = 4P_1^0(\cos \theta)P_1^{-1}(\cos \theta) \left\langle \beta_{l,1}^0\beta_{p,1}^{-1} \right\rangle \\
+ 4P_1^0(\cos \theta)P_1^0(\cos \theta) \left\langle \beta_{l,1}^0\beta_{p,1}^0 \right\rangle \\
+ 4P_1^0(\cos \theta)P_1^1(\cos \theta) \left\langle \beta_{l,1}^0\beta_{p,1}^1 \right\rangle \\
C_{l,p} = 2P_0^0(\cos \theta)P_1^{-1}(\cos \theta) \left\langle \alpha_{l,0}^0\beta_{p,1}^{-1} \right\rangle \\
+ 2P_0^0(\cos \theta)P_1^0(\cos \theta) \left\{ \left\langle \alpha_{l,0}^0\beta_{p,1}^0 \right\rangle + \left\langle \alpha_{p,0}^0\beta_{l,1}^0 \right\rangle \right\} \\
+ 2P_0^0(\cos \theta)P_1^1(\cos \theta) \left\langle \alpha_{l,0}^0\beta_{p,1}^1 \right\rangle \\
+ 2P_0^0(\cos \theta)P_1^2(\cos \theta) \left\langle \alpha_{l,0}^0\beta_{p,2}^0 \right\rangle \\
+ 2P_2^0(\cos \theta)P_2^{-1}(\cos \theta) \left\langle \alpha_{p,2}^{-1}\beta_{l,1}^0 \right\rangle \\
+ 2P_1^0(\cos \theta)P_2^0(\cos \theta) \left\{ \left\langle \alpha_{p,2}^0\beta_{l,1}^0 \right\rangle + \left\langle \alpha_{l,2}^0\beta_{p,1}^0 \right\rangle \right\} \\
+ 2P_1^0(\cos \theta)P_2^1(\cos \theta) \left\langle \alpha_{p,2}^0\beta_{l,1}^1 \right\rangle \\
+ 2P_1^0(\cos \theta)P_2^2(\cos \theta) \left\langle \alpha_{p,2}^0\beta_{l,1}^2 \right\rangle \\
+ 2P_2^0(\cos \theta)P_1^{-1}(\cos \theta) \left\langle \alpha_{l,2}^{-1}\beta_{p,1}^0 \right\rangle \\
+ 2P_2^0(\cos \theta)P_1^1(\cos \theta) \left\langle \alpha_{l,2}^0\beta_{p,1}^1 \right\rangle
Similarly we can find $A_{l,p}$, $B_{l,p}$, $C_{l,p}$ for $l = 5$, $p \in \{2, 4, 8\}$ and for $l \in \{1, 3, 7\}$, $p = 6$. and,

$$
\alpha^m_{1,n} = \Re \left\{ ak^* \tilde{B}^{(0)}_{nm,j_n'} (ak^*) e^{im\phi e^{i\omega t}} \right\}
$$

$$
\beta^m_{1,n} = \Im \left\{ ak^* \tilde{B}^{(0)}_{nm,j_n'} (ak^*) e^{im\phi e^{i\omega t}} \right\}
$$

$$
\alpha^m_{2,n} = \Re \left\{ ak^* \left( \tilde{B}^{(1)}_{nm,j_n'} (ak^*) + ak^* \delta(\theta, \phi) \tilde{B}^{(0)}_{nm,j_n'} (ak^*) \right) e^{im\phi e^{i\omega t}} \right\}
$$

$$
\beta^m_{2,n} = \Im \left\{ ak^* \left( \tilde{B}^{(1)}_{nm,j_n'} (ak^*) + ak^* \delta(\theta, \phi) \tilde{B}^{(0)}_{nm,j_n'} (ak^*) \right) e^{im\phi e^{i\omega t}} \right\}
$$

$$
\alpha^m_{3,n} = \Re \left\{ im \tilde{B}^{(0)}_{nm,j_n} (ak^*) e^{im\phi e^{i\omega t}} \right\}
$$

$$
\beta^m_{3,n} = \Im \left\{ im \tilde{B}^{(0)}_{nm,j_n} (ak^*) e^{im\phi e^{i\omega t}} \right\}
$$

$$
\alpha^m_{4,n} = \Re \left\{ im \left( \tilde{B}^{(1)}_{nm,j_n} (ak^*) + ak^* \delta(\theta, \phi) \tilde{B}^{(0)}_{nm,j_n} (ak^*) \right) e^{im\phi e^{i\omega t}} \right\}
$$

$$
\beta^m_{4,n} = \Im \left\{ im \left( \tilde{B}^{(1)}_{nm,j_n} (ak^*) + ak^* \delta(\theta, \phi) \tilde{B}^{(0)}_{nm,j_n} (ak^*) \right) e^{im\phi e^{i\omega t}} \right\}
$$

$$
\alpha^m_{5,n} = \Re \left\{ \tilde{B}^{(0)}_{nm,j_n} (ak^*) e^{im\phi e^{i\omega t}} \right\}
$$

$$
\beta^m_{5,n} = \Im \left\{ \tilde{B}^{(0)}_{nm,j_n} (ak^*) e^{im\phi e^{i\omega t}} \right\}
$$

$$
\alpha^m_{6,n} = \Re \left\{ \left( \tilde{B}^{(1)}_{nm,j_n} (ak^*) + a \tilde{B}^{(0)}_{nm,k} \delta(\theta, \phi) j_n'' (ak^*) \right) e^{im\phi e^{i\omega t}} \right\}
$$

$$
\beta^m_{6,n} = \Im \left\{ \left( \tilde{B}^{(1)}_{nm,j_n} (ak^*) + a \tilde{B}^{(0)}_{nm,k} \delta(\theta, \phi) j_n'' (ak^*) \right) e^{im\phi e^{i\omega t}} \right\}
$$

$$
\alpha^m_{7,n} = \Re \left\{ \tilde{B}^{(0)}_{nm,j_n} (ak^*) e^{im\phi e^{i\omega t}} \right\}
$$

$$
\beta^m_{7,n} = \Im \left\{ \tilde{B}^{(0)}_{nm,j_n} (ak^*) e^{im\phi e^{i\omega t}} \right\}
$$

$$
\alpha^m_{8,n} = \Re \left\{ \left( \tilde{B}^{(1)}_{nm,j_n} (ak^*) + ak^* \tilde{B}^{(0)}_{nm} \delta(\theta, \phi) j_n'' (ak^*) \right) e^{im\phi e^{i\omega t}} \right\}
$$

$$
\beta^m_{8,n} = \Im \left\{ \left( \tilde{B}^{(1)}_{nm,j_n} (ak^*) + ak^* \tilde{B}^{(0)}_{nm} \delta(\theta, \phi) j_n'' (ak^*) \right) e^{im\phi e^{i\omega t}} \right\}
$$
Notice that,

\[
\sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_n^m(\cos \theta) M_{l,n}^m = \sum_{i=0}^{\infty} S_{l,i} \quad \forall \ l = \{1, 2, 3, 4, 7, 8\}
\]

where,

\[
S_{l,0} = P_0^0(\cos \theta) M_{l,0}^0
\]

\[
S_{l,1} = P_1^{-1}(\cos \theta) M_{l,1}^{-1} + P_1^0(\cos \theta) M_{l,1}^0 + P_1^1(\cos \theta) M_{l,1}^1
\]

\[
S_{l,2} = P_2^{-2}(\cos \theta) M_{l,2}^{-2} + P_2^{-1}(\cos \theta) M_{l,2}^{-1} + P_2^0(\cos \theta) M_{l,2}^0 + P_2^1(\cos \theta) M_{l,2}^1 + P_2^2(\cos \theta) M_{l,2}^2
\]

\[
\langle S_{1,j} S_{2,k} \rangle = \frac{1}{T} \int_0^T (S_{1,j} S_{2,k})dt, \quad T = \frac{2\pi}{\omega}
\]

The \( S_{l,j} \ \forall \ l = \{1, 2, 3, 4, 7, 8\} \ j = \{0, 1, 2,...\} \) may be written in the form:

\[
S_{l,j} = S_{l,j[1]} \cos(\omega t) + S_{l,j[2]} \sin(\omega t)
\]

In particular, \( S_{l,0} \) can be written as

\[
S_{l,0} = S_{l,0[1]} \cos(\omega t) + S_{l,0[2]} \sin(\omega t)
\]

\[
= P_0^0(\cos \theta) M_{l,0[1]}^0 \cos(\omega t) + P_0^0(\cos \theta) M_{l,0[2]}^0 \sin(\omega t)
\]

\[
= P_0^0(\cos \theta) \left[ M_{l,0[1]}^0 \cos(\omega t) + M_{l,0[2]}^0 \sin(\omega t) \right]
\]

Similarly, we can write \( S_{l,1} \) and \( S_{l,2} \).

Now from,

\[
S_{1,j} = S_{1,j[1]} \cos(\omega t) + S_{1,j[2]} \sin(\omega t)
\]

\[
S_{2,k} = S_{2,k[1]} \cos(\omega t) + S_{2,k[2]} \sin(\omega t)
\]

\[
S_{1,j} S_{2,k} = S_{1,j[1]} S_{2,k[1]} \cos^2(\omega t) + S_{1,j[1]} S_{2,k[2]} \cos(\omega t) \sin(\omega t)
\]

\[
+ S_{2,k[2]} S_{2,k[2]} \sin^2(\omega t) + S_{1,j[2]} S_{2,k[1]} \cos(\omega t) \sin(\omega t)
\]
it follows that
\[ \langle S_{1,j} S_{2,k} \rangle = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} (S_{1,j} S_{2,k}) dt \]

Using the following result
\[ \int_0^{2\pi/\omega} \cos^2(\omega t) dt = \frac{\pi}{\omega} \]
\[ \int_0^{2\pi/\omega} \sin(\omega t) \cos(\omega t) dt = 0 \]
\[ \int_0^{2\pi/\omega} \sin^2(\omega t) dt = \frac{\pi}{\omega}, \]
we obtain
\[ \langle S_{1,j} S_{2,k} \rangle = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} S_{1,j} S_{2,k} dt \]
\[ = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} S_{1,j[1]} S_{2,k[1]} \cos^2(\omega t) \]
\[ + \frac{\omega}{2\pi} \int_0^{2\pi/\omega} S_{1,j[1]} S_{2,k[2]} \cos(\omega t) \sin(\omega t) \]
\[ + \frac{\omega}{2\pi} \int_0^{2\pi/\omega} S_{2,j[2]} S_{2,k[2]} \sin^2(\omega t) \]
\[ + \frac{\omega}{2\pi} \int_0^{2\pi/\omega} S_{1,j[2]} S_{2,k[1]} \cos(\omega t) \sin(\omega t) \]
\[ = \frac{\omega}{2\pi} \left[ S_{1,j[1]} S_{2,k[1]} \frac{\pi}{\omega} + S_{1,j[2]} S_{2,k[2]} \frac{\pi}{\omega} \right] \]

Hence,
\[ 2 \langle S_{1,j} S_{2,k} \rangle = S_{1,j[1]} S_{2,k[1]} + S_{1,j[2]} S_{2,k[2]}, \]
where
\[ S_{1,j[1]} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} S_{1,j} \cos(\omega t) dt \]
\[ S_{1,j[2]} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} S_{1,j} \sin(\omega t) dt \]

Similarly we can prove that the above results hold \( \forall l = \{1, 2, 3, 4, 7, 8\} \)
Figure 2.3 2D-slice of obstacle placed in a plane standing acoustic-wave field.
CHAPTER 3

LAMB’S GENERAL SOLUTION OF STOKES EQUATION

For the purpose of computing the drag, we may assume that the governing equations for the fluid motion are given by the following Stokes’ equations for small transitional and/or angular Reynolds numbers ([5])

\[
\nabla^2 \mathbf{v} = \frac{1}{\mu} \nabla p
\]

\[
\nabla \cdot \mathbf{v} = 0
\]  \(\text{(3.1)}\)

where \(\mathbf{v}\) is the local velocity field, \(\mu\) is dynamic viscosity and \(p\) is the hydrodynamic pressure.

Based on ([5]) Lamb’s general solution of Stokes equation (3.1) in terms of spherical harmonics is given as (in vector form)

\[
\mathbf{v} = \sum_{n=-\infty}^{\infty} \left[ \nabla \times (r \chi_n) + \nabla \phi_n + \frac{(n+3)}{2(n+1)(2n+3)\mu} \times r^2 \nabla p_n - r \frac{n}{(n+1)(2n+3)\mu} p_n \right]
\]  \(\text{(3.2)}\)

\[
p = \sum_{n=-\infty}^{\infty} p_n
\]  \(\text{(3.3)}\)

where \(\chi_n, \phi_n\) and \(p_n\) are solid spherical harmonics of degree \(n\).

We now outline the general scheme for obtaining the harmonic functions in (3.2) for an arbitrary prescribed set of boundary condition to be satisfied on the surface of a sphere (of radius \(a\), say). This summary is sufficient since the details are given in [5].

According to Euler’s theorem for a homogeneous function \(h_n\) in \(r^n\), we have

\[
r \frac{\partial h_n}{\partial r} = nh_n
\]  \(\text{(3.4)}\)
A solid spherical harmonic $h_n$ satisfies

\[ [h_n]_{r=a} = \left( \frac{a}{r} \right)^n h_n \]  

(3.5)

Using Equations (3.4) and (3.5) we have the following identities:

\[
\left[ \frac{r \cdot v}{r} \right]_{r=a} = \frac{r}{r} \cdot v(a, \theta, \phi)
= \sum_{n=-\infty}^{\infty} \left[ \frac{n a}{2(2n + 3) \mu} \left( \frac{a}{r} \right)^n p_n + \frac{n}{a} \left( \frac{a}{r} \right)^n \phi_n \right] 
= \sum_{n=1}^{\infty} X_n(\theta, \phi)
\]  

(3.6)

(3.7)

(3.8)

\[
\left[ (r \cdot \nabla) \left( \frac{r \cdot v}{r} \right) - r \nabla \cdot v \right]_{r=a} = -r \nabla \cdot v(a, \theta, \phi)
= \sum_{n=-\infty}^{\infty} \left[ \frac{n(n + 1) a}{2(2n + 3) \mu} \left( \frac{a}{r} \right)^n p_n + \frac{n(n - 1)}{a} \left( \frac{a}{r} \right)^n \phi_n \right] 
= \sum_{n=1}^{\infty} Y_n(\theta, \phi)
\]  

(3.9)

(3.10)

(3.11)

\[
[r \cdot \nabla \times v]_{r=a} = r \cdot \nabla \times v(a, \theta, \phi)
= \sum_{n=-\infty}^{\infty} n(n + 1) \left( \frac{a}{r} \right)^n \chi_n 
= \sum_{n=1}^{\infty} Z_n(\theta, \phi)
\]  

(3.12)

(3.13)

(3.14)

where $X_n(\theta, \phi)$, $Y_n(\theta, \phi)$, and $Z_n(\theta, \phi)$ are spherical harmonics.

For exterior problems, in which the fluid motion takes place in the infinite space external to the sphere $r = a$, the condition that the fluid be at rest at infinity requires that

\[ p_n = \phi_n = \chi_n = 0 \quad n \geq 1, \]
so only the negative harmonic functions will survive and we have

\[
p_{-(n+1)} = \frac{(2n-1)\mu}{(n+1)a} \left( \frac{a}{r} \right)^{n+1} [(n+2)X_n + Y_n] \tag{3.15}
\]

\[
\phi_{-(n+1)} = \frac{a}{2(n+1)} \left( \frac{a}{r} \right)^{n+1} [nX_n + Y_n] \tag{3.16}
\]

\[
\chi_{-(n+1)} = \frac{1}{n(n+1)} \left( \frac{a}{r} \right)^{n+1} Z_n \tag{3.17}
\]

In this notation Lamb’s solution to Equations (3.2) and (3.3) may be written as:

\[
\bar{u} = \sum_{n=1}^{\infty} \left[ \bar{\nabla} \times \left( \bar{r} \chi_{-(n+1)} \right) + \bar{\nabla} \phi_{-(n+1)} - \frac{(n-2)}{2n(2n-1)\mu} r^2 \nabla p_{-(n+1)} + \frac{(n+1)}{n(2n-1)\mu} p_{-(n+1)} \right] \tag{3.18}
\]

and

\[
p = \sum_{n=1}^{\infty} p_{-(n+1)}, \tag{3.19}
\]

and the hydrodynamic force exerted by the fluid on the particle is given by ([5])

\[
F = -4\pi \nabla (r^3 p_{-2}) \tag{3.20}
\]

### 3.1 Drag on a Perturbed sphere

We now proceed to find the solution of Equation (3.1) satisfying the following boundary conditions.

\[
\bar{u} = \bar{0} \quad \text{on } S_p \tag{3.21}
\]

\[
\bar{u} = \bar{U} \quad \text{at } r = \infty \tag{3.22}
\]

where \( S_p \) is the surface of a perturbed sphere of radius \( r = a(1+\epsilon\delta(\theta, \phi)) \) and \( \delta(\theta, \phi) \) is an arbitrary smooth perturbation function \( \pi \)-periodic in \( \theta \) and \( 2\pi \)-periodic in \( \phi \). To this end we will assume that the velocity and pressure fields can be expanded in
powers of $\epsilon$ as

$$\tilde{V} = \sum_{i=0}^{\infty} \epsilon^i \tilde{V}^{(i)} \quad (3.23)$$

$$p = \sum_{i=0}^{\infty} \epsilon^i p^{(i)} \quad (3.24)$$

Substituting Equation (3.23) and Equation (3.24) in (3.1) and equating like powers of $\epsilon$, we find that the individual perturbation fields $(\tilde{V}^{(i)}, p^{(i)})$ satisfy Stokes’ equations

$$\nabla^2 \tilde{V}^{(i)} = \frac{1}{\mu} \nabla p^{(i)} \quad (3.25)$$

$$\tilde{V} \cdot \nabla \tilde{V}^{(i)} = 0 \quad (3.26)$$

Similarly substituting Equation (3.23) in Equation (3.22) and equating like powers of $\epsilon$ we find that

$$\tilde{V}^{(0)} = \tilde{U} \quad \text{at} \quad r = \infty \quad (3.27)$$

$$\tilde{V}^{(i)} = \tilde{0} \quad \text{at} \quad r = \infty \quad (i = 1, 2, 3, \ldots) \quad (3.28)$$

Finally, the boundary condition (3.22) is rewritten as

$$\sum_{i=0}^{\infty} \epsilon^i \tilde{V}^{(i)} = \tilde{0} \quad \text{on} \quad S_p \quad (3.29)$$

A Taylor series expansion of $\tilde{V}^{(i)}$ about $r = a$ is

$$\tilde{V}^{(i)} = (\tilde{V}^{(i)})_{r=a} + \sum_{j=1}^{\infty} \frac{1}{j!} (r-a)^j \left( \frac{\partial^{(j)} \tilde{V}^{(i)}}{\partial r^{(j)}} \right)_{r=a}$$

As, $r - a = \epsilon a \delta(\theta, \phi)$ on $S_p$, the boundary condition (3.29) reduces to

$$\sum_{i=0}^{\infty} \epsilon^i \left[ \tilde{V}^{(i)} + \sum_{j=1}^{\infty} \frac{1}{j!} \epsilon^j a^j \delta^j(\theta, \phi) \left( \frac{\partial^{(j)} \tilde{V}^{(i)}}{\partial r^{(j)}} \right) \right] = 0, \quad \text{at} \quad r = a$$

Rearranging the terms involving like powers of $\epsilon$, we obtain

$$\tilde{V}^{(0)} + \sum_{i=1}^{\infty} \epsilon^i \left[ \tilde{V}^{(i)} \sum_{j=1}^{\infty} \frac{1}{j!} \delta^j(\theta, \phi) \left( \frac{\partial^{(j)} \tilde{V}^{(i-j)}}{\partial r^{(j)}} \right) \right] = 0, \quad \text{at} \quad r = a$$
Thus, the boundary condition on the deformed sphere may be satisfied to any order in \( \varepsilon \) by requiring that the various perturbation fields satisfy the following conditions

\[
\tilde{v}^{(0)} = 0 \quad \text{at} \quad r = a \quad (3.30)
\]

and for \( i = 1, 2, 3, \ldots \),

\[
\tilde{v}^i = -\sum_{j=1}^{\infty} \frac{1}{j!} a^j \delta^j(\theta, \phi) \left( \frac{\partial^j \tilde{v}^{(i-j)}}{\partial \tau^j} \right) \quad \text{at} \quad r = a \quad (3.31)
\]

The leading term in the expansion, \((\tilde{v}^{(0)}, p^{(0)})\), satisfying boundary conditions (3.27) and (3.30) is, of course, Stokes' solution for streaming flow past the undeformed sphere. This solution is ([23])

\[
\tilde{v}^{(0)} = \tilde{U} + \nabla \phi^{(0)} + \frac{1}{2\mu} r^2 \nabla p^{(0)} + \frac{2}{\mu} r \tilde{p}^{(0)}_{-2} \quad (3.32)
\]

\[
p^{(0)} = p^{(0)}_{-2} \quad (3.33)
\]

where

\[
p^{(0)}_{-2} = -\frac{3a\mu \vec{r} \cdot \tilde{U}}{2r^3} \quad (3.34)
\]

\[
\phi^{(0)}_{-2} = -\frac{a^3 \vec{r} \cdot \tilde{U}}{4r^3} \quad (3.35)
\]

The boundary conditions to be satisfied by the next perturbation, \( \tilde{v}^{(1)} \), are given by the Equations (3.28) and (3.31) for \( i = 1 \). Since \( \delta(\theta, \phi) = \sum_{k=0}^{\infty} \delta_k(\theta, \phi) \), the linearity of Stokes' equations and boundary conditions allows us to write

\[
\tilde{v}^{(1)} = \sum_{k=0}^{\infty} \tilde{v}_k^{(1)} \quad (3.36)
\]

\[
p^{(1)} = \sum_{k=0}^{\infty} p_k^{(1)} \quad (3.37)
\]
where each \( \vec{v}_k^{(1)} (k = 0, 1, 2, \cdots) \) satisfies Stokes equations and the boundary conditions

\[
\vec{v}_k^{(1)} = 0 \quad \text{at} \quad r = \infty \quad (3.38)
\]
\[
\psi_k^{(1)} (a, \theta, \phi) = -\frac{3}{2} \vec{U} \cdot \left( I - \frac{r^2}{r^2} \right) \delta_k (\theta, \phi) \quad (3.39)
\]

Using the general methods from the previous section we can find the fields \((\psi_k^{(1)}, p_k^{(1)})\) satisfying the above boundary conditions. From [Brenner] we have

\[
k p_{-(n+1)}^{(1)} = \left\{ \begin{array}{ll}
\frac{3(k-1)(2k-3)a^{k-1}}{2k(2k+1)} r^{-(2k+1)} \times (\vec{U} \cdot \vec{\nabla})r^k \delta_k & \text{for } n = k - 1; \\
\frac{3}{2} a^{k+1} (\vec{U} \cdot \vec{\nabla})r^{-(k+1)} \delta_k & \text{for } n = k + 1; \\
0 & \text{for all other } n.
\end{array} \right. \quad (3.40)
\]

\[
k \psi_{-(n+1)}^{(1)} = \left\{ \begin{array}{ll}
\frac{3(k-1)a^{k+1}}{4k(2k+1)} r^{-(2k-1)} \times (\vec{U} \cdot \vec{\nabla})r^k \delta_k & \text{for } n = k - 1; \\
\frac{3a^{k+3}}{4(2k+1)} (\vec{U} \cdot \vec{\nabla})r^{-(k+1)} \delta_k & \text{for } n = k + 1; \\
0 & \text{for all other } n.
\end{array} \right. \quad (3.41)
\]

\[
k \chi_{-(n+1)}^{(1)} = \left\{ \begin{array}{ll}
\frac{3a^{k+1}}{2k(2k+1)} r^{-(k+1)} \times (\vec{U} \cdot \vec{\nabla}) \times (\vec{r} \delta_k) & \text{for } n = k; \\
0 & \text{for all other } n.
\end{array} \right. \quad (3.42)
\]

Using Equations (3.18) and (3.19) the above expression will provide the velocity and pressure fields \((\psi_k^{(1)}, p_k^{(1)})\). When the latter are summed over \(k\) in accordance with Equations (3.36) and (3.37), we obtain expressions for \((\vec{v}^{(1)}, p^{(1)})\).

The force on the deformed sphere may now be obtained from (3.20). Write

\[
\vec{F} = \vec{F}^{(0)} + \epsilon \vec{F}^{(1)} + O(\epsilon^2) \quad (3.43)
\]

where \(\vec{F}^{(0)} = 6\pi \mu a \vec{U}\) is the Stokes force on the undeformed sphere and

\[
\vec{F}^{(1)} = -4\pi \nabla \left( r^3 \sum_{k=-2}^{\infty} kp_{-k}^{(1)} \right) \quad (3.44)
\]
But from (3.40), since \( n = 1 \), only the terms \( k = 0 \) and \( k = 2 \) contribute to this infinite sum. Hence, since \( \delta_0 = \text{constant} \), we obtain

\[
\sum_{k=0}^{\infty} k p_{-2}^{(1)} = -\frac{3}{2} \mu a r^{-3} \left[ \delta_0 \bar{U} \cdot \bar{r} - \frac{1}{10} (\bar{U} \cdot \nabla) (r^2 \delta_2) \right]; 
\tag{3.45}
\]

whence,

\[
\bar{F} = 6\pi \mu a \bar{U} + 6\pi \mu a \epsilon \left[ \bar{U} \delta_0 - \frac{1}{10} (\bar{U} \cdot \nabla) \nabla (r^2 \delta_2) \right] + O(\epsilon^2) 
\tag{3.46}
\]

\[
= 6\pi \mu a \bar{T} \cdot \bar{U}
\]

where \( \bar{T} = I + \epsilon [I \delta_0 - \frac{1}{10} \nabla \nabla (r^2 \delta_2)] + O(\epsilon^2) \)

is a dimensionless Stokes resistance dyadic and \( \nabla \nabla (r^2 \delta_2) \) is a constant dyadic.

As the idempotent dyadic \( I \) and the dyadic operator \( \nabla \nabla \) are symmetric so is dyadic \( \bar{T} \).
CHAPTER 4

FORCES AND DYNAMIC EQUATION FOR PARTICLE

The goal is to find the trajectories of slightly perturbed spherical particles suspended in stationary sound wave fields in a low viscosity fluid. The equation of motion of a micron-sized perturbed sphere particle under the influence of a plane stationary acoustic field and fluid flow for a dilute suspension that is consistent with the mathematical model proposed by Aboobaker et al. [1, 2] is:

$$m_v \frac{d\vec{v}}{dt} = \vec{F}_{AC} + \vec{F}_D + \vec{F}_B$$

For dilute suspension, the secondary forces are neglected. The motion of the particles is observed with reference to the global Cartesian coordinates \((x, y, z)\) with the unit vectors \(\hat{e}_x, \hat{e}_y, \hat{e}_z\). Each particle has also has its own local spherical coordinate system \((r, \theta, \phi)\), the origin of which is at its equilibrium center. Thus the distance variable \(h\) in the force terms computed in the preceding chapters will be replaced by \(z\) when expressed in term of the global fixed Cartesian reference frame. Thus, \(z\) is the position of the migrating particle in the \(z\)-direction between a transducer and a reflector separated by one half wavelength at the given frequency.
\[ F_{AC} = (F_x, F_y, F_z) \] the acoustic radiation force on perturbed sphere

\[ F_i \approx F_{10cc} \cos^2(kz) + F_{10ss} \sin^2(kz) + F_{10sc} \sin(2kz) \]

\[ + \epsilon (F_{11cc} \cos^2(kz) + F_{11ss} \sin^2(kz) + F_{11sc} \sin(2kz)) + O(\epsilon^2) \quad \forall \ i \in \{x, y, z\} \]

\[ \ddot{g} = \text{gravitational force} \]

\[ \vec{\nu} = \text{velocity of the particle} = (\dot{x}, \dot{y}, \dot{z}) \]

\[ V_p = \text{volume of the perturbed spherical particle} \]

\[ V_p = \int_0^{2\pi} \int_0^\pi \int_0^{\alpha(1+\epsilon \delta(\theta, \phi))} r^2 \sin(\theta) dr d\theta d\phi \]

\[ = V_p^0 + \epsilon V_p^1 + \epsilon^2 V_p^2 + \epsilon^3 V_p^3 \]

\[ V_p^0 = \frac{4}{3} \pi a^3 = \text{volume of the unperturbed spherical particle} \]

\[ V_p^1 = \int_0^{2\pi} \int_0^\pi \alpha^3 \sin(\theta) \delta(\theta, \phi) d\theta d\phi \]

\[ V_p^2 = \int_0^{2\pi} \int_0^\pi \alpha^3 \sin(\theta) \delta(\theta, \phi)^2 d\theta d\phi \]

\[ V_p^3 = \int_0^{2\pi} \int_0^\pi \frac{1}{3} \alpha^3 \sin(\theta) \delta(\theta, \phi)^3 d\theta d\phi \]

\[ m_\nu = (\rho^* + 0.5 \rho_0) V_p = \text{virtual mass of the perturbed particle} \]

\[ = m_\nu^0 + \epsilon m_\nu^1 + \epsilon^2 m_\nu^2 + \epsilon^3 m_\nu^3 \]

\[ m_\nu^0 = (\rho^* + 0.5 \rho_0) V_p^0 = \text{virtual mass of the unperturbed sphere} \]

\[ m_\nu^i = (\rho^* + 0.5 \rho_0) V_p^i \quad \forall \ i \in \{1, 2, 3\} \]

\[ \vec{F}_B = V_p (\rho^* - \rho_0) \ddot{g} = \text{Net buoyancy force on the perturbed sphere} \]

\[ = F_B^0 + \epsilon F_B^1 + \epsilon^2 F_B^2 + \epsilon^3 F_B^3 \]

\[ F_B^0 = V_p^0 (\rho^* - \rho_0) \ddot{g} = \text{Net buoyancy force on the unperturbed sphere} \]

\[ F_B^i = V_p^i (\rho^* - \rho_0) \ddot{g} \quad \forall \ i \in \{1, 2, 3\} \]
\[ F_D = \text{approximate drag force on perturbed spherical particle} \]
\[ = 6\pi \mu a \left[ I + \epsilon T + O(\epsilon^2) \right] \dot{v} \]

where \( T = [I \delta_0 - \frac{1}{10} \nabla \nabla (r^2 \delta_2)] \) which is a constant symmetric matrix,

Hence,

\[
F_D = 6\pi \mu a
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} + 6\pi \mu a \epsilon
\begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}
\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}
\]

\[
T_{ii} = \delta_0 - \frac{1}{10} \frac{\partial^2 (r^2 \delta_2)}{\partial i^2} \quad \forall \ i \in \{x, y, z\}
\]

\[
T_{ij} = T_{ji} = -\frac{1}{10} \frac{\partial^2 (r^2 \delta_2)}{\partial i \partial j} \quad \forall \ i \in \{x, y, z\}
\]

In Cartesian coordinates, we have

\[
m_v \ddot{x} = F_x - 6\pi \mu a \dot{x} - 6\pi a \mu \epsilon [T_{xx} \dot{x} + T_{xy} \dot{y} + T_{xz} \dot{z}] - V_0 (\rho^* - \rho) g
\]

\[
m_v \ddot{y} = F_y - 6\pi \mu a \dot{y} - 6\pi a \mu \epsilon [T_{yx} \dot{x} + T_{yy} \dot{y} + T_{yz} \dot{z}]
\]

\[
m_v \ddot{z} = F_z - 6\pi \mu a \dot{z} - 6\pi a \mu \epsilon [T_{zx} \dot{x} + T_{zy} \dot{y} + T_{zz} \dot{z}]
\]

Notice, the effect of viscosity is to introduce a force opposing the motion. Negative sign is taken along buoyancy force because we have taken our coordinate system with x-axis pointing upwards. The above equations of motion reduces to

\[
\ddot{x} = X_0 + \frac{\epsilon}{(m_v)^2} X_1 + O(\epsilon^2)
\]

\[
\ddot{y} = Y_0 + \frac{\epsilon}{(m_v)^2} Y_1 + O(\epsilon^2)
\]

\[
\ddot{z} = Z_0 + \frac{\epsilon}{(m_v)^2} Z_1 + O(\epsilon^2)
\]

(4.1)
where,

\[ X_0 = \frac{1}{m_0} \left[ F_{z0cc} \cos^2(kz) + F_{z0ss} \sin^2(kz) + F_{z0sc} \sin(2kz) - 6\pi \mu a \hat{x} - F_B^0 \right] \]

\[ X_1 = -m_0^1 \left[ F_{z1cc} \cos^2(kz) + F_{z1ss} \sin^2(kz) + F_{z1sc} \sin(2kz) - 6\pi \mu a \hat{x} - F_B^1 \right] \]

\[ + m_0^0 \left[ F_{z1cc} \cos^2(kz) + F_{z1ss} \sin^2(kz) + F_{z1sc} \sin(2kz) \right] \]

\[ - m_0^0 \left[ 6\pi \alpha \mu [T_{zz} \hat{x} + T_{xy} \hat{y} + T_{xz} \hat{z}] + F_B^1 \right] \]

\[ Y_0 = \frac{1}{m_0} \left[ F_{y0cc} \cos^2(kz) + F_{y0ss} \sin^2(kz) + F_{y0sc} \sin(2kz) - 6\pi \mu a \hat{y} \right] \]

\[ Y_1 = -m_0^1 \left[ F_{y1cc} \cos^2(kz) + F_{y1ss} \sin^2(kz) + F_{y1sc} \sin(2kz) - 6\pi \mu a \hat{y} \right] \]

\[ + m_0^0 \left[ F_{y1cc} \cos^2(kz) + F_{y1ss} \sin^2(kz) + F_{y1sc} \sin(2kz) \right] \]

\[ - m_0^0 \left[ 6\pi \alpha \mu [T_{yz} \hat{y} + T_{yy} \hat{y} + T_{yz} \hat{z}] \right] \]

\[ Z_0 = \frac{1}{m_0} \left[ F_{z0cc} \cos^2(kz) + F_{z0ss} \sin^2(kz) + F_{z0sc} \sin(2kz) - 6\pi \mu a \hat{z} \right] \]

\[ Z_1 = -m_0^1 \left[ F_{z1cc} \cos^2(kz) + F_{z1ss} \sin^2(kz) + F_{z1sc} \sin(2kz) - 6\pi \mu a \hat{z} \right] \]

\[ + m_0^0 \left[ F_{z1cc} \cos^2(kz) + F_{z1ss} \sin^2(kz) + F_{z1sc} \sin(2kz) \right] \]

\[ - m_0^0 \left[ 6\pi \alpha \mu [T_{zz} \hat{z} + T_{zy} \hat{z} + T_{zz} \hat{z}] \right] \]

### 4.1 Approximate Analysis of Equation of Motion

The equations of motion (4.1) tend to be quite stiff as shown in the example in the next section; so that even numerical schemes designed for stiff ordinary differential are not particularly effective. Therefore, we shall extend the approximate method used in [1, 2] to this higher dimensional system.
Up to a first order expansion in the small (radius variation) parameter $\epsilon$, the equations of motion have the form:

$$
\begin{align*}
\ddot{x} &= A - B\dot{x} + \epsilon \left\{ DA - [e_{11}\dot{x} + e_{12}\dot{y} + e_{13}\dot{z}] \right\} \\
&\quad + \epsilon \left[ c_{11} \sin(2kz) + c_{12} \sin^2(kz) + c_{13} \cos^2(kz) \right] \\
&\quad + \epsilon \left[ -B\dot{y} + \epsilon \left\{ -[e_{21}\dot{x} + e_{22}\dot{y} + e_{23}\dot{z}] \right\} \\
\ddot{y} &= -B\dot{y} + \epsilon \left\{ -[e_{21}\dot{x} + e_{22}\dot{y} + e_{23}\dot{z}] \right\} \\
&\quad + \epsilon \left[ c_{21} \sin(2kz) + c_{22} \sin^2(kz) + c_{23} \cos^2(kz) \right] \\
&\quad + \epsilon \left[ -B\dot{z} + C \sin(2kz) + \epsilon \left\{ -[e_{31}\dot{x} + e_{32}\dot{y} + e_{33}\dot{z}] \right\} \\
\ddot{z} &= -B\dot{z} + C \sin(2kz) + \epsilon \left\{ -[e_{31}\dot{x} + e_{32}\dot{y} + e_{33}\dot{z}] \right\} \\
&\quad + \epsilon \left[ c_{31} \sin(2kz) + c_{32} \sin^2(kz) + c_{33} \cos^2(kz) \right]
\end{align*}
$$

where $E = (e_{ij})$ is a real symmetric matrix, $A$ is a constant, $B, C, k$ are positive constants, $D$ is a negative constant, and the coefficients $c_{ij}$ are all real constants. These equations have no closed form solutions, and they are very stiff for physical constants corresponding to the phenomena under consideration. They are usually so stiff that even numerical schemes developed for such systems appear to be unable to produce good approximate solutions in reasonable computation times. Therefore, we shall extend the approximate methods employed in Aboobaker et al. [2, 27] to treat perfectly spherical particles.

As finding $z = z(t)$ for (4.2) is of primary importance, we shall focus on the last of Equations (4.2), but we shall also need to deal with the first two equations because of the obvious interdependence on all of the variables. In particular, we seek approximate solutions expanded in powers of the small parameter $\epsilon$ up to order one; namely, we assume

$$
\begin{align*}
x &= x_0 + \epsilon x_1 + \cdots \\
y &= y_0 + \epsilon y_1 + \cdots \\
z &= z_0 + \epsilon z_1 + \cdots
\end{align*}
$$

(4.3)
Substituting Equation (4.3) into Equation (4.2) and equating like powers of \( \epsilon \), we obtain

\[
\begin{align*}
\ddot{x}_0 &= A - B\dot{x}_0 \\
\ddot{y}_0 &= -B\dot{y}_0 \\
\ddot{z}_0 &= -B\dot{z}_0 + C \sin(2kz_0)
\end{align*}
\]

(4.4)

It may be assumed without loss of generality that all of the physical initial values are contained in the above zeroth-order equations, so we add the following to Equation (4.4):

\[
\begin{align*}
x_0(0) &= x_*, & y_0(0) &= y_*, & z_0(0) &= z_* \\
\dot{x}_0(0) &= \dot{x}_*, & \dot{y}_0(0) &= \dot{y}_*, & \dot{z}_0(0) &= \dot{z}_*
\end{align*}
\]

(4.5)

representing the initial position \((x_*, y_*, z_*)\) and initial velocity \((\dot{x}_*, \dot{y}_*, \dot{z}_*)\) of the (center of mass of the) particle. The first two of Equation (4.4) subject to (4.5) are easily solved, and they yield

\[
\begin{align*}
x_0(t) &= x_* - B^{-2}(A - B\dot{x}_*) + B^{-2}(A - B\dot{x}_*)e^{-Bt} + \frac{A}{B}t \\
y_0(t) &= y_* + \frac{\dot{y}_*}{B} - \frac{\dot{y}_*}{B}e^{-Bt}
\end{align*}
\]

(4.6)

Unfortunately, the last of Equations (4.4) does not appear to be solvable in closed form. Here, we can simply adapt the analysis of Aboobaker et al. [2, 27] to deal with

\[
\ddot{z}_0 + B\dot{z}_0 - C \sin(2kz_0) = 0; \quad z_0(0) = z_*; \quad \dot{z}_0(0) = \dot{z}_*
\]

(4.7)

The idea in obtaining an excellent approximate solution of (4.7) is to consider the approximate perturbation of the singular perturbation solution, that is the solution of

\[
B\dot{z}_0 - C \sin(2kz_0) = 0
\]

(4.8)
subject to the first of the initial conditions. As $B$ and $C$ are typically several orders of magnitude larger than one ($1 \ll B, C$), one expects the singular perturbation solution, which is

$$z_0 \simeq z_s(t) = \frac{1}{k} \tan^{-1} \left[ \tan k z_e \right] e^{2kCt}, \quad (4.9)$$

to produce a reasonably good approximation to the solution.

This forces the velocity to have the form

$$\dot{z}_s = \frac{C}{B} \sin (2k z_e(t)) \quad (4.10)$$

Then, a correction is added to Equation (4.9) to obtain a better approximate solution, and this is

$$z_0(t) \simeq \frac{1}{k} \tan^{-1} \left[ \tan k z_e \right] e^{2kCt} + \frac{C}{B \sqrt{B^2 - 8kC}} \left[ e^{-r_1 t} - e^{-r_2 t} \right], \quad (4.11)$$

with

$$\dot{z}_0(t) \simeq \frac{C}{B} \sin (2k z_0(t)) - \frac{C}{B \sqrt{B^2 - 8kC}} \left[ r_1 e^{-r_1 t} - r_2 e^{-r_2 t} \right], \quad (4.12)$$

where

$$r_1 := \frac{B + \sqrt{B^2 - 8kC}}{2}, \quad \text{and}$$

$$r_2 := \frac{B - \sqrt{B^2 - 8kC}}{2}, \quad (4.13)$$

Note we have assumed for convenience that $\dot{z}_0(0) = 0$ in these calculations.

Now that we have found $x_0(t), y_0(t)$ and $z_0(t)$, we can find $z_1(t)$. For this, we substitute the expressions (4.3) into the third of the Equations (4.2) and equate first-order terms in $\epsilon$, after expanding $\sin(2k(z_0 + \epsilon z_1))$ as

$$\sin 2k \left[ \frac{\pi}{2k} - \tilde{z}_0 + \epsilon z_1 \right] = \sin 2k \left[ \frac{\pi}{2k} - (\tilde{z}_0 - \epsilon z_1) \right] \simeq -2k[\epsilon z_1 - \tilde{z}_0]$$
leads to
\[
\ddot{z}_1 = -B\dot{z}_1 - kC z_1 - e_{31}\dot{x}_0(t) - e_{32}\dot{y}_0(t) - e_{33}\dot{z}_0(t) + c_{31}\sin(2kz_0(t)) \\
+ c_{32}\sin^2(kz_0(t)) + c_{33}\cos^2(kz_0(t))
\] (4.14)

satisfying the initial conditions

\[
z_1(0) = 0, \quad \dot{z}_1(0) = 0
\] (4.15)

To solve Equation (4.14), it is convenient to rewrite it as

\[
\ddot{z}_1 + B\dot{z}_1 + 2kCz_1 = f(t),
\] (4.16)

where \( f \) is the known function defined as

\[
f(t) := -e_{31}\dot{x}_0(t) - e_{32}\dot{y}_0(t) - e_{33}\dot{z}_0(t) + c_{31}\sin(2kz_0(t)) + c_{32}\sin^2(kz_0(t)) \\
+ c_{33}\cos^2(kz_0(t)),
\]

which using Equations (4.6) and (4.9) can be shown to be

\[
f(t) := -e_{31}\left[\frac{A}{B} - B^{-1}(A - B\dot{x}_*)e^{-Bt}\right] - e_{32}\dot{y}_*e^{-Bt} \\
- e_{33}\left[\frac{C}{B}\sin(2kz_0(t)) - \frac{C}{B\sqrt{B^2 - 8kC}}(r_1e^{-r_1t} - r_2e^{r_2t})\right] \\
+ c_{31}\sin\left[2k\left(\frac{1}{k}\tan^{-1}\left[(\tan kz_*e^{2kCt})\right]\right) + \frac{C}{B\sqrt{B^2 - 8kC}}(e^{-r_1t} - e^{r_2t})\right] \\
+ c_{32}\sin^2(kz_0(t)) + c_{33}\cos^2(kz_0(t))
\] (4.17)
Equation (4.14) subject to the initial conditions (4.15) can be solved by variation of constants, which yields

\[ z_1 = [B^2 - 8kC]^{-1/2} \int_0^t \left[ e^{-(r_2\tau + r_1\tau)} - e^{-(r_1\tau + r_2\tau)} \right] e^{B\tau} f(\tau) d\tau \]  \hspace{1cm} (4.18)

Thus, it follows that

\[ z(t) \approx z_0(t) + \varepsilon z_1(t) = \frac{1}{k} \tan^{-1} \left[ \tan \left( \frac{kz_0}{B} \right) e^{\frac{2kC}{B}t} \right] + \frac{C}{B \sqrt{B^2 - 8kC}} \left[ e^{-r_1t} - e^{-r_2t} \right] + \frac{\varepsilon}{\sqrt{B^2 - 8kC}} \int_0^t \left[ e^{-(r_2\tau + r_1\tau)} - e^{-(r_1\tau + r_2\tau)} \right] e^{B\tau} f(\tau) d\tau \]  \hspace{1cm} (4.19)

Observe that it follows from Equation (4.11) that \( z(t) \to \frac{\pi}{2k} \) and \( \dot{z}(t) \to 0 \) exponentially and uniformly as \( t \to \infty \) over any compact set of initial conditions, which are both expected and desired properties of the approximate solutions of Equations (4.2).

### 4.2 An Example

For the particular perturbation \( \delta(\theta, \phi) \) defined as

\[ \delta(\theta, \phi) = Y_0^0(\theta, \phi) + \frac{1}{2} \left( Y_2^2(\theta, \phi) + Y_2^{-2}(\theta, \phi) \right) \]  \hspace{1cm} (4.20)

\[ = \frac{1}{2\sqrt{\pi}} + \frac{1}{4} \sqrt{\frac{\pi}{2\pi}} \sin^2(\theta) \cos(2\phi) \]

\[ = \frac{1}{2\sqrt{\pi}} + \frac{1}{4} \sqrt{\frac{15}{2\pi}} \left( \frac{x^2 - y^2}{r^2} \right) \]
Figure 4.1 Perturbed sphere for $\epsilon = 0.2$.

Figure 4.2 Perturbed sphere for $\epsilon = 0.9$. 
The acoustic radiation force experienced by perturbed particle in the direction of the $x-$, $y-$, and $z-$ axis with perturbation defined above reduces to (see Appendix D for details):

\[
F_x \approx 0
\]

\[
F_y \approx 0
\]

\[
F_z \approx F_{z0sc} \sin(2kz) + \epsilon \{ F_{z1sc} \cos^2(kz) + F_{z1ss} \sin^2(kz) + F_{z1sc} \sin(2kz) \} + O(\epsilon^2)
\]

where,

\[
F_{z0sc} = -\frac{4\pi \rho_0}{(ak*)^3} \left( R\tilde{B}_{00} \Re \tilde{B}_{10} - \Im \tilde{B}_{00} \Im \tilde{B}_{10} \right) (-\sin(ak*) + \cos(ak*)ak*)
\]

\[
\times (2 \cos(ak*)ak* + \sin(ak*) (-2 + (ak*)^2))
\]

\[
- \frac{8\pi \rho_0}{5} \left( R\tilde{B}_{10} \Re \tilde{B}_{20} - \Im \tilde{B}_{10} \Im \tilde{B}_{20} \right) (j_1 (ak*) - j_2 (ak*) ak*)
\]

\[
\times (2j_2 (ak*) - j_3 (ak*) ak*)
\]

\[
- \frac{8\pi \lambda^2 \rho_0}{5} \left( R\tilde{B}_{10} \Re \tilde{B}_{20} - \Im \tilde{B}_{10} \Im \tilde{B}_{20} \right) j_1 (ak*) j_2 (ak*)
\]

\[
+ \frac{4a^2 \pi \lambda^2 \omega^2 \rho_0}{3c_0^2} \left( R\tilde{B}_{00} \Re \tilde{B}_{10} - \Im \tilde{B}_{00} \Im \tilde{B}_{10} \right) j_0 (ak*) j_1 (ak*)
\]

\[
+ \frac{8a^2 \pi \lambda^2 \omega^2 \rho_0}{15c_0^2} \left( R\tilde{B}_{10} \Re \tilde{B}_{20} - \Im \tilde{B}_{10} \Im \tilde{B}_{20} \right) j_1 (ak*) j_2 (ak*)
\]

\[
- \frac{8\pi \lambda \rho_0}{5} \left( R\tilde{B}_{10} \Re \tilde{B}_{20} - \Im \tilde{B}_{10} \Im \tilde{B}_{20} \right) j_2 (ak*) (j_1 (ak*) - j_2 (ak*) ak*)
\]

\[
- \frac{8\pi \lambda \rho_0}{3(ak*)^3} \left( R\tilde{B}_{00} \Re \tilde{B}_{10} - \Im \tilde{B}_{00} \Im \tilde{B}_{10} \right) (\sin(ak*) - \cos(ak*)ak*)^2
\]

\[
+ \frac{8\pi \lambda \rho_0}{15} \left( R\tilde{B}_{10} \Re \tilde{B}_{20} - \Im \tilde{B}_{10} \Im \tilde{B}_{20} \right) j_1 (ak*) (2j_2 (ak*) - j_3 (ak*) ak*)
\]
\[ F_{z1cc} = \frac{\lambda \rho_0 \pi^{3/2}}{(\mathbf{k}^*)^2} \left( R \tilde{B}^{(0)}_{00} R \tilde{B}^{(0)}_{00} + \Im \tilde{B}^{(0)}_{00} \Im \tilde{B}^{(0)}_{00} \right) (\sin(\mathbf{k}^*) - \cos(\mathbf{k}^*)\mathbf{k}^*)^2 \\
- \frac{\lambda \rho_0 \pi^{3/2} \mathbf{k}^*}{4} \left( R \tilde{B}^{(0)}_{00} R \tilde{B}^{(0)}_{20} + \Im \tilde{B}^{(0)}_{00} \Im \tilde{B}^{(0)}_{20} \right) j_1(\mathbf{k}^*) (-2j_2(\mathbf{k}^*) + j_3(\mathbf{k}^*) \mathbf{k}^*) \\
+ \frac{5 \lambda \rho_0 \pi^{3/2}}{32} \left( R \tilde{B}^{(0)}_{20} R \tilde{B}^{(0)}_{20} + \Im \tilde{B}^{(0)}_{20} \Im \tilde{B}^{(0)}_{20} \right) (-2j_2(\mathbf{k}^*) + j_3(\mathbf{k}^*) \mathbf{k}^*)^2 \\
+ \frac{\pi^2 \lambda \rho_0}{(\mathbf{k}^*)^2} \left( R \tilde{B}^{(0)}_{00} R \tilde{B}^{(1)}_{00} + \Im \tilde{B}^{(0)}_{00} \Im \tilde{B}^{(1)}_{00} \right) (-1 + \cos(2\mathbf{k}^*) + \sin(2\mathbf{k}^*)\mathbf{k}^*) \\
- \frac{\lambda \rho_0 \pi}{4} \left( R \tilde{B}^{(0)}_{00} R \tilde{B}^{(0)}_{00} + \Im \tilde{B}^{(0)}_{00} \Im \tilde{B}^{(0)}_{00} \right) j_0(\mathbf{k}^*) (2j_2(\mathbf{k}^*) - j_3(\mathbf{k}^*) \mathbf{k}^*) \\
+ \frac{\lambda \rho_0 \pi^{3/2} \mathbf{k}^*}{4} \left( R \tilde{B}^{(0)}_{00} R \tilde{B}^{(1)}_{00} + \Im \tilde{B}^{(0)}_{00} \Im \tilde{B}^{(1)}_{00} \right) j_1(\mathbf{k}^*) j_2(\mathbf{k}^*) \\
+ \frac{5 \lambda \rho_0 \pi^{3/2}}{16} \left( R \tilde{B}^{(0)}_{20} R \tilde{B}^{(1)}_{20} + \Im \tilde{B}^{(0)}_{20} \Im \tilde{B}^{(1)}_{20} \right) j_2(\mathbf{k}^*) (2j_2(\mathbf{k}^*) - j_3(\mathbf{k}^*) \mathbf{k}^*) \\
F_{z1ss} = \frac{\lambda \rho_0 \pi^{3/2}}{4} \left( R \tilde{B}^{(0)}_{10} R \tilde{B}^{(0)}_{10} + \Im \tilde{B}^{(0)}_{10} \Im \tilde{B}^{(0)}_{10} \right) (j_1(\mathbf{k}^*) - j_2(\mathbf{k}^*) \mathbf{k}^*)^2 \\
+ \frac{\lambda \rho_0 \pi}{2} \left( R \tilde{B}^{(0)}_{10} R \tilde{B}^{(1)}_{10} + \Im \tilde{B}^{(0)}_{10} \Im \tilde{B}^{(1)}_{10} \right) j_1(\mathbf{k}^*) (j_1(\mathbf{k}^*) - j_2(\mathbf{k}^*) \mathbf{k}^*) \\
F_{z1sc} = -\frac{2 \rho_0 \pi^{1/2}}{(\mathbf{k}^*)^3} \left( R \tilde{B}^{(0)}_{00} \Im \tilde{B}^{(0)}_{00} - \Im \tilde{B}^{(0)}_{00} R \tilde{B}^{(0)}_{00} \right) (-\sin(\mathbf{k}^*) + \cos(\mathbf{k}^*)\mathbf{k}^*) \\
\times (\cos(\mathbf{k}^*)\mathbf{k}^*) (-6 + (\mathbf{k}^*)^2) - 3 \sin(\mathbf{k}^*) (-2 + (\mathbf{k}^*)^2)) \\
+ \frac{4 \pi^{1/2} \rho_0}{5 (\mathbf{k}^*)^5} \left( R \tilde{B}^{(0)}_{20} \Im \tilde{B}^{(0)}_{10} - \Im \tilde{B}^{(0)}_{20} R \tilde{B}^{(0)}_{10} \right) \\
\times (\sin(\mathbf{k}^*) (9 - 4(\mathbf{k}^*)^2) + \cos(\mathbf{k}^*)\mathbf{k}^* (-9 + (\mathbf{k}^*)^2)) \\
\times (\cos(\mathbf{k}^*)\mathbf{k}^*) (-6 + (\mathbf{k}^*)^2) - 3 \sin(\mathbf{k}^*) (-2 + (\mathbf{k}^*)^2)) \\
+ \frac{2 \rho_0 \pi^{1/2}}{(\mathbf{k}^*)^5} \left( R \tilde{B}^{(0)}_{10} \Im \tilde{B}^{(0)}_{00} - \Im \tilde{B}^{(0)}_{10} R \tilde{B}^{(0)}_{00} \right) \\
\times (2 \cos(\mathbf{k}^*)\mathbf{k}^* + \sin(\mathbf{k}^*) (-2 + (\mathbf{k}^*)^2))^2 \\
- \frac{\rho_0 \pi^{1/2}}{(\mathbf{k}^*)^7} \left( R \tilde{B}^{(0)}_{10} \Im \tilde{B}^{(0)}_{20} - \Im \tilde{B}^{(0)}_{10} R \tilde{B}^{(0)}_{20} \right) \\
\times (2 \cos(\mathbf{k}^*)\mathbf{k}^* + \sin(\mathbf{k}^*) (-2 + (\mathbf{k}^*)^2)) \\
\times (\cos(\mathbf{k}^*)\mathbf{k}^* (-36 + 5(\mathbf{k}^*)^2) + \sin(\mathbf{k}^*) (36 - 17 (\mathbf{k}^*)^2 + (\mathbf{k}^*)^4)) \\
- \frac{4 \pi \rho_0}{(\mathbf{k}^*)^3} \left( R \tilde{B}^{(0)}_{10} \Im \tilde{B}^{(1)}_{00} - \Im \tilde{B}^{(0)}_{10} R \tilde{B}^{(1)}_{00} \right) (\sin(\mathbf{k}^*) - \cos(\mathbf{k}^*)\mathbf{k}^*)
\begin{align*}
&\times (2 \cos(ak^*)ak^* + \sin(ak^*) (-2 + (ak^*)^2)) \\
&+ \frac{8\pi \rho_0}{5} \left( \mathcal{R} \tilde{B}^{(0)}_{10} \Im \tilde{B}^{(1)}_{20} - \Im \tilde{B}^{(0)}_{10} \mathcal{R} \tilde{B}^{(1)}_{20} \right) (-j_1 (ak^*) + j_2 (ak^*) ak^*) \\
&\times (-2j_2 (ak^*) + j_3 (ak^*) ak^*) \\
&- \frac{4\pi \rho_0}{(ak^*)^3} \left( \mathcal{R} \tilde{B}^{(0)}_{00} \Im \tilde{B}^{(1)}_{10} - \Im \tilde{B}^{(0)}_{00} \mathcal{R} \tilde{B}^{(1)}_{10} \right) (-\sin(ak^*) + \cos(ak^*)ak^*) \\
&\times (2 \cos(ak^*)ak^* + \sin(ak^*) (-2 + (ak^*)^2)) \\
&- \frac{8\pi \rho_0}{5} \left( \mathcal{R} \tilde{B}^{(0)}_{20} \Im \tilde{B}^{(1)}_{10} - \Im \tilde{B}^{(0)}_{20} \mathcal{R} \tilde{B}^{(1)}_{10} \right) (-j_1 (ak^*) + j_2 (ak^*) ak^*) \\
&\times (-2j_2 (ak^*) + j_3 (ak^*) ak^*) \\
&- \frac{4\lambda^2 \rho_0 \pi^{1/2}}{5} \left( \mathcal{R} \tilde{B}^{(0)}_{20} \Im \tilde{B}^{(1)}_{10} - \Im \tilde{B}^{(0)}_{20} \mathcal{R} \tilde{B}^{(1)}_{10} \right) j_2 (ak^*) (j_1 (ak^*) - j_2 (ak^*) ak^*) \\
&- \frac{4\lambda^2 \rho_0 \pi^{1/2}}{5} \left( \mathcal{R} \tilde{B}^{(0)}_{10} \Im \tilde{B}^{(1)}_{20} - \Im \tilde{B}^{(0)}_{10} \mathcal{R} \tilde{B}^{(1)}_{20} \right) \\
&\times j_1 (ak^*) (-2j_2 (ak^*) + j_3 (ak^*) ak^*) \\
&+ \frac{8\pi \lambda^2 \rho_0}{5} j_1 (ak^*) j_2 (ak^*) \left( \mathcal{R} \tilde{B}^{(0)}_{10} \Im \tilde{B}^{(1)}_{20} - \Im \tilde{B}^{(0)}_{10} \mathcal{R} \tilde{B}^{(1)}_{20} \right) \\
&- \frac{8\pi \lambda^2 \rho_0}{5} j_1 (ak^*) j_2 (ak^*) \left( \mathcal{R} \tilde{B}^{(0)}_{20} \Im \tilde{B}^{(1)}_{10} - \Im \tilde{B}^{(0)}_{20} \mathcal{R} \tilde{B}^{(1)}_{10} \right) \\
&+ \frac{2a^2 \lambda^2 \omega^2 \rho_0 \pi^{1/2}}{3c_0^2} \left( \mathcal{R} \tilde{B}^{(0)}_{00} \Im \tilde{B}^{(1)}_{10} - \Im \tilde{B}^{(0)}_{00} \mathcal{R} \tilde{B}^{(1)}_{10} \right) \\
&\times j_0 (ak^*) (j_1 (ak^*) - j_2 (ak^*) ak^*) \\
&+ \frac{4\pi^{1/2} a^2 \lambda^2 \omega^2 \rho_0}{15c_0^2} \left( \mathcal{R} \tilde{B}^{(0)}_{20} \Im \tilde{B}^{(1)}_{10} - \Im \tilde{B}^{(0)}_{20} \mathcal{R} \tilde{B}^{(1)}_{10} \right) \\
&\times j_2 (ak^*) (j_1 (ak^*) - j_2 (ak^*) ak^*) \\
&+ \frac{2a^2 \pi^{1/2} \lambda^2 \omega^2 \rho_0}{3c_0^2 (ak^*)^3} \left( \mathcal{R} \tilde{B}^{(0)}_{10} \Im \tilde{B}^{(1)}_{00} - \Im \tilde{B}^{(0)}_{10} \mathcal{R} \tilde{B}^{(1)}_{00} \right) \\
&\times (\sin(ak^*) - \cos(ak^*) ak^*)^2 \\
&+ \frac{4a^2 \pi^{1/2} \lambda^2 \omega^2 \rho_0}{15c_0^2} \left( \mathcal{R} \tilde{B}^{(0)}_{10} \Im \tilde{B}^{(1)}_{20} - \Im \tilde{B}^{(0)}_{10} \mathcal{R} \tilde{B}^{(1)}_{20} \right)
\end{align*}
\[ \times j_1 (ak^*) (-2j_2 (ak^*) + j_3 (ak^*) ak^*) \]

\[- \frac{4a^2\pi\lambda^2\omega^2\rho_0}{3c_0^2} \left( \Re \tilde{B}^{(0)} \Im \tilde{B}^{(1)} - \Im \tilde{B}^{(0)} \Re \tilde{B}^{(1)} \right) j_0 (ak^*) j_1 (ak^*) \]

\[- \frac{8a^2\pi\lambda^2\omega^2\rho_0}{15c_0^2} \left( \Re \tilde{B}^{(0)} \Im \tilde{B}^{(2)} - \Im \tilde{B}^{(0)} \Re \tilde{B}^{(2)} \right) j_1 (ak^*) j_2 (ak^*) \]

\[+ \frac{4a^2\pi\lambda^2\omega^2\rho_0}{3c_0^2} \left( \Re \tilde{B}^{(0)} \Im \tilde{B}^{(1)} - \Im \tilde{B}^{(0)} \Re \tilde{B}^{(1)} \right) j_0 (ak^*) j_1 (ak^*) \]

\[+ \frac{8a^2\pi\lambda^2\omega^2\rho_0}{15c_0^2} \left( \Re \tilde{B}^{(0)} \Im \tilde{B}^{(1)} - \Im \tilde{B}^{(0)} \Re \tilde{B}^{(1)} \right) j_1 (ak^*) j_2 (ak^*) \]

\[- \frac{4\pi^{1/2}\rho_0}{\sin(ak^*) + \cos(ak^*) ak^*) \left( \Re \tilde{B}^{(0)} \Im \tilde{B}^{(1)} - \Im \tilde{B}^{(0)} \Re \tilde{B}^{(1)} \right) \left( -2 + (ak^*)^2 \right) \]

\[- \frac{8\pi^{1/2}\rho_0}{5} \left( \Re \tilde{B}^{(0)} \Im \tilde{B}^{(2)} - \Im \tilde{B}^{(0)} \Re \tilde{B}^{(2)} \right) \left( j_1 (ak^*) - j_2 (ak^*) ak^* \right) \]

\[\times \left( 2\cos(ak^*) ak^* + \sin(ak^*) \right) \left( -2 + (ak^*)^2 \right) \]

\[- \frac{8\pi^{1/2}\rho_0}{5} \left( \Re \tilde{B}^{(0)} \Im \tilde{B}^{(2)} - \Im \tilde{B}^{(0)} \Re \tilde{B}^{(2)} \right) \left( j_1 (ak^*) - j_2 (ak^*) ak^* \right) \]

\[\times \left( 2j_2 (ak^*) - j_3 (ak^*) ak^* \right) \]

\[+ \frac{4a^2\pi^{1/2}\lambda^2\omega^2\rho_0}{3c_0^2} \left( \Re \tilde{B}^{(0)} \Im \tilde{B}^{(1)} - \Im \tilde{B}^{(0)} \Re \tilde{B}^{(1)} \right) j_0 (ak^*) j_1 (ak^*) \]

\[+ \frac{8a^2\pi^{3/2}\lambda^2\omega^2\rho_0}{15c_0^2} \left( \Re \tilde{B}^{(0)} \Im \tilde{B}^{(1)} - \Im \tilde{B}^{(0)} \Re \tilde{B}^{(1)} \right) j_1 (ak^*) j_2 (ak^*) \]

\[- \frac{4\pi^{1/2}\lambda\rho_0}{5} \left( \Re \tilde{B}^{(0)} \Im \tilde{B}^{(2)} - \Im \tilde{B}^{(0)} \Re \tilde{B}^{(2)} \right) j_2 (ak^*) \left( j_1 (ak^*) - j_2 (ak^*) ak^* \right) \]

\[- \frac{4\pi^{3/2}\lambda\rho_0}{3(ak^*)^3} \left( \Re \tilde{B}^{(0)} \Im \tilde{B}^{(1)} - \Im \tilde{B}^{(0)} \Re \tilde{B}^{(1)} \right) \left( \sin(ak^*) - \cos(ak^*) ak^* \right)^2 \]

\[+ \frac{4\pi^{1/2}\lambda\rho_0}{15\omega} \left( \Re \tilde{B}^{(0)} \Im \tilde{B}^{(2)} - \Im \tilde{B}^{(0)} \Re \tilde{B}^{(2)} \right) \left( 2j_2 (ak^*) - j_3 (ak^*) \right) \]

The drag on the perturbed particle with perturbation defined above reduces to:

\[\vec{F}_D = 6\pi\mu a \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} + 6\pi\mu a e \begin{bmatrix} \frac{1}{2\sqrt{\pi}} - \frac{1}{26\sqrt{\frac{15}{2\pi}}} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{\pi}} + \frac{1}{26\sqrt{\frac{15}{2\pi}}} & 0 \\ 0 & 0 & \frac{1}{2\sqrt{\pi}} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \]
Employing the properties of the particles and fluid that was used in [2]
Simplifying Equation (4.1) with values in Table 4.1 for

\[ \delta(\theta, \phi) = Y_0^0(\theta, \phi) + \frac{1}{2} \left( Y_2^2(\theta, \phi) + Y_2^{-2}(\theta, \phi) \right) \]

yields:

<table>
<thead>
<tr>
<th>Description</th>
<th>Solid (Silica fumes)</th>
<th>Medium (DI Water)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Particle size, Diameter (micrometer)</td>
<td>2</td>
<td>-</td>
</tr>
<tr>
<td>Density ((\rho)) (kg/m(^3))</td>
<td>2.649E+3</td>
<td>1000</td>
</tr>
<tr>
<td>Frequency of sound in medium ((f)) (kHz)</td>
<td>-</td>
<td>333</td>
</tr>
<tr>
<td>Speed of sound (m/sec)</td>
<td>3,750</td>
<td>1,500</td>
</tr>
<tr>
<td>Viscosity of medium ((\mu)) (N – sec/m(^2))</td>
<td>-</td>
<td>9.98E-4</td>
</tr>
</tbody>
</table>

\[
V_P = 4.18879 + 3.54491\epsilon + 1.50000\epsilon^2 + 0.235079\epsilon^3
\]

\[
m_v = 1.31905 \times 10^{-14} + 1.11629 \times 10^{-14}\epsilon + 4.7235 \times 10^{-15}\epsilon^2 + 7.40264 \times 10^{-16}\epsilon^3
\]

\[
F_B = 6.77376 \times 10^{-8} + 5.73253 \times 10^{-8}\epsilon + 2.42567 \times 10^{-8}\epsilon^2 + 3.8015 \times 10^{-9}\epsilon^3
\]
Finally Equation 4.1 reduces to:

\[
\begin{align*}
T_{xx} &= \frac{1}{2\sqrt{\pi}} - \frac{1}{20}\sqrt{\frac{15}{2\pi}} = 0.20484 \\
T_{yy} &= \frac{1}{2\sqrt{\pi}} + \frac{1}{20}\sqrt{\frac{15}{2\pi}} = 0.35935 \\
T_{zz} &= \frac{1}{2\sqrt{\pi}} = 0.282095 \\
T_{xy} &= T_{yx} = T_{xz} = T_{zx} = T_{yz} = T_{zy} = 0 \\
F_x &\approx 0, \quad F_y \approx 0 \\
F_z &\approx 1.3573 \times 10^{-6} \sin(2kz) + 2.64899 \times 10^{-4} \sin(2kz) \\
&\quad - 2.8663 \times 10^{-28} \cos^2(kz) - 1.111 \times 10^{-21} \sin^2(kz) + O(\epsilon^2)
\end{align*}
\]

Finally Equation 4.1 reduces to:

\[
\begin{align*}
\ddot{x} &= -1.42617 \times 10^6 \dot{x} - 5.13533 \times 10^6 + 914804 \dot{x} + O(\epsilon^2) \\
\ddot{y} &= -1.42617 \times 10^6 \dot{y} + 694447 \dot{y} + O(\epsilon^2) \\
\ddot{z} &= -1.42617 \times 10^6 \dot{z} + 1.029 \times 10^8 \sin(2kz) + 804626 \dot{z} \\
&+ \epsilon[-2.17301 \times 10^{-14} \cos^2(kz) - 8.42271 \times 10^{-8} \sin^2(kz) + 1.9692 \times 10^{10} \sin(2kz)] \\
&+ O(\epsilon^2)
\end{align*}
\] (4.21)

4.3 Approximate Analysis of an Example

Let us now apply the above analysis to the example for which the specific dynamical equations have been computed as above: By Simply following the general approach above for the specific Equations (4.21), one finds the following:

Using the expansions (4.3), we compute according to Equations (4.6) that

\[
\begin{align*}
x_0(t) &= x_* + 4.91652 \times 10^{-13}(-5.13533 \times 10^6 + 1.42617 \times 10^6 \dot{x}_*) \\
&\quad + (-1.42617 \times 10^6)^{-2}(-5.13533 \times 10^6 + 1.42617 \times 10^6 \dot{x}_*) e^{1426170t} \\
&\quad + 3.60078 \epsilon \\
y_0(t) &= y_* + (-7.01179 \times 10^{-7}) y_* + 7.01179 \times 10^{-7} y_* e^{1426170t}
\end{align*}
\] (4.22)
Then, using the third equation of Equation (4.21)- having specific coefficient value - one finds from Equation (4.11) that

\[
\begin{align*}
    z_0(t) &= \frac{1}{k} \tan^{-1} \left[(\tan kz_*) e^{14.303kt}\right] \\
           & \quad + \frac{72.1513 \sin(2kz_*)}{\sqrt{2.03396 \times 10^{12} - 8.232 \times 10^{8}k}} \left[e^{-r_1 t} - e^{-r_2 t}\right], \\
    & \quad (4.23)
\end{align*}
\]

where

\[
\begin{align*}
    r_1 &= (0.5) \left[-1.42617 \times 10^6 + \sqrt{2.03396 \times 10^{12} - 8.232 \times 10^{8}k}\right] \\
    r_2 &= (0.5) \left[-1.42617 \times 10^6 - \sqrt{2.03396 \times 10^{12} - 8.232 \times 10^{8}k}\right] \\
    & \quad (4.24)
\end{align*}
\]

Finally, one can proceed directly to the computation of \( z_1 \) for this example, and using Equations (4.17)-(4.19) compute that

\[
\begin{align*}
    f(t) &= 804626z_0(t) + (1.9692 \times 10^{10}) \sin(2kz_0(t)) + (-8.42271 \times 10^{-8}) \sin^2(kz_0(t)) \\
           & \quad + (-2.17301 \times 10^{-14}) \cos^2(kz_0(t)), \\
    & \quad (4.25)
\end{align*}
\]

and

\[
\begin{align*}
    z_1(t) &= \frac{1}{\sqrt{2.03396 \times 10^{12} - 8.232 \times 10^{8}k}} \\
            & \quad \times \int_{0}^{t} \left[e^{-(r_2 t + r_1 \tau)} - e^{-(r_1 t + r_2 \tau)}\right] e^{1.42617 \times 10^{8} \tau} f(\tau) \, d\tau, \\
    & \quad (4.26)
\end{align*}
\]
where \( r_1 \) and \( r_2 \) as in Equation (4.24). Whence, we obtain from Equation (4.19) the approximate formula

\[
\begin{align*}
 z(t) & \simeq z_0(t) + \epsilon z_1(t) \\
 & = \frac{1}{k} \tan^{-1} \left[ (\tan kz_\ast) e^{144.303kt} \right] \\
 & + \frac{72.1513 \sin(2kz_\ast)}{\sqrt{2.03396 \times 10^{12} - 8.232 \times 10^8 k}} \left[ e^{-r_1 t} - e^{-r_2 t} \right] \\
 & + \frac{1}{\sqrt{2.03396 \times 10^{12} - 8.232 \times 10^8 k}} \\
 & \times \int_{0}^{t} \left[ e^{-(r_2 t + r_1 \tau)} - e^{-(r_1 t + r_2 \tau)} \right] e^{1.42617 \times 10^6 \tau} f(\tau) d\tau \\
& \quad (4.27)
\end{align*}
\]

\[\text{Figure 4.3} \quad \text{Particle displacement from pressure antinode vs time.}\]
The behavior of particle movement is shown in Figure 4.3 using the trajectory Equation (4.27). In the graph the horizontal axis represents the time in seconds for the sound field treatment. The values on the vertical axis represents the position of particles at the corresponding times. The distances are in micrometers and for convenience taken at $\lambda_k/16$ of wavelength intervals from the first pressure antinode at the face of transducer. All the particles move towards the pressure node as expected. Particles of 2 micrometers diameter take about 20 seconds to reach the node. The value 11.126 micrometers in the graph represents the first pressure node towards which particles are supposed to move.

One cannot help but be struck by the very large values of the first-order perturbation of the acoustic forces, which has an order of magnitude that is roughly twice that of the zeroth-order acoustic term. This very large perturbation is most likely a result of the fact that the radius perturbation has partial derivatives that are very much larger than $\delta$ itself over significant portions of the boundary of the particle. This result then, if it is typical of other perturbed spherical particles treated in applications, which we believe is the case, makes a compelling case for the necessity of considering perturbed spherical particles in applications, and may in fact account for the quantitative discrepancies between predictions and experiments observed when using the perfect spherical model analysis.
CHAPTER 5

CONCLUSIONS AND FUTURE RESEARCH

In this chapter we conclude this dissertation by summarizing results obtained in the course of our research, with a special emphasis on the dynamic motion of a perturbed sphere under acoustic and fluid flows.

We began this dissertation by citing some of the famous and widely used applications of acoustic standing wave fields; these were used to segregate or fractionate fine particles by Meegoda et al. A mathematical model was developed to calculate the trajectories and concentration of the micron size particles suspended in a fluid with acoustic standing waves that resulted in excellent qualitative but only fair to good quantitative agreement with experiments. The reason for the discrepancy between the theory and experiments might be due to the assumption of spherical shaped object particles in the theory. So it was proposed to reanalyze the theory for different shaped particles which motivated us to take into consideration irregular shaped geometries, and in particular, those having a perturbed spherical shape.

In Chapter 2, we introduced a few basic definitions and calculated the acoustic radiation force on the perturbed compressible spherical object immersed in non-viscous fluids in plane stationary waves with in the long wavelength limits, based on integration of the time-averaged momentum flux tensor over the equilibrium surface of the perturbed sphere. In order to accomplish this, we extended Yosioka and Kawasima’s [40] approach to non-axisymmetric objects by developing a simple and straightforward method for treating irregular shaped spherical objects. The perturbed solution is generated starting with the wave equation. This method includes the expansion of the boundary condition in Taylor series, which in effect transforms the boundary condition at the irregular boundary into a succession of boundary
conditions to be satisfied at the surface of a sphere. A detailed formulation is carried 
out to the first order in $\epsilon$, which is a small parameter representing variations in 
the radius. We found that the first order acoustic correction terms appear to be 
generally several orders of magnitude larger than the zeroth order terms- due primarily 
to large variations in derivatives of the perturbation. Naturally, this underscores 
the importance of including these correction terms in the analysis. Procedures to 
obtain higher terms are also indicated. To the best of our knowledge, this type of 
perturbation technique and loss of axial symmetry does not appear in the acoustic 
scattering problems in the literature, but there are some analogs in the literature of 
electromagnetics (see e.g. [13, 14]).

In Chapter 3, we revisited Lamb's general solution of Stokes equation that has 
been used by Brenner to find the Stokes resistance on a slightly deformed sphere. 
The perturbation technique was developed by Brenner for fluid flow past a slightly 
deformed no slip sphere considering only the first- order expansion in the small 
deformation parameter.

Finally in Chapter 4, we developed the particle equations of motion under 
acoustic standing waves and fluid flows using the momentum principle. We evaluated 
the complete formulations of acoustic radiation forces and drag on a perturbed sphere 
for a particular perturbation and used the mathematical model for this example 
to calculate the approximate particle trajectories. A combination of phase space 
and asymptotic analysis turns out to be far more useful in obtaining approximate 
solutions. We would like to in future use our new first order approach to investigate 
the dynamics of collections of such particles with a particular eye to estimating the 
evolution of the concentration of the particles.

Our investigation has also raised a number of interesting questions that should 
be studied in future research, among which are the following: Can our techniques 
be used to obtain a particle trajectory when the parameter $\epsilon$ is fairly large, say of
the order of a quarter of the particle diameter? We have ignored possible rotational motion of the irregular shaped particle in our approach. How can our method be extended to include rotational dynamics? What happens to the trajectory if an object is placed under dynamic radiation force for a continuous wave-field whose intensity varies slowly with time? How can our approach be extended to take into account the effects of viscosity and heat conduction in the host fluid? We focused here on the geometry of the particle and not on material properties. Hence, it is natural to ask if the approach can be extended to more general types of objects and liquids, such as viscoelastic fluids. These are some of the problems that we would like to study in the future.
Here we describe the scattering of stationary acoustic waves by a compressible perturbed spherical object. We consider the obstacle to be a compressible perturbed sphere of mean radius $a$ whose center is at distance $h$ from the nearest pressure antinode of a plane stationary wave field. Assuming that the incident wave is axially symmetric and its wave fronts are normal to the $z$-axis, The incident velocity potential referred to the equilibrium position of the center as origin of a rectangular coordinate system $(x, y, z)$ can be written as

$$\Phi_i = e^{i\omega t} [e^{ik(z+h)} + e^{-ik(z+h)}] \quad (A.1)$$

where the first term and second term represent the plane progressive waves propagating in the direction of the $-z$ and $+z$ axes respectively.

In a system of spherical coordinates $(r, \theta, \phi)$, $z = r \cos \theta$, we have,

$$e^{-ikz} = e^{-ikr \cos \theta} = e^{-ikr \mu} = e^{-iz\mu} \quad \text{where} \quad \mu = \cos \theta, \ Z = kr$$

Let,

$$e^{-iz\mu} = \sum_{m=0}^{\infty} A_m P_m(\mu) \quad (A.2)$$

where $P_m(\mu)$ is a spherical harmonic (Legendre polynomial) of degree $m$.

Multiplying both sides of Equation A.2 by $P_n(\mu)$ and integrating with respect to $\mu$
from -1 to 1, we obtain

\[ A_m = \frac{2m + 1}{2} \int_{-1}^{1} e^{-iz\mu} P_n(\mu) \, d\mu \]

\[ = \left(\frac{2m + 1}{2}\right) 2(-i)^m \sqrt{\frac{\pi}{2Z}} J_{m+\frac{1}{2}}(Z) \]

\[ = (2m + 1)(-i)^m j_m(Z) \]

Here we have used

\[ \int_{-1}^{1} P_m(\mu) P_n(\mu) \, d\mu = \frac{2}{2m + 1} \delta_{nm} \]

\[ \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x) = j_n(x) \]

where \( \delta_{nm} \) is Kronecker delta, \( J_{m+\frac{1}{2}} \) is ordinary Bessel function of order \( m + \frac{1}{2} \), and \( j_m(Z) \) is the spherical Bessel function of order \( m \).

Therefore,

\[ e^{(i\omega t - kx)} = e^{i\omega t} \sum_{m=0}^{\infty} A_m P_m(\mu) \]

\[ = e^{i\omega t} \sum_{m=0}^{\infty} (2m + 1)(-i)^m j_m(Z) P_m(\mu) \]

\[ = e^{i\omega t} \sum_{n=0}^{\infty} (2n + 1)(-i)^n j_n(kr) P_n(\cos \theta). \]

Hence,

\[ \Phi_i = \sum_{n=0}^{\infty} (2n + 1)(-i)^n \delta_n j_n(kr) P_n(\cos \theta) e^{i\omega t} \]
where

\[ \delta_n = e^{-ikh} + (-1)^n e^{ikh}, \]

\( j_n(\cdot) \): is the spherical Bessel function of order \( n \),

\[ k \equiv \frac{\omega}{c_0} \] is the wave number in the surrounding fluid

\( P_n(\cdot) \): is the Legendre polynomial of order \( n \).
APPENDIX B

SPHERICAL COMPONENTS OF TERMS IN ACOUSTIC FORCE

It is convenient to express $\vec{F}$, $F_x$, $F_y$ and $F_z$ in terms of $\Phi^*$, and by using the boundary conditions, we have the following expressions evaluated at $r = r_p$:

$$-\frac{1}{2} \rho_0 |\Phi^*|^2 \vec{n} \bigg|_{r=r_p}$$ components

$$\frac{1}{2} e^{2i\omega t} \sin(\theta) \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( ak^* \tilde{b}_{n,m,j_n}(ak^*) Y_n^m(\theta, \phi) + ak^* Y_n^m(\theta, \phi) \right) \left( \tilde{b}_{n,m,j_n}(ak^*) + ak^* \delta(\theta, \phi) \tilde{b}_{n,m,j_n}(ak^*) \right) \right)^2$$

$$+ \frac{1}{2} e^{2i\omega t} \csc(\theta) \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( \tilde{b}_{n,m,j_n}(ak^*) Y_{n\phi}^m(\theta, \phi) + \delta(\theta, \phi) \tilde{b}_{n,m,j_n}(ak^*) \right) \right)^2$$

$$+ \frac{1}{2} e^{2i\omega t} \sin(\theta) \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( ak^* \tilde{b}_{n,m,j_n}(ak^*) Y_n^m(\theta, \phi) + ak^* Y_n^m(\theta, \phi) \right) \left( \tilde{b}_{n,m,j_n}(ak^*) + ak^* \delta(\theta, \phi) \tilde{b}_{n,m,j_n}(ak^*) \right) \right)^2$$

$$\times \epsilon \delta(\theta, \phi),$$

$$- \frac{1}{2} e^{2i\omega t} \sin(\theta) \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( ak^* \tilde{b}_{n,m,j_n}(ak^*) Y_n^m(\theta, \phi) + ak^* Y_n^m(\theta, \phi) \right) \left( \tilde{b}_{n,m,j_n}(ak^*) + ak^* \delta(\theta, \phi) \tilde{b}_{n,m,j_n}(ak^*) \right) \right)^2$$

$$\times \epsilon \delta(\theta, \phi),$$

$$- \frac{1}{2} e^{2i\omega t} \csc(\theta) \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( \tilde{b}_{n,m,j_n}(ak^*) Y_{n\phi}^m(\theta, \phi) + \delta(\theta, \phi) \tilde{b}_{n,m,j_n}(ak^*) \right) \right)^2$$

$$\times \epsilon \delta(\theta, \phi),$$

$$- \frac{1}{2} e^{2i\omega t} \sin(\theta) \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( ak^* \tilde{b}_{n,m,j_n}(ak^*) Y_n^m(\theta, \phi) + ak^* Y_n^m(\theta, \phi) \right) \left( \tilde{b}_{n,m,j_n}(ak^*) + ak^* \delta(\theta, \phi) \tilde{b}_{n,m,j_n}(ak^*) \right) \right)^2$$

$$\times \epsilon \delta(\theta, \phi),$$

$$+ \frac{1}{2} e^{2i\omega t} \csc(\theta) \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( \tilde{b}_{n,m,j_n}(ak^*) Y_{n\phi}^m(\theta, \phi) + \delta(\theta, \phi) \tilde{b}_{n,m,j_n}(ak^*) \right) \right)^2$$

$$\times \epsilon \delta(\theta, \phi),$$

$$+ \frac{1}{2} e^{2i\omega t} \sin(\theta) \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( ak^* \tilde{b}_{n,m,j_n}(ak^*) Y_n^m(\theta, \phi) + ak^* Y_n^m(\theta, \phi) \right) \left( \tilde{b}_{n,m,j_n}(ak^*) + ak^* \delta(\theta, \phi) \tilde{b}_{n,m,j_n}(ak^*) \right) \right)^2$$

$$\times \epsilon \delta(\theta, \phi).$$

89
\[ \begin{align*}
- \frac{1}{2}e^{2iu_0} & \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( a_k^{*} b_{n,m}^{(0)} (a_k^{*}) Y_n^m (\theta, \phi) + a_k^{*} Y_n^m (\theta, \phi) \left( b_{n,m}^{(1)} (a_k^{*}) + a_k^{*} \delta (\theta, \phi) b_{n,m}^{(0)} (a_k^{*}) \right) \right)^2 
\times \delta_\phi (\theta, \phi) 
\right)
\end{align*} \]

\[ \begin{align*}
- \frac{\lambda^2}{2} e^{2iu_t} & \rho_0 \csc (\theta)^2 \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( b_{n,m}^{(0)} (a_k^{*}) Y_n^m (\theta, \phi) + a_k^{*} \delta (\theta, \phi) b_{n,m}^{(0)} (a_k^{*}) \right) Y_n^m (\theta, \phi) \right)^2 
\times \delta_\phi (\theta, \phi) 
\right)
\end{align*} \]

\[ \begin{align*}
- \frac{1}{2} e^{2iu_0} & \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( a_k^{*} b_{n,m}^{(0)} (a_k^{*}) Y_n^m (\theta, \phi) + a_k^{*} \delta (\theta, \phi) b_{n,m}^{(0)} (a_k^{*}) \right) Y_n^m (\theta, \phi) \right)^2 
\times \delta_\phi (\theta, \phi) 
\right)
\end{align*} \]

\[ \begin{align*}
+ \frac{\lambda^2}{2} e^{2iu_t} & \csc (\theta)^2 \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( b_{n,m}^{(0)} (a_k^{*}) Y_n^m (\theta, \phi) + a_k^{*} \delta (\theta, \phi) b_{n,m}^{(0)} (a_k^{*}) \right) Y_n^m (\theta, \phi) \right)^2 
\times \delta_\phi (\theta, \phi) 
\right)
\end{align*} \]

\[ \begin{align*}
+ \frac{\lambda^2}{2} e^{2iu_0} & \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( b_{n,m}^{(0)} (a_k^{*}) Y_n^m (\theta, \phi) + a_k^{*} \delta (\theta, \phi) b_{n,m}^{(0)} (a_k^{*}) \right) Y_n^m (\theta, \phi) \right)^2 
\times \delta_\phi (\theta, \phi) 
\right)
\end{align*} \]

\[ \left( \lambda^2 \rho_0 \frac{\partial \phi^*}{\partial t} \right)^2 \eta^l \bigg|_{l \rho_p} \text{ components} \]

\[ \begin{align*}
- \lambda^2 \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( b_{n,m}^{(0)} (a_k^{*}) Y_n^m (\theta, \phi) + a_k^{*} \delta (\theta, \phi) b_{n,m}^{(0)} (a_k^{*}) \right) + 0 (\epsilon)^2 \right)^2 
\times (a^2 \sin (\theta) + 2 a^2 \sin (\theta) + a^2 \sin (\theta))^2 \theta (\theta, \phi) \right)
\end{align*} \]

\[ \begin{align*}
- \lambda^2 \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( b_{n,m}^{(0)} (a_k^{*}) Y_n^m (\theta, \phi) + a_k^{*} \delta (\theta, \phi) b_{n,m}^{(0)} (a_k^{*}) \right) + 0 (\epsilon)^2 \right)^2 
\times (-a^2 \sin (\theta) \delta_\phi (\theta, \phi) - a^2 \sin (\theta) \delta_\phi (\theta, \phi)) \right)
\end{align*} \]

\[ \begin{align*}
- \lambda^2 \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( b_{n,m}^{(0)} (a_k^{*}) Y_n^m (\theta, \phi) + a_k^{*} \delta (\theta, \phi) b_{n,m}^{(0)} (a_k^{*}) \right) + 0 (\epsilon)^2 \right)^2 
\times (-a^2 \delta_\phi (\theta, \phi) - a^2 \delta_\phi (\theta, \phi)) \right)
\end{align*} \]
\[
(\vec{u} \cdot \vec{n}) = -\alpha e^{iw} \sin(\theta) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( a_k^* \bar{b}_{nm}^{(0)} Y_n^m(\theta, \phi) j_n^m(ak^*) + a_k Y_n^m(\theta, \phi) \left( \bar{b}_{nm}^{(1)} j_n^m(ak^*) + a_k^* \delta(\theta, \phi) \bar{b}_{nm}^{(0)} j_n^m(ak^*) \right) \right) e + O(e^2)
\]
\[
+ \alpha e^{iw} \csc(\theta) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( \bar{b}_{nm}^{(0)} Y_n^m(\theta, \phi) + \left( \bar{b}_{nm}^{(1)} j_n^m(ak^*) + a_k^* \delta(\theta, \phi) \bar{b}_{nm}^{(0)} j_n^m(ak^*) \right) Y_n^m(\theta, \phi) e + O(e^2) \right) \times e \delta(\theta, \phi)
\]
\[
+ \alpha e^{iw} \sin(\theta) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( \bar{b}_{nm}^{(0)} Y_n^m(\theta, \phi) + \left( \bar{b}_{nm}^{(1)} j_n^m(ak^*) + a_k^* \delta(\theta, \phi) \bar{b}_{nm}^{(0)} j_n^m(ak^*) \right) Y_n^m(\theta, \phi) e + O(e^2) \right) \times e \delta(\theta, \phi)
\]
\[
- 2\alpha e^{iw} \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( a_k^* \bar{b}_{nm}^{(0)} Y_n^m(\theta, \phi) j_n^m(ak^*) + a_k Y_n^m(\theta, \phi) \left( \bar{b}_{nm}^{(1)} j_n^m(ak^*) + a_k^* \delta(\theta, \phi) \bar{b}_{nm}^{(0)} j_n^m(ak^*) \right) \right) e + O(e^2) \right)
\]
\[
\times e \delta(\theta, \phi) + O(e \delta(\theta, \phi))^2
\]

\[
\vec{u} = -\frac{\alpha e^{iw} a}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( a_k^{*} \bar{b}_{nm}^{(0)} Y_n^m(\theta, \phi) j_n^m(ak^*) + a_k Y_n^m(\theta, \phi) \left( \bar{b}_{nm}^{(1)} j_n^m(ak^*) + a_k^* \delta(\theta, \phi) \bar{b}_{nm}^{(0)} j_n^m(ak^*) \right) \right) e + O(e^2)
\]
\[
- \frac{\Delta \delta}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( \bar{b}_{nm}^{(0)} Y_n^m(\theta, \phi) + \left( \bar{b}_{nm}^{(1)} j_n^m(ak^*) + a_k^* \delta(\theta, \phi) \bar{b}_{nm}^{(0)} j_n^m(ak^*) \right) Y_n^m(\theta, \phi) e + O(e^2) \right) \times e \delta(\theta, \phi)
\]
\[
+ O(e \delta(\theta, \phi))^2,
\]
\[
- \frac{\Delta \delta}{a} \csc(\theta) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( \bar{b}_{nm}^{(0)} Y_n^m(\theta, \phi) + \left( \bar{b}_{nm}^{(1)} j_n^m(ak^*) + a_k^* \delta(\theta, \phi) \bar{b}_{nm}^{(0)} j_n^m(ak^*) \right) Y_n^m(\theta, \phi) e + O(e^2) \right) \times e \delta(\theta, \phi)
\]
\[
+ O(e \delta(\theta, \phi))^2
\]

From above we obtain the \( \rho_0(\vec{u} \cdot \vec{n}) \) components.
Let $F_{ur}$ be the $r$-component of $\frac{1}{2} \rho_0 |\vec{\nabla} \Phi^*|^2 \hat{\eta}^2_{rr}$.

\[
\frac{1}{2} \rho_0 \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right)^2 \\
+ \frac{\lambda^2}{2} \rho_0 \csc(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right)^2 \\
+ \frac{\lambda^2}{2} \rho_0 \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) \right)^2 \\
+ \rho_0 \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right) \epsilon \delta(\theta, \phi)
\]

Let $F_{u\theta}$ be the $\theta$-component of $\frac{1}{2} \rho_0 |\vec{\nabla} \Phi^*|^2 \hat{\eta}^2_{\theta\theta}$.

\[
- \frac{1}{2} \rho_0 \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right) \epsilon \delta_\theta(\theta, \phi) \\
- \frac{\lambda^2}{2} \rho_0 \csc(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right) \epsilon \delta_\theta(\theta, \phi) \\
- \frac{\lambda^2}{2} \rho_0 \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) \right) \epsilon \delta_\theta(\theta, \phi) \\
- \frac{1}{2} \rho_0 \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right) \epsilon \delta(\theta, \phi) \epsilon \delta_\theta(\theta, \phi) \\
+ \frac{\lambda^2}{2} \rho_0 \csc(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right) \epsilon \delta(\theta, \phi) \epsilon \delta_\theta(\theta, \phi) \\
+ \frac{\lambda^2}{2} \rho_0 \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) \right) \epsilon \delta(\theta, \phi) \epsilon \delta_\theta(\theta, \phi)
\]
Let $F_{u\phi}$ be the $\phi$-component of $\frac{-1}{2}\rho_0 \left| \nabla \Phi^* \right|^2 \vec{n} \bigg|_{r=r_p}$

$$\frac{-1}{2} \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right)^2 e \delta_{\phi}(\theta, \phi)$$

$$\frac{-\lambda^2}{2} \rho_0 \csc^2(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right)^2 e \delta_{\phi}(\theta, \phi)$$

$$\frac{-\lambda^2}{2} \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) \right)^2 e \delta_{\phi}(\theta, \phi)$$

$$\frac{1}{2} \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right)^2 e \delta(\theta, \phi) e \delta_{\phi}(\theta, \phi)$$

$$\frac{\lambda^2}{2} \rho_0 \csc^2(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right)^2 \delta(\theta, \phi) e \delta_{\phi}(\theta, \phi)$$

$$\frac{\lambda^2}{2} \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) \right)^2 e \delta(\theta, \phi) e \delta_{\phi}(\theta, \phi)$$

Let $F_{tr}$ be the $r$-component of $\frac{\lambda^2 \rho_0}{2c_0^2} \frac{\partial \Phi^*}{\partial t} \vec{n} \bigg|_{r=r_p}$

$$- \lambda^2 \rho_0 \frac{\omega^2}{2c_0^2} \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_7 + \epsilon I_8) \right)^2$$

$$\times \left( a^2 \sin(\theta) + 2a^2 \sin(\theta) e \delta(\theta, \phi) + a^2 \sin(\theta) e^2 \delta(\theta, \phi)^2 \right)$$

Let $F_{t\theta}$ be the $\theta$-component of $\frac{\lambda^2 \rho_0}{2c_0^2} \frac{\partial \Phi^*}{\partial t} \vec{n} \bigg|_{r=r_p}$

$$- \lambda^2 \rho_0 \frac{\omega^2}{2c_0^2} \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_7 + \epsilon I_8) \right)^2$$

$$\times \left(-a^2 \sin(\theta) e \delta_{\theta}(\theta, \phi) - a^2 \sin(\theta) e \delta(\theta, \phi) e \delta_{\phi}(\theta, \phi) \right)$$

Let $F_{t\phi}$ be the $\phi$-component of $\frac{\lambda^2 \rho_0}{2c_0^2} \frac{\partial \Phi^*}{\partial t} \vec{n} \bigg|_{r=r_p}$

$$- \lambda^2 \rho_0 \frac{\omega^2}{2c_0^2} \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_7 + \epsilon I_8) \right)^2$$

$$\times \left(-a^2 e \delta_{\phi}(\theta, \phi) - a^2 e \delta(\theta, \phi) e \delta_{\phi}(\theta, \phi) \right)$$
Let \( F_{unr} \) be the \( r \)-component of \( \rho_0(\vec{\nabla} \Phi^*(\vec{\nabla} \Phi^* \cdot \vec{n})) \) components

\[
= -\rho_0 \left( -\frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) + O(\epsilon^2) \right) \times a \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right) \\
+ \rho_0 \left( -\frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) + O(\epsilon^2) \right) \times \lambda a \csc(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right) \epsilon \delta_\phi(\theta, \phi) \\
+ \rho_0 \left( -\frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) + O(\epsilon^2) \right) \times \lambda a \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) \right) \epsilon \delta_\theta(\theta, \phi) \\
- \rho_0 \left( -\frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) + O(\epsilon^2) \right) \times 2a \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right) \epsilon \delta(\theta, \phi) \\
+ O(\epsilon^2 \delta(\theta, \phi)^2)
\]

Let \( F_{ang} \) be the \( \theta \)-component of \( \rho_0(\vec{\nabla} \Phi^*(\vec{\nabla} \Phi^* \cdot \vec{n})) \) components

\[
= -\lambda \rho_0 \left( -\frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) + \frac{\epsilon \delta(\theta, \phi)}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) + O(\epsilon^2 \delta(\theta, \phi)^2) \right) \\
\times a \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right) \\
+ \lambda \rho_0 \left( -\frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) + \frac{\epsilon \delta(\theta, \phi)}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) + O(\epsilon^2 \delta(\theta, \phi)^2) \right) \\
\times \lambda a \csc(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right) \epsilon \delta_\phi(\theta, \phi) \\
+ \lambda \rho_0 \left( -\frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) + \frac{\epsilon \delta(\theta, \phi)}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) + O(\epsilon^2 \delta(\theta, \phi)^2) \right) \\
\times \lambda a \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) \right) \epsilon \delta_\theta(\theta, \phi) \\
- \lambda \rho_0 \left( -\frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) + \frac{\epsilon \delta(\theta, \phi)}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) + O(\epsilon^2 \delta(\theta, \phi)^2) \right) \\
\times 2a \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right) \epsilon \delta(\theta, \phi) + O(\epsilon^2 \delta(\theta, \phi)^2)
\]
Let $F_{\nu\phi}$ be the $\phi$-component of $\rho_0(\nabla^* \Phi (\nabla^* \Phi \cdot \vec{n}))$ components

$$= -\lambda \rho_0 \left( -\frac{1}{a} \csc(\theta) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right)$$

$$- \lambda \rho_0 \left( \frac{\epsilon \delta(\theta, \phi)}{a} \csc(\theta) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) + O(\epsilon^2 \delta(\theta, \phi)^2) \right)$$

$$\times a^2 \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right)$$

$$+ \lambda \rho_0 \left( -\frac{1}{a} \csc(\theta) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right)$$

$$+ \lambda \rho_0 \left( \frac{\epsilon \delta(\theta, \phi)}{a} \csc(\theta) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) + O(\epsilon^2 \delta(\theta, \phi)^2) \right)$$

$$\times \lambda \csc(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right) \epsilon \delta(\theta, \phi)$$

$$+ \lambda \rho_0 \left( -\frac{1}{a} \csc(\theta) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right)$$

$$+ \lambda \rho_0 \left( \frac{\epsilon \delta(\theta, \phi)}{a} \csc(\theta) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) + O(\epsilon^2 \delta(\theta, \phi)^2) \right)$$

$$\times \lambda \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) \right) \epsilon \delta(\theta, \phi)$$

$$- \lambda \rho_0 \left( -\frac{1}{a} \csc(\theta) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right)$$

$$- \lambda \rho_0 \left( \frac{\epsilon \delta(\theta, \phi)}{a} \csc(\theta) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) + O(\epsilon^2 \delta(\theta, \phi)^2) \right)$$

$$\times 2a^2 \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right) \epsilon \delta(\theta, \phi) + O(\epsilon^2 \delta(\theta, \phi)^2)$$
where,

\[ I_1 = ak^* b_{nm}^{(0)} Y_n^m(\theta, \phi) j_n'(ak^*) e^{i\omega t} \]

\[ I_2 = ak^* Y_n^m(\theta, \phi) \left( b_{nm}^{(1)} j_n'(ak^*) + ak^* \delta(\theta, \phi) b_{nm}^{(0)} j_n''(ak^*) \right) e^{i\omega t} \]

\[ I_3 = b_{nm}^{(0)} j_n(ak^*) Y_n^m(\theta, \phi) e^{i\omega t} \]

\[ I_4 = Y_{n\phi}^m(\theta, \phi) \left( b_{nm}^{(1)} j_n'(ak^*) + ak^* \delta(\theta, \phi) b_{nm}^{(0)} j_n''(ak^*) \right) e^{i\omega t} \]

\[ I_5 = b_{nm}^{(0)} j_n(ak^*) Y_{n\phi}^m(\theta, \phi) e^{i\omega t} \]

\[ I_6 = Y_{n\phi}^m(\theta, \phi) \left( b_{nm}^{(1)} j_n'(ak^*) + ak^* \delta(\theta, \phi) b_{nm}^{(0)} j_n''(ak^*) \right) e^{i\omega t} \]

\[ I_7 = b_{nm}^{(0)} j_n(ak^*) Y_n^m(\theta, \phi) e^{i\omega t} \]

\[ I_8 = Y_n^m(\theta, \phi) \left( b_{nm}^{(1)} j_n'(ak^*) + ak^* \delta(\theta, \phi) b_{nm}^{(0)} j_n''(ak^*) \right) e^{i\omega t} \]
Simplifying various components yields

\[ F_{ur} = \frac{1}{2} \rho_0 \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right)^2 \]
\[ + \frac{\lambda^2}{2} \rho_0 \csc(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right)^2 \]
\[ + \frac{\lambda^2}{2} \rho_0 \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) \right)^2 \]
\[ + \rho_0 \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right)^2 \epsilon \delta(\theta, \phi) \]

\[ = \frac{1}{2} \rho_0 \sin(\theta) \left( J_1 J_1 + 2\epsilon J_1 J_2 + O(\epsilon^2) \right) \]
\[ + \frac{\lambda^2}{2} \rho_0 \csc(\theta) \left( J_3 J_3 + 2\epsilon J_3 J_4 + O(\epsilon^2) \right) \]
\[ + \frac{\lambda^2}{2} \rho_0 \sin(\theta) \left( J_5 J_5 + 2\epsilon J_5 J_6 + O(\epsilon^2) \right) \]
\[ + \rho_0 \sin(\theta) \left( J_1 J_1 + 2\epsilon J_1 J_2 + O(\epsilon^2) \right) \epsilon \delta(\theta, \phi) \]

\[ = \frac{1}{2} \rho_0 \sin(\theta) J_1 J_1 + \frac{\lambda^2}{2} \rho_0 \csc(\theta) J_3 J_3 + \frac{\lambda^2}{2} \rho_0 \sin(\theta) J_5 J_5 \]
\[ + \epsilon \rho_0 \sin(\theta) J_1 J_2 + \epsilon \lambda^2 \rho_0 \csc(\theta) J_3 J_4 \]
\[ + \epsilon \lambda^2 \rho_0 \sin(\theta) J_5 J_6 + \epsilon \rho_0 \sin(\theta) J_1 J_1 \delta(\theta, \phi) \]
\[ + O(\epsilon^2) \]
\[ F_{u\theta} = -\frac{1}{2} \rho_0 \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right)^2 \epsilon \delta \theta (\theta, \phi) \]

\[ -\frac{\lambda^2}{2} \rho_0 \csc(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right)^2 \epsilon \delta \theta (\theta, \phi) \]

\[ -\frac{\lambda^2}{2} \rho_0 \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) \right)^2 \epsilon \delta \theta (\theta, \phi) \]

\[ -\frac{1}{2} \rho_0 \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right)^2 \epsilon \delta \theta (\theta, \phi) \epsilon \delta \theta (\theta, \phi) \]

\[ +\frac{\lambda^2}{2} \rho_0 \csc(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right)^2 \epsilon \delta \theta (\theta, \phi) \epsilon \delta \theta (\theta, \phi) \]

\[ +\frac{\lambda^2}{2} \rho_0 \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) \right)^2 \epsilon \delta \theta (\theta, \phi) \epsilon \delta \theta (\theta, \phi) \]

\[ = -\frac{1}{2} \rho_0 \sin(\theta) \left( J_1 J_1 + 2 \epsilon J_1 J_2 + 0(\epsilon^2) \right) \epsilon \delta \theta (\theta, \phi) \]

\[ -\frac{\lambda^2}{2} \rho_0 \csc(\theta) \left( J_3 J_3 + 2 \epsilon J_3 J_4 + 0(\epsilon^2) \right) \epsilon \delta \theta (\theta, \phi) \]

\[ -\frac{\lambda^2}{2} \rho_0 \sin(\theta) \left( J_5 J_5 + 2 \epsilon J_5 J_6 + 0(\epsilon^2) \right) \epsilon \delta \theta (\theta, \phi) + O(\epsilon^2) \]

\[ = -\frac{\epsilon}{2} \rho_0 \sin(\theta) J_1 J_1 \delta \theta (\theta, \phi) - \frac{\epsilon \lambda^2}{2} \rho_0 \csc(\theta) J_3 J_3 \delta \theta (\theta, \phi) \]

\[ -\frac{\epsilon \lambda^2}{2} \rho_0 \sin(\theta) J_5 J_5 \delta \theta (\theta, \phi) + O(\epsilon^2) \]
\[ F_{\phi} = -\frac{1}{2} \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right)^2 \times \delta_{\phi}(\theta, \phi) \]

\[-\frac{\lambda^2}{2} \rho_0 \csc^2(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right)^2 \times \delta_{\phi}(\theta, \phi) \]

\[-\frac{\lambda^2}{2} \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_6 + \epsilon I_5) \right)^2 \times \delta_{\phi}(\theta, \phi) \]

\[-\frac{1}{2} \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right)^2 \times \delta(\theta, \phi) \delta_{\phi}(\theta, \phi) \]

\[+\frac{\lambda^2}{2} \rho_0 \csc^2(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right)^2 \times \delta(\theta, \phi) \delta_{\phi}(\theta, \phi) \]

\[+\frac{\lambda^2}{2} \rho_0 \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_6 + \epsilon I_5) \right)^2 \times \delta(\theta, \phi) \delta_{\phi}(\theta, \phi) \]

\[= -\frac{1}{2} \rho_0 \left( J_1 J_1 + 2\epsilon J_1 J_2 + O(\epsilon)^2 \right) \delta_{\phi}(\theta, \phi) \]

\[-\frac{\lambda^2}{2} \rho_0 \csc^2(\theta) \left( J_3 J_3 + 2\epsilon J_3 J_4 + O(\epsilon)^2 \right) \delta_{\phi}(\theta, \phi) \]

\[-\frac{\lambda^2}{2} \rho_0 \left( J_5 J_5 + 2\epsilon J_5 J_6 + O(\epsilon)^2 \right) \delta_{\phi}(\theta, \phi) + O(\epsilon)^2 \]

\[= -\frac{\epsilon}{2} \rho_0 J_1 J_1 \delta_{\phi}(\theta, \phi) - \frac{\epsilon \lambda^2}{2} \rho_0 \csc^2(\theta) J_3 J_3 \delta_{\phi}(\theta, \phi) \]

\[-\frac{\epsilon \lambda^2}{2} \rho_0 J_5 J_5 \delta_{\phi}(\theta, \phi) + O(\epsilon)^2 \]
\[ F_{tr} = -\lambda^2 \rho_0 \frac{\omega^2}{2c_0^2} \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_7 + \epsilon I_8) \right)^2 \times (a^2 \sin(\theta) + 2a^2 \sin(\theta) \epsilon \delta(\theta, \phi) + a^2 \sin(\theta) \epsilon^2 \delta(\theta, \phi)^2) \]

\[ = -\lambda^2 \rho_0 \frac{\omega^2}{2c_0^2} \left( J_7 J_7 + 2\epsilon J_7 J_8 + 0(\epsilon^2) \right) \times (a^2 \sin(\theta) + 2a^2 \sin(\theta) \epsilon \delta(\theta, \phi) + a^2 \sin(\theta) \epsilon^2 \delta(\theta, \phi)^2) \]

\[ = -\lambda^2 \rho_0 \frac{\omega^2}{2c_0^2} J_7 J_7 a^2 \sin(\theta) - \epsilon \lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} 2J_7 J_8 \sin(\theta) \]

\[ - \epsilon \lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} 2J_7 J_7 \sin(\theta) \delta(\theta, \phi) + 0(\epsilon^2) \]

\[ F_{\theta \theta} = -\lambda^2 \rho_0 \frac{\omega^2}{2c_0^2} \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_7 + \epsilon I_8) \right)^2 \times (-a^2 \sin(\theta) \epsilon \delta_{\theta} (\theta, \phi) - a^2 \sin(\theta) \epsilon \delta(\theta, \phi) \epsilon \delta_{\theta} (\theta, \phi)) \]

\[ = -\lambda^2 \rho_0 \frac{\omega^2}{2c_0^2} \left( J_7 J_7 + 2\epsilon J_7 J_8 + 0(\epsilon^2) \right) \times (-a^2 \sin(\theta) \epsilon \delta_{\theta} (\theta, \phi) - a^2 \sin(\theta) \epsilon \delta(\theta, \phi) \epsilon \delta_{\theta} (\theta, \phi)) \]

\[ = \epsilon \lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} J_7 J_7 \sin(\theta) \delta_{\theta} (\theta, \phi) + 0(\epsilon^2) \]
\[ F_{t\phi} = -\lambda^2 \rho_0 \frac{\omega^2}{2c_0^2} \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_7 + \varepsilon I_8) \right)^2 \times (-a^2 \varepsilon \delta_{\phi}(\theta, \phi) - a^2 \varepsilon \delta(\theta, \phi) \varepsilon \delta_{\phi}(\theta, \phi)) \]

\[ = -\lambda^2 \rho_0 \frac{\omega^2}{2c_0^2} (J_7 J_7 + 2 \varepsilon J_7 J_8 + 0(\varepsilon)^2) \times (-a^2 \varepsilon \delta_{\phi}(\theta, \phi) - a^2 \varepsilon \delta(\theta, \phi) \varepsilon \delta_{\phi}(\theta, \phi)) \]

\[ = \varepsilon \lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} J_7 J_7 \delta_{\phi}(\theta, \phi) + 0(\varepsilon)^2 \]
\[ F_{unr} = -\rho_0 \left( -\frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) + O(\epsilon^2) \right) \times a \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right) + \rho_0 \left( -\frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) + O(\epsilon^2) \right) \times \lambda a \csc(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right) \epsilon \delta(\theta, \phi) + \rho_0 \left( -\frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) + O(\epsilon^2) \right) \times \lambda a \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) \right) \epsilon \delta(\theta, \phi) - \rho_0 \left( -\frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) + O(\epsilon^2) \right) \times 2a \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right) \epsilon \delta(\theta, \phi) + O(\epsilon^2 \delta(\theta, \phi)^2) \]

\[ = \rho_0 \sin(\theta) (J_1 + \epsilon J_2) (J_1 + \epsilon J_2) - \epsilon \lambda \rho_0 \csc(\theta) (J_1 + \epsilon J_2) (J_3 + \epsilon J_4) \delta(\theta, \phi) - \epsilon \lambda \rho_0 \sin(\theta) (J_1 + \epsilon J_2) (J_5 + \epsilon J_6) \delta(\theta, \phi) + \epsilon^2 \rho_0 \sin(\theta) (J_1 + \epsilon J_2) (J_1 + \epsilon J_2) \delta(\theta, \phi) + O(\epsilon^2) \]

\[ = \rho_0 \sin(\theta) (J_1 J_1 + \epsilon^2 J_1 J_2 + O(\epsilon^2)) - \epsilon \lambda \rho_0 \csc(\theta) (J_1 J_3 + \epsilon (J_{1,j} J_{4,k}) + \epsilon J_2 J_3 + O(\epsilon^2)) \delta(\theta, \phi) - \epsilon \lambda \rho_0 \sin(\theta) (J_1 J_5 + \epsilon J_1 J_6 + \epsilon J_2 J_5 + \epsilon^2 J_2 J_6 + O(\epsilon^2)) \delta(\theta, \phi) + \epsilon^2 \rho_0 \sin(\theta) (J_1 J_1 + \epsilon^2 J_1 J_2 + O(\epsilon^2)) \delta(\theta, \phi) + O(\epsilon^2) \]

\[ = \rho_0 \sin(\theta) J_1 J_1 + \epsilon^2 \rho_0 \sin(\theta) J_1 J_2 - \epsilon \lambda \rho_0 \csc(\theta) J_1 J_3 \delta(\theta, \phi) - \epsilon \lambda \rho_0 \sin(\theta) J_1 J_5 \delta(\theta, \phi) + \epsilon^2 \rho_0 \sin(\theta) J_1 J_1 \delta(\theta, \phi) + O(\epsilon^2) \]
\[ F_{un\theta} = -\lambda \rho_0 \left( -\frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) + \frac{\epsilon \delta(\theta, \phi)}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) + O(\epsilon \delta(\theta, \phi))^2 \right) \times \\
\times a \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right) + \\
+ \lambda \rho_0 \left( -\frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) + \frac{\epsilon \delta(\theta, \phi)}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) + O(\epsilon \delta(\theta, \phi))^2 \right) \times \\
\times \lambda a \csc(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right) \epsilon \delta(\theta, \phi) + \\
+ \lambda \rho_0 \left( -\frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) + \frac{\epsilon \delta(\theta, \phi)}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) + O(\epsilon \delta(\theta, \phi))^2 \right) \times \\
\times \lambda a \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) \right) \epsilon \delta(\theta, \phi) + \\
- \lambda \rho_0 \left( -\frac{1}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) + \frac{\epsilon \delta(\theta, \phi)}{a} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_5 + \epsilon I_6) + O(\epsilon \delta(\theta, \phi))^2 \right) \times \\
\times 2a \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right) \epsilon \delta(\theta, \phi) + O(\epsilon^2) = \\
-\lambda \rho_0 \left( -\frac{1}{a} (J_5 + \epsilon J_6) + \frac{\epsilon \delta(\theta, \phi)}{a} (J_5 + \epsilon J_6) \right) \times a \sin(\theta) (J_1 + \epsilon J_2) + \\
+ \lambda \rho_0 \left( -\frac{1}{a} (J_5 + \epsilon J_6) + \frac{\epsilon \delta(\theta, \phi)}{a} (J_5 + \epsilon J_6) \right) \times \lambda a \csc(\theta) (J_3 + \epsilon J_4) \epsilon \delta(\theta, \phi) + \\
+ \lambda \rho_0 \left( -\frac{1}{a} (J_5 + \epsilon J_6) + \frac{\epsilon \delta(\theta, \phi)}{a} (J_5 + \epsilon J_6) \right) \times \lambda a \sin(\theta) (J_5 + \epsilon J_6) \epsilon \delta(\theta, \phi) + \\
- \lambda \rho_0 \left( -\frac{1}{a} (J_5 + \epsilon J_6) + \frac{\epsilon \delta(\theta, \phi)}{a} (J_5 + \epsilon J_6) \right) \times 2a \sin(\theta) (J_1 + \epsilon J_2) \epsilon \delta(\theta, \phi) + \\
+ O(\epsilon^2) \]
\[ = \lambda \rho_0 \sin(\theta) (J_6 + \epsilon J_6) (J_1 + \epsilon J_2) - \epsilon \lambda \rho_0 \sin(\theta) (J_5 + \epsilon J_5) (J_1 + \epsilon J_2) \delta(\theta, \phi) \\
- \epsilon \lambda^2 \rho_0 \csc(\theta) (J_5 + \epsilon J_5) (J_3 + \epsilon J_4) \delta_\phi(\theta, \phi) \\
- \epsilon \lambda^2 \rho_0 \sin(\theta) (J_5 + \epsilon J_5) (J_6 + \epsilon J_6) \delta_\theta(\theta, \phi) \\
+ \epsilon \lambda^2 \rho_0 \sin(\theta) (J_5 + \epsilon J_5) (J_1 + \epsilon J_2) \delta(\theta, \phi) + O(\epsilon^2) \]

\[ = \lambda \rho_0 \sin(\theta) (J_1 J_5 + \epsilon J_1 J_6 + \epsilon J_2 J_5 + O(\epsilon^2)) \\
- \epsilon \lambda \rho_0 \sin(\theta) (J_1 J_5 + \epsilon J_1 J_6 + \epsilon J_2 J_5 + O(\epsilon^2)) \delta(\theta, \phi) \\
- \epsilon \lambda^2 \rho_0 \csc(\theta) (J_3 J_5 + \epsilon J_3 J_6 + \epsilon J_4 J_5 + O(\epsilon^2)) \delta_\phi(\theta, \phi) \\
- \epsilon \lambda^2 \rho_0 \sin(\theta) (J_5 J_5 + \epsilon J_5 J_6 + O(\epsilon^2)) \delta_\theta(\theta, \phi) \\
+ \epsilon \lambda^2 \rho_0 \sin(\theta) (J_1 J_5 + \epsilon J_1 J_6 + \epsilon J_2 J_5 + O(\epsilon^2)) \delta(\theta, \phi) + O(\epsilon^2) \]

\[ = \lambda \rho_0 \sin(\theta) J_1 J_5 + \epsilon \lambda \rho_0 \sin(\theta) J_1 J_6 - \epsilon \lambda \rho_0 \sin(\theta) J_1 J_5 \delta(\theta, \phi) \\
- \epsilon \lambda^2 \rho_0 \csc(\theta) J_3 J_5 \delta_\phi(\theta, \phi) - \epsilon \lambda^2 \rho_0 \sin(\theta) J_5 J_5 \delta_\theta(\theta, \phi) \\
+ \epsilon \lambda^2 \rho_0 \sin(\theta) J_1 J_5 \delta(\theta, \phi) + O(\epsilon^2) \]
\[ F_{\text{un}\phi} = -\lambda \rho_0 \left( -\frac{1}{a} \csc(\theta) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) + \frac{\epsilon \delta(\theta, \phi)}{a} \csc(\theta) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) + O(\epsilon^2 \delta(\theta, \phi)^2) \right) \times \\
\frac{a \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right)}{\lambda a^{\frac{1}{2}} \csc(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right) \epsilon \delta(\theta, \phi)} \\
+ \lambda \rho_0 \left( -\frac{1}{a} \csc(\theta) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) + \frac{\epsilon \delta(\theta, \phi)}{a} \csc(\theta) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) + O(\epsilon^2 \delta(\theta, \phi)^2) \right) \times \\
\frac{\lambda a \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_3 + \epsilon I_4) \right) \epsilon \delta(\theta, \phi)}{2 a \sin(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (I_1 + \epsilon I_2) \right) \epsilon \delta(\theta, \phi) + O(\epsilon^2 \delta(\theta, \phi)^2)} \]

\[ = -\lambda \rho_0 \left( -\frac{1}{a} \csc(\theta) (J_3 + \epsilon J_4) + \frac{\epsilon \delta(\theta, \phi)}{a} \csc(\theta) (J_3 + \epsilon J_4) \right) \times \frac{\lambda a \sin(\theta) (J_1 + \epsilon J_2)}{\lambda a \csc(\theta) (J_1 + \epsilon J_2) \epsilon \delta(\theta, \phi)} \\
+ \lambda \rho_0 \left( -\frac{1}{a} \csc(\theta) (J_3 + \epsilon J_4) + \frac{\epsilon \delta(\theta, \phi)}{a} \csc(\theta) (J_3 + \epsilon J_4) \right) \times \lambda a \sin(\theta) (J_3 + \epsilon J_4) \epsilon \delta(\theta, \phi) \\
+ \lambda \rho_0 \left( -\frac{1}{a} \csc(\theta) (J_3 + \epsilon J_4) + \frac{\epsilon \delta(\theta, \phi)}{a} \csc(\theta) (J_3 + \epsilon J_4) \right) \times \lambda a \sin(\theta) (J_3 + \epsilon J_4) \epsilon \delta(\theta, \phi) \\
- \lambda \rho_0 \left( -\frac{1}{a} \csc(\theta) (J_3 + \epsilon J_4) + \frac{\epsilon \delta(\theta, \phi)}{a} \csc(\theta) (J_3 + \epsilon J_4) \right) \times 2 a \sin(\theta) (J_1 + \epsilon J_2) \epsilon \delta(\theta, \phi) + O(\epsilon^2) \]
\[ = \lambda \rho_0 \left( J_3 + \varepsilon J_4 \right) \left( J_1 + \varepsilon J_2 \right) - \varepsilon \lambda \rho_0 \left( J_3 + \varepsilon J_4 \right) \left( J_1 + \varepsilon J_2 \right) \delta(\theta, \phi) \\
- \varepsilon \lambda^2 \rho_0 \csc^2(\theta) \left( J_3 + \varepsilon J_4 \right) \left( J_1 + \varepsilon J_2 \right) \delta(\theta, \phi) \\
- \varepsilon \lambda^2 \rho_0 \left( J_3 + \varepsilon J_4 \right) \left( J_5 + \varepsilon J_6 \right) \delta(\theta, \phi) \\
+ \varepsilon \lambda^2 \rho_0 \left( J_3 + \varepsilon J_4 \right) \left( J_1 + \varepsilon J_2 \right) \delta(\theta, \phi) + O(\varepsilon^2) \]

\[ = \lambda \rho_0 \left( J_1 J_3 + \varepsilon J_{1, j} J_{4, k} + \varepsilon J_2 J_3 + O(\varepsilon^2) \right) \\
- \varepsilon \lambda \rho_0 \left( J_1 J_3 + \varepsilon J_{1, j} J_{4, k} + \varepsilon J_2 J_3 + O(\varepsilon^2) \right) \delta(\theta, \phi) \\
- \varepsilon \lambda^2 \rho_0 \csc^2(\theta) \left( J_3 J_3 + \varepsilon J_{3, j} J_{4, k} \right) \delta(\theta, \phi) \\
- \varepsilon \lambda^2 \rho_0 \left( J_3 J_5 + \varepsilon J_{4, j} J_{5, k} + \varepsilon J_3 J_6 + O(\varepsilon^2) \right) \delta(\theta, \phi) \\
+ \varepsilon \lambda^2 \rho_0 \left( J_1 J_3 + \varepsilon J_{1, j} J_{4, k} + \varepsilon J_2 J_3 + O(\varepsilon^2) \right) \delta(\theta, \phi) + O(\varepsilon^2) \]

\[ = \lambda \rho_0 J_1 J_3 + \varepsilon \lambda \rho_0 J_{1, j} J_{4, k} + \varepsilon \lambda \rho_0 J_2 J_3 - \varepsilon \lambda \rho_0 J_1 J_3 \delta(\theta, \phi) \\
- \varepsilon \lambda^2 \rho_0 \csc^2(\theta) J_3 J_3 \delta(\theta, \phi) - \varepsilon \lambda^2 \rho_0 J_3 J_5 \delta(\theta, \phi) \\
+ \varepsilon \lambda^2 \rho_0 J_1 J_3 \delta(\theta, \phi) + O(\varepsilon^2) \]

where,

\[ J_i = \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} I_i \right) \]
APPENDIX C

ACOUSTIC FORCE ON A PERTURBED SPHERE

\[
F_x = - \left\langle \int_0^{2\pi} \int_0^{\pi} \mathcal{R} \left\{ F_{ur} \cos(\phi) \sin(\theta) + F_{u\theta} \cos(\theta) \cos(\phi) - F_{u\phi} \sin(\phi) \right\} d\theta d\phi \right\rangle \\
\quad - \left\langle \int_0^{2\pi} \int_0^{\pi} \mathcal{R} \left\{ F_{tr} \cos(\phi) \sin(\theta) + F_{t\theta} \cos(\theta) \cos(\phi) - F_{t\phi} \sin(\phi) \right\} d\theta d\phi \right\rangle \\
\quad - \left\langle \int_0^{2\pi} \int_0^{\pi} \mathcal{R} \left\{ F_{unr} \cos(\phi) \sin(\theta) + F_{un\theta} \cos(\theta) \cos(\phi) - F_{un\phi} \sin(\phi) \right\} d\theta d\phi \right\rangle \\
\]

\[
F_y = - \left\langle \int_0^{2\pi} \int_0^{\pi} \mathcal{R} \left\{ F_{ur} \sin(\phi) \sin(\theta) + F_{u\theta} \cos(\theta) \sin(\phi) + F_{u\phi} \cos(\phi) \right\} d\theta d\phi \right\rangle \\
\quad - \left\langle \int_0^{2\pi} \int_0^{\pi} \mathcal{R} \left\{ F_{tr} \sin(\phi) \sin(\theta) + F_{t\theta} \cos(\theta) \sin(\phi) + F_{t\phi} \cos(\phi) \right\} d\theta d\phi \right\rangle \\
\quad - \left\langle \int_0^{2\pi} \int_0^{\pi} \mathcal{R} \left\{ F_{unr} \sin(\phi) \sin(\theta) + F_{un\theta} \cos(\theta) \sin(\phi) + F_{un\phi} \cos(\phi) \right\} d\theta d\phi \right\rangle \\
\]

\[
F_z = - \left\langle \int_0^{2\pi} \int_0^{\pi} \mathcal{R} \left\{ F_{ur} \cos(\theta) - F_{u\theta} \sin(\theta) \right\} d\theta d\phi \right\rangle \\
\quad - \left\langle \int_0^{2\pi} \int_0^{\pi} \mathcal{R} \left\{ F_{tr} \cos(\theta) - F_{t\theta} \sin(\theta) \right\} d\theta d\phi \right\rangle \\
\quad - \left\langle \int_0^{2\pi} \int_0^{\pi} \mathcal{R} \left\{ F_{unr} \cos(\theta) - F_{un\theta} \sin(\theta) \right\} d\theta d\phi \right\rangle \\
\]
Whence,

\[
F_x = -\left\langle \int_0^{2\pi} \int_0^{\pi} \mathbb{R}\{F_{ur} + F_{tr} + F_{ unr}\} \cos(\phi) \sin(\theta) d\theta d\phi \right\rangle \\
- \left\langle \int_0^{2\pi} \int_0^{\pi} \mathbb{R}\{F_{u\theta} + F_{t\theta} + F_{ unr\theta}\} \cos(\theta) \cos(\phi) d\theta d\phi \right\rangle \\
+ \left\langle \int_0^{2\pi} \int_0^{\pi} \mathbb{R}\{F_{u\phi} + F_{t\phi} + F_{ unr\phi}\} \sin(\phi) d\theta d\phi \right\rangle \\
F_y = -\left\langle \int_0^{2\pi} \int_0^{\pi} \mathbb{R}\{F_{ur} + F_{tr} + F_{ unr}\} \sin(\theta) \sin(\phi) d\theta d\phi \right\rangle \\
- \left\langle \int_0^{2\pi} \int_0^{\pi} \mathbb{R}\{F_{u\theta} + F_{t\theta} + F_{ unr\theta}\} \cos(\phi) d\theta d\phi \right\rangle \\
- \left\langle \int_0^{2\pi} \int_0^{\pi} \mathbb{R}\{F_{u\phi} + F_{t\phi} + F_{ unr\phi}\} \cos(\phi) d\theta d\phi \right\rangle \\
F_z = -\left\langle \int_0^{2\pi} \int_0^{\pi} \mathbb{R}\{F_{ur} + F_{tr} + F_{ unr}\} \cos(\theta) d\theta d\phi \right\rangle \\
+ \left\langle \int_0^{2\pi} \int_0^{\pi} \mathbb{R}\{F_{u\theta} + F_{t\theta} + F_{ unr\theta}\} \sin(\theta) d\theta d\phi \right\rangle
\]

Consider,

\[
\left\langle \int_0^{2\pi} \int_0^{\pi} \mathbb{R}\{F_{ur}\} \cos(\phi) \sin(\theta) d\theta d\phi \right\rangle \\
= \int_0^{2\pi} \int_0^{\pi} \frac{1}{2} a^2 \rho_0 \cos(\phi) \sin^2(\theta) \left( \mathbb{R}\{J_1\} \mathbb{R}\{J_1\} \right) d\theta d\phi + ... \\
= \int_0^{2\pi} \int_0^{\pi} \frac{1}{2} a^2 \rho_0 \cos(\phi) \sin^2(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathbb{R}(I_{1,n}^m) \right) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathbb{R}(I_{1,n}^m) \right) d\theta d\phi + ... \\
= \int_0^{2\pi} \int_0^{\pi} \frac{1}{2} a^2 \rho_0 \cos(\phi) \sin^2(\theta) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_n(\theta, \phi) M_{1,n}^m \right) \left( \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_n(\theta, \phi) M_{1,n}^m \right) d\theta d\phi
\]

where,

\[
M_{1,n}^m = \mathbb{R}\{ak^*E_nB_{nm}\delta_n(ak^*) e^{im\phi} e^{i\omega t}\} \\
M_{2,n}^m = \mathbb{R}\{ak^*E_n\delta_n(ak^*) + ak^*E_n\delta_n(ak^*) \} e^{im\phi} e^{i\omega t}\} \\
M_{3,n}^m = \mathbb{R}\{imE_n\delta_n(ak^*) e^{im\phi} e^{i\omega t}\} \\
M_{4,n}^m = \mathbb{R}\{im\delta_n(ak^*) e^{im\phi} e^{i\omega t}\} \\
\]
\[ M_{5,n}^m = \Re \left\{ E_n \tilde{B}_{nm}^{(0)} j_n(ak^*) e^{im\phi} e^{i\omega t} \right\} \]
\[ M_{6,n}^m = \Re \left\{ (E_n \tilde{B}_{nm}^{(1)} j_n(ak^*) + ak^* E_n \tilde{B}_{nm}^{(0)} \delta(\theta, \phi) j_n'(ak^*) ) e^{im\phi} e^{i\omega t} \right\} \]
\[ M_{7,n}^m = \Re \left\{ E_n \tilde{B}_{nm}^{(0)} j_n(ak^*) e^{im\phi} e^{i\omega t} \right\} \]
\[ M_{8,n}^m = \Re \left\{ (E_n \tilde{B}_{nm}^{(1)} j_n(ak^*) + ak^* E_n \tilde{B}_{nm}^{(0)} \delta(\theta, \phi) j_n'(ak^*) ) e^{im\phi} e^{i\omega t} \right\} \] (C.1)

The infinite double series can be written in terms of a single series as

\[ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_n^m (\cos \theta) M_{l,n}^m = \sum_{i=0}^{\infty} S_{l,i} , \forall l = \{1, 2, 3, 4, 7, 8\} \]

where,

\[ S_{l,0} = P_0^0 (\cos \theta) M_{l,0}^0 \]
\[ S_{l,1} = P_1^{-1} (\cos \theta) M_{l,1}^{-1} + P_1^0 (\cos \theta) M_{l,1}^0 + P_1^1 (\cos \theta) M_{l,1}^1 \]
\[ S_{l,2} = P_2^{-2} (\cos \theta) M_{l,2}^{-2} + P_2^{-1} (\cos \theta) M_{l,2}^{-1} + P_2^0 (\cos \theta) M_{l,2}^0 + P_2^1 (\cos \theta) M_{l,2}^1 + P_2^2 (\cos \theta) M_{l,2}^2 \] (C.2)

\[ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_{n\theta}^m (\cos \theta) M_{l,n}^m = \sum_{i=0}^{\infty} S_{l,i} , \forall l = \{5, 6\} \]

where,

\[ S_{l,0} = P_{\theta}^0 (\cos \theta) M_{l,0}^0 \]
\[ S_{l,1} = P_{1\theta}^{-1} (\cos \theta) M_{l,1}^{-1} + P_{1\theta}^0 (\cos \theta) M_{l,1}^0 + P_{1\theta}^1 (\cos \theta) M_{l,1}^1 \]
\[ S_{l,2} = P_{2\theta}^{-2} (\cos \theta) M_{l,2}^{-2} + P_{2\theta}^{-1} (\cos \theta) M_{l,2}^{-1} + P_{2\theta}^0 (\cos \theta) M_{l,2}^0 + P_{2\theta}^1 (\cos \theta) M_{l,2}^1 + P_{2\theta}^2 (\cos \theta) M_{l,2}^2 \] (C.3)
Using the Cauchy product, we obtain
\[
\left\langle \left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_n^m(\cos \theta) M_{1,n}^m \right\} \left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_n^m(\cos \theta) M_{5,n}^m \right\} \right\rangle = \left\langle \left\{ \sum_{i=0}^{\infty} S_{1,i} \right\} \left\{ \sum_{i=0}^{\infty} S_{5,i} \right\} \right\rangle \\
= \sum_{i=0}^{\infty} \left\{ \sum_{j+k=i} S_{1,j} S_{5,k} \right\} \\
= \sum_{i=0}^{\infty} \left\{ \sum_{j+k=i} (S_{1,j} S_{5,k}) \right\}
\]

Therefore,
\[
\int_0^{2\pi} \int_0^{\pi} \frac{1}{2} a^2 \rho_0 \cos(\phi) \sin^2(\theta) \left\langle \left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_n^m(\cos \theta) M_{1,n}^m \right\} \left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_n^m(\cos \theta) M_{1,n}^m \right\} \right\rangle \, d\theta d\phi + ..
\]
\[
= \int_0^{2\pi} \int_0^{\pi} \frac{1}{2} a^2 \rho_0 \cos(\phi) \sin^2(\theta) \sum_{i=0}^{\infty} \left\{ \sum_{j+k=i} (S_{1,j} S_{1,k}) \right\} \, d\theta d\phi + ..
\]
\[
= \sum_{i=0}^{\infty} \left\{ \sum_{j+k=i} \int_0^{2\pi} \int_0^{\pi} \frac{1}{2} a^2 \rho_0 \cos(\phi) \sin^2(\theta) (S_{1,j} S_{1,k}) \, d\theta d\phi \right\} + ..
\]
\[
\approx \sum_{0 \leq j,k \leq 2} \int_0^{2\pi} \int_0^{\pi} \frac{1}{2} a^2 \rho_0 \cos(\phi) \sin^2(\theta) (S_{1,j} S_{1,k}) \, d\theta d\phi + ..
\]

Applying the above technique and assuming that
\[
\Re \left\{ F_{ur} + F_{tr} + F_{ unr} \right\} = \sum_{i=0}^{\infty} \sum_{j+k=i} F_r
\]
\[
\Re \left\{ F_{u\theta} + F_{t\theta} + F_{ unr\theta} \right\} = \sum_{i=0}^{\infty} \sum_{j+k=i} F_\theta
\]
\[
\Re \left\{ F_{u\phi} + F_{t\phi} + F_{ unr\phi} \right\} = \sum_{i=0}^{\infty} \sum_{j+k=i} F_\phi
\]
we get,

\[
F_x = - \sum_{i=0}^{\infty} \left\{ \sum_{j+k=i}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} F_r \cos(\phi) \sin(\theta) d\theta d\phi \right\} \\
- \sum_{i=0}^{\infty} \left\{ \sum_{j+k=i}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} F_{\theta} \cos(\theta) \cos(\phi) d\theta d\phi \right\} \\
+ \sum_{i=0}^{\infty} \left\{ \sum_{j+k=i}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} F_{\phi} \sin(\phi) d\theta d\phi \right\}
\]

\[
F_y = - \sum_{i=0}^{\infty} \left\{ \sum_{j+k=i}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} F_r \sin(\theta) \sin(\phi) d\theta d\phi \right\} \\
- \sum_{i=0}^{\infty} \left\{ \sum_{j+k=i}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} F_{\theta} \cos(\theta) \sin(\phi) d\theta d\phi \right\} \\
- \sum_{i=0}^{\infty} \left\{ \sum_{j+k=i}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} F_{\phi} \cos(\phi) d\theta d\phi \right\}
\]

\[
F_z = - \sum_{i=0}^{\infty} \left\{ \sum_{j+k=i}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} F_r \cos(\theta) d\theta d\phi \right\} \\
+ \sum_{i=0}^{\infty} \left\{ \sum_{j+k=i}^{\infty} \int_{0}^{2\pi} \int_{0}^{\pi} F_{\theta} \sin(\theta) d\theta d\phi \right\}
\]

where

\[
F_r = F_{r0} + \epsilon F_{r1} + \epsilon F_{r2} \delta(\theta, \phi) + \epsilon F_{r3} \delta(\theta, \phi) + \epsilon F_{r4} \delta(\theta, \phi)
\]

\[
F_{r0} = \frac{3}{2} \rho_0 \sin(\theta) \langle S_{1,j} S_{1,k} \rangle + \frac{\lambda^2}{2} \rho_0 \csc(\theta) \langle S_{3,j} S_{3,k} \rangle \\
+ \frac{\lambda^2}{2} \rho_0 \sin(\theta) \langle S_{5,j} S_{5,k} \rangle - \frac{\lambda^2 a^2 \rho_0}{2c_0^2} \omega^2 \sin(\theta) \langle S_{7,j} S_{7,k} \rangle
\]

\[
F_{r1} = 3 \rho_0 \sin(\theta) \langle S_{1,j} S_{2,k} \rangle + \lambda^2 \rho_0 \csc(\theta) \langle S_{3,j} S_{4,k} \rangle \\
+ \lambda^2 \rho_0 \sin(\theta) \langle S_{5,j} S_{6,k} \rangle - \lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) \langle S_{7,j} S_{8,k} \rangle
\]

\[
F_{r2} = 3 \rho_0 \sin(\theta) \langle S_{1,j} S_{1,k} \rangle - \epsilon \lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) \langle S_{7,j} S_{7,k} \rangle
\]

\[
F_{r3} = -\lambda \rho_0 \sin(\theta) \langle S_{1,j} S_{5,k} \rangle
\]

\[
F_{r4} = -\lambda \rho_0 \csc(\theta) \langle S_{1,j} S_{3,k} \rangle
\] (C.4)
\[ F_\theta = F_{\theta 0} + \epsilon F_{\theta 1} + \epsilon F_{\theta 2} \delta(\theta, \phi) + \epsilon F_{\theta 3} \delta(\theta, \phi) + \epsilon F_{\theta 4} \delta(\theta, \phi) \]

\[ F_{\theta 0} = \lambda \rho_0 \sin(\theta) \langle S_{1,j}S_{5,k} \rangle \]

\[ F_{\theta 1} = \lambda \rho_0 \sin(\theta) \langle S_{1,j}S_{6,k} \rangle \]

\[ F_{\theta 2} = \lambda \rho_0 \sin(\theta) \langle S_{1,j}S_{5,k} \rangle \]

\[ F_{\theta 3} = -\frac{1}{2} \rho_0 \sin(\theta) \langle S_{1,j}S_{1,k} \rangle - \frac{\lambda^2}{2} \rho_0 \csc(\theta) \langle S_{3,j}S_{3,k} \rangle - \frac{3\lambda^2}{2} \rho_0 \sin(\theta) \langle S_{5,j}S_{5,k} \rangle + \lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} \langle S_{7,j}S_{7,k} \rangle \sin(\theta) \]

\[ F_{\theta 4} = -\lambda^2 \rho_0 \csc(\theta) \langle S_{3,j}S_{5,k} \rangle \quad (C.5) \]

\[ F_\phi = F_{\phi 0} + \epsilon F_{\phi 1} + \epsilon F_{\phi 2} \delta(\theta, \phi) + \epsilon F_{\phi 3} \delta(\theta, \phi) + \epsilon F_{\phi 4} \delta(\theta, \phi) \]

\[ F_{\phi 0} = \lambda \rho_0 \langle S_{1,j}S_{3,k} \rangle \]

\[ F_{\phi 1} = \lambda \rho_0 \langle S_{1,j}S_{4,k} \rangle + \lambda \rho_0 \langle S_{2,j}S_{3,k} \rangle \]

\[ F_{\phi 2} = \lambda \rho_0 \langle S_{1,j}S_{3,k} \rangle \]

\[ F_{\phi 3} = -\lambda^2 \rho_0 \langle S_{3,j}S_{5,k} \rangle \]

\[ F_{\phi 4} = -\frac{1}{2} \rho_0 \langle S_{1,j}S_{1,k} \rangle - \frac{3\lambda^2}{2} \rho_0 \csc^2(\theta) \langle S_{3,j}S_{3,k} \rangle - \frac{\lambda^2}{2} \rho_0 \langle S_{5,j}S_{5,k} \rangle + \lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} \langle S_{7,j}S_{7,k} \rangle \quad (C.6) \]

Whence,

\[ F_x \approx \sum_{0 \leq j,k \leq 2} \int_0^{2\pi} \int_0^{\pi} \{-F_r\sin(\theta)\cos(\phi) - F_\theta\cos(\theta)\cos(\phi) + F_\phi\sin(\phi)\} \, d\theta d\phi \]

\[ F_y \approx \sum_{0 \leq j,k \leq 2} \int_0^{2\pi} \int_0^{\pi} \{-F_r\sin(\theta)\sin(\phi) - F_\theta\cos(\theta)\sin(\phi) - F_\phi\cos(\phi)\} \, d\theta d\phi \]

\[ F_z \approx \sum_{0 \leq j,k \leq 2} \int_0^{2\pi} \int_0^{\pi} \{-F_r\cos(\theta) + F_\theta\sin(\theta)\} \, d\theta d\phi \]

\( F_r, \ F_\theta, \) and \( F_\phi \) are given by (C.4), (C.5), (C.6).
Consider,

\[ E_n = e^{-ikh} + (-1)^n e^{ikh} \]

\[ = \begin{cases} 
2 \cos kh & \text{if } n \text{ is even;} \\
-2i \sin kh & \text{if } n \text{ is odd.}
\end{cases} \]

\[ M_{1,n}^m = \Re \left\{ E_n a_k^* \tilde{B}_{n,m}^{(0)} e^{im\phi} j_n (ak^*) e^{i\omega t} \right\} \]

\[ = \begin{cases} 
2 \cos(kh) \Re \left\{ a_k^* \tilde{B}_{nm}^{(0)} j_n (ak^*) e^{im\phi} e^{i\omega t} \right\} & \text{if } n \text{ is even;} \\
2 \sin(kh) \Im \left\{ a_k^* \tilde{B}_{nm}^{(0)} j_n (ak^*) e^{im\phi} e^{i\omega t} \right\} & \text{if } n \text{ is odd.}
\end{cases} \]

\[ = \begin{cases} 
2 \cos(kh) \alpha_{1,n}^m & \text{if } n \text{ is even;} \\
2 \sin(kh) \beta_{1,n}^m & \text{if } n \text{ is odd.}
\end{cases} \]

where,

\[ \alpha_{1,n}^m = \Re \left\{ a_k^* \tilde{B}_{nm}^{(0)} j_n (ak^*) e^{im\phi} e^{i\omega t} \right\} \]

\[ \beta_{1,n}^m = \Im \left\{ a_k^* \tilde{B}_{nm}^{(0)} j_n (ak^*) e^{im\phi} e^{i\omega t} \right\} \]

\( \Re \) and \( \Im \) indicates the real and imaginary parts.

Similarly, we can define any \( M_{l,n}^m \) for \( l \in 2, 3, 4, 5, 6, 7, 8 \) as

\[ M_{l,n}^m = \begin{cases} 
2 \cos(kh) \alpha_{l,n}^m & \text{if } n \text{ is even;} \\
2 \sin(kh) \beta_{l,n}^m & \text{if } n \text{ is odd.}
\end{cases} \]
where,

\[
\alpha_{1,n}^m = \Re \left\{ ak^* \tilde{B}_{nm,j_n}^{(0)} (ak^*) e^{im\phi} e^{i\omega t} \right\}
\]
\[
\beta_{1,n}^m = \Im \left\{ ak^* \tilde{B}_{nm,j_n}^{(0)} (ak^*) e^{im\phi} e^{i\omega t} \right\}
\]
\[
\alpha_{2,n}^m = \Re \left\{ ak^* \left( \tilde{B}_{nm,j_n}^{(1)} (ak^*) + ak^* \delta(\theta, \phi) \tilde{B}_{nm,j_n}^{(0)} (ak^*) \right) e^{im\phi} e^{i\omega t} \right\}
\]
\[
\beta_{2,n}^m = \Im \left\{ ak^* \left( \tilde{B}_{nm,j_n}^{(1)} (ak^*) + ak^* \delta(\theta, \phi) \tilde{B}_{nm,j_n}^{(0)} (ak^*) \right) e^{im\phi} e^{i\omega t} \right\}
\]
\[
\alpha_{3,n}^m = \Re \left\{ im \tilde{B}_{nm,j_n}^{(0)} (ak^*) e^{im\phi} e^{i\omega t} \right\}
\]
\[
\beta_{3,n}^m = \Im \left\{ im \tilde{B}_{nm,j_n}^{(0)} (ak^*) e^{im\phi} e^{i\omega t} \right\}
\]
\[
\alpha_{4,n}^m = \Re \left\{ im \left( \tilde{B}_{nm,j_n}^{(1)} (ak^*) + ak^* \delta(\theta, \phi) \tilde{B}_{nm,j_n}^{(0)} (ak^*) \right) e^{im\phi} e^{i\omega t} \right\}
\]
\[
\beta_{4,n}^m = \Im \left\{ im \left( \tilde{B}_{nm,j_n}^{(1)} (ak^*) + ak^* \delta(\theta, \phi) \tilde{B}_{nm,j_n}^{(0)} (ak^*) \right) e^{im\phi} e^{i\omega t} \right\}
\]
\[
\alpha_{5,n}^m = \Re \left\{ \tilde{B}_{nm,j_n}^{(0)} (ak^*) e^{im\phi} e^{i\omega t} \right\}
\]
\[
\beta_{5,n}^m = \Im \left\{ \tilde{B}_{nm,j_n}^{(0)} (ak^*) e^{im\phi} e^{i\omega t} \right\}
\]
\[
\alpha_{6,n}^m = \Re \left\{ \left( \tilde{B}_{nm,j_n}^{(1)} (ak^*) + a \tilde{B}_{nm,j_n}^{(0)} k^* \delta(\theta, \phi) j_n (ak^*) \right) e^{im\phi} e^{i\omega t} \right\}
\]
\[
\beta_{6,n}^m = \Im \left\{ \left( \tilde{B}_{nm,j_n}^{(1)} (ak^*) + a \tilde{B}_{nm,j_n}^{(0)} k^* \delta(\theta, \phi) j_n (ak^*) \right) e^{im\phi} e^{i\omega t} \right\}
\]
\[
\alpha_{7,n}^m = \Re \left\{ \tilde{B}_{nm,j_n}^{(0)} (ak^*) e^{im\phi} e^{i\omega t} \right\}
\]
\[
\beta_{7,n}^m = \Im \left\{ \tilde{B}_{nm,j_n}^{(0)} (ak^*) e^{im\phi} e^{i\omega t} \right\}
\]
\[
\alpha_{8,n}^m = \Re \left\{ \left( \tilde{B}_{nm,j_n}^{(1)} (ak^*) + ak^* \tilde{B}_{nm,j_n}^{(0)} \delta(\theta, \phi) j_n (ak^*) \right) e^{im\phi} e^{i\omega t} \right\}
\]
\[
\beta_{8,n}^m = \Im \left\{ \left( \tilde{B}_{nm,j_n}^{(1)} (ak^*) + ak^* \tilde{B}_{nm,j_n}^{(0)} \delta(\theta, \phi) j_n (ak^*) \right) e^{im\phi} e^{i\omega t} \right\}
\]

(C.7)

Since, $\tilde{B}_{nm}$ exist only for $m = 0$. 
Therefore for \( l = \{1, 3, 7\} \) \( j = \{0, 1, 2\} \), \((C.2)\), \(S_{l,j}\) reduces to:

\[
S_{l,0} = 2 \cos(kh)P_0^0(\cos \theta)\alpha_{l,0}^0 \\
S_{l,1} = 2 \sin(kh)P_1^0(\cos \theta)\beta_{l,1}^0 \\
S_{l,2} = 2 \cos(kh)P_2^0(\cos \theta)\alpha_{l,2}^0
\]

Similarly for \( l = \{5\} \) \( j = \{0, 1, 2\} \), \((C.3)\), \(S_{l,j}\) reduces to:

\[
S_{5,0} = 2 \cos(kh)P_{0\theta}^0(\cos \theta)\alpha_{5,0}^0 \\
S_{5,1} = 2 \sin(kh)P_{1\theta}^0(\cos \theta)\beta_{5,1}^0 \\
S_{5,2} = 2 \cos(kh)P_{2\theta}^0(\cos \theta)\alpha_{5,2}^0
\]

For \( l, p \in \{1, 2, 3, 4, 5, 6, 7, 8\} \) we have,

\[
\sum_{0 \leq j, k \leq 2} \langle S_{l,j}S_{p,k} \rangle = \langle S_{l,0}S_{p,0} + S_{l,0}S_{p,1} + S_{l,0}S_{p,2} \rangle \\
+ \langle S_{l,1}S_{p,0} + S_{l,1}S_{p,1} + S_{l,1}S_{p,2} \rangle \\
+ \langle S_{l,2}S_{p,0} + S_{l,2}S_{p,1} + S_{l,2}S_{p,2} \rangle
\]

Therefore, for \( l, p \in \{1, 3, 7\} \) we have,

\[
\sum_{0 \leq j, k \leq 2} \langle S_{l,j}S_{p,k} \rangle = A_{l,p} \cos^2(kh) + B_{l,p} \sin^2(kh) + C_{l,p} \sin(2kh)
\]

where

\[
A_{l,p} = 4P_0^0(\cos \theta)P_0^0(\cos \theta) \langle \alpha_{l,0}^0\alpha_{p,0}^0 \rangle \\
+ 4P_0^0(\cos \theta)P_2^0(\cos \theta) \{ \langle \alpha_{l,0}^0\alpha_{p,2}^0 \rangle + \langle \alpha_{l,2}^0\alpha_{p,0}^0 \rangle \} \\
+ 4P_2^0(\cos \theta)P_2^0(\cos \theta) \langle \alpha_{l,2}^0\alpha_{p,2}^0 \rangle
\]

\[
B_{l,p} = 4P_1^0(\cos \theta)P_1^0(\cos \theta) \langle \beta_{l,1}^0\beta_{p,1}^0 \rangle
\]

\[
C_{l,p} = 2P_0^0(\cos \theta)P_1^0(\cos \theta) \{ \langle \alpha_{l,0}^0\beta_{p,1}^0 \rangle + \langle \alpha_{p,0}^0\beta_{l,1}^0 \rangle \} \\
+ 2P_1^0(\cos \theta)P_2^0(\cos \theta) \{ \langle \alpha_{p,2}^0\beta_{l,1}^0 \rangle + \langle \alpha_{l,2}^0\beta_{p,1}^0 \rangle \}
\]  \( \text{(C.8)} \)
For \( l \in \{1, 3, 7\} \) \( p = \{5\} \) we have,

\[
\sum_{0 \leq j, k \leq 2} \langle S_{l,j} S_{5,k} \rangle = A_5 \cos^2(kh) + B_{l,5} \sin^2(kh) + C_{l,5} \sin(2kh)
\]

where,

\[
A_{l,5} = 4 P_0^0(\cos \theta) P_{00}^0(\cos \theta) \langle \alpha_{l,0}^0 \alpha_{5,0}^0 \rangle \\
+ 4 P_0^0(\cos \theta) P_{20}^0(\cos \theta) \langle \alpha_{l,0}^0 \alpha_{5,2}^0 \rangle \\
+ 4 P_2^0(\cos \theta) P_{00}^0(\cos \theta) \langle \alpha_{l,2}^0 \alpha_{5,0}^0 \rangle \\
+ 4 P_2^0(\cos \theta) P_{20}^0(\cos \theta) \langle \alpha_{l,2}^0 \alpha_{5,2}^0 \rangle
\]

\[
B_{l,5} = 4 P_1^0(\cos \theta) P_{1\theta}^0(\cos \theta) \langle \beta_{l,1}^0 \beta_{5,1}^0 \rangle
\]

\[
C_{l,5} = 2 P_0^0(\cos \theta) P_{1\theta}^0(\cos \theta) \langle \alpha_{l,0}^0 \beta_{5,1}^0 \rangle \\
+ 2 P_1^0(\cos \theta) P_{0\theta}^0(\cos \theta) \langle \beta_{l,1}^0 \alpha_{5,0}^0 \rangle \\
+ 2 P_1^0(\cos \theta) P_{2\theta}^0(\cos \theta) \langle \beta_{l,1}^0 \alpha_{5,2}^0 \rangle \\
+ 2 P_2^0(\cos \theta) P_{1\theta}^0(\cos \theta) \langle \alpha_{l,2}^0 \beta_{5,1}^0 \rangle
\] (C.9)
For \( l \in \{1, 3, 7\} \) and \( p \in \{2, 4, 8\} \) we have,

\[
\sum_{0 \leq j, k \leq 2} \langle S_{l,j} S_{p,k} \rangle = A_{l,p} \cos^2(kh) + B_{l,p} \sin^2(kh) + C_{l,p} \sin(2kh)
\]

where,

\[
A_{l,p} = 4P_0^0(\cos \theta)P_0^0(\cos \theta) \left( \alpha_{l,0}^0 \alpha_{p,0}^0 \right)
\]

\[
+ 4P_0^0(\cos \theta)P_2^{-2}(\cos \theta) \left( \alpha_{l,0}^0 \alpha_{p,2}^{-2} \right)
\]

\[
+ 4P_0^0(\cos \theta)P_2^{-1}(\cos \theta) \left( \alpha_{l,0}^0 \alpha_{p,2}^{-1} \right)
\]

\[
+ 4P_0^0(\cos \theta)P_2^0(\cos \theta) \left( \alpha_{l,0}^0 \alpha_{p,2}^0 \right)
\]

\[
+ 4P_0^0(\cos \theta)P_2^1(\cos \theta) \left( \alpha_{l,0}^0 \alpha_{p,2}^1 \right)
\]

\[
+ 4P_0^0(\cos \theta)P_2^2(\cos \theta) \left( \alpha_{l,0}^0 \alpha_{p,2}^2 \right)
\]

\[
B_{l,p} = 4P_1^0(\cos \theta)P_1^{-1}(\cos \theta) \left( \beta_{l,1}^0 \beta_{p,1}^{-1} \right)
\]

\[
+ 4P_1^0(\cos \theta)P_1^0(\cos \theta) \left( \beta_{l,1}^0 \beta_{p,1}^0 \right)
\]

\[
+ 4P_1^0(\cos \theta)P_1^1(\cos \theta) \left( \beta_{l,1}^0 \beta_{p,1}^1 \right)
\]

\[
C_{l,p} = 2P_0^0(\cos \theta)P_1^{-1}(\cos \theta) \left( \alpha_{l,0}^0 \beta_{p,1}^{-1} \right)
\]

\[
+ 2P_0^0(\cos \theta)P_1^0(\cos \theta) \left( \alpha_{l,0}^0 \beta_{p,1}^0 \right)
\]

\[
+ 2P_0^0(\cos \theta)P_1^1(\cos \theta) \left( \alpha_{l,0}^0 \beta_{p,1}^1 \right)
\]

\[
+ 2P_0^0(\cos \theta)P_2^{-2}(\cos \theta) \left( \alpha_{p,2}^{-2} \beta_{l,1}^0 \right)
\]

\[
+ 2P_0^0(\cos \theta)P_2^{-1}(\cos \theta) \left( \alpha_{p,2}^{-1} \beta_{l,1}^0 \right)
\]

\[
+ 2P_0^0(\cos \theta)P_2^0(\cos \theta) \left( \alpha_{p,2}^0 \beta_{l,1}^0 \right)
\]

\[
+ 2P_0^0(\cos \theta)P_2^1(\cos \theta) \left( \alpha_{p,2}^0 \beta_{l,1}^1 \right)
\]

\[
+ 2P_0^0(\cos \theta)P_2^2(\cos \theta) \left( \alpha_{p,2}^0 \beta_{l,1}^2 \right)
\]

\[
+ 2P_0^0(\cos \theta)P_1^{-1}(\cos \theta) \left( \alpha_{l,0}^0 \beta_{p,1}^{-1} \right)
\]

\[
\tag{C.10}
\]
Similarly we can find

\[ \sum_{0 \leq j,k \leq 2} \langle S_{l,j} S_{p,k} \rangle \]

for \( l = 5, \ p \in \{2, 4, 8\} \) and for \( l \in \{1, 3, 7\}, \ p = 6. \)

Using the above results \( F_r, \ F_\theta, \ F_\phi \) can be rewritten as:

\[
\sum_{0 \leq j,k \leq 2} F_r = F_{\text{rcs}} \cos^2(kh) + F_{\text{rsc}} \sin^2(kh) + F_{\text{rsc}} \sin(2kh)
\]

\[
\sum_{0 \leq j,k \leq 2} F_\theta = F_{\text{thh}} \cos^2(kh) + F_{\text{thh}} \sin^2(kh) + F_{\text{thh}} \sin(2kh)
\]

\[
\sum_{0 \leq j,k \leq 2} F_\phi = F_{\text{phi}} \cos^2(kh) + F_{\text{phi}} \sin^2(kh) + F_{\text{phi}} \sin(2kh)
\]

where,

\[ F_{\text{rcs}} = F_{\text{r0c}} + \epsilon F_{\text{r1c}} + \epsilon F_{\text{r2c}} \delta(\theta, \phi) + \epsilon F_{\text{r3c}} \delta(\theta, \phi) + \epsilon F_{\text{r4c}} \delta(\theta, \phi) \]

\[ F_{\text{r0c}} = \frac{3}{2} \rho_0 \sin(\theta) A_{1,1} + \frac{\lambda^2}{2} \rho_0 \csc(\theta) A_{3,3} \]

\[ + \frac{\lambda^2}{2} \rho_0 \sin(\theta) A_{5,5} - \lambda^2 a^2 \frac{\omega^2}{2c_0^2} \sin(\theta) A_{7,7} \]

\[ F_{\text{r1c}} = 3 \rho_0 \sin(\theta) A_{1,2} + \lambda^2 \rho_0 \csc(\theta) A_{3,4} \]

\[ + \lambda^2 \rho_0 \sin(\theta) A_{5,6} - \lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) A_{7,8} \]

\[ F_{\text{r2c}} = 3 \rho_0 \sin(\theta) A_{1,1} - \lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) A_{7,7} \]

\[ F_{\text{r3c}} = -\lambda \rho_0 \sin(\theta) A_{1,5} \]

\[ F_{\text{r4c}} = -\lambda \rho_0 \csc(\theta) A_{1,3} \]
\[ F_{r_{ss}} = F_{r_{0ss}} + \epsilon F_{r_{1ss}} + \epsilon F_{r_{2ss}} \delta(\theta, \phi) + \epsilon F_{r_{3ss}} \phi(\theta, \phi) + \epsilon F_{r_{4ss}} \phi(\theta, \phi) \]

\[ F_{r_{0ss}} = \frac{3}{2} \rho_0 \sin(\theta) B_{1,1} + \frac{\lambda^2}{2} \rho_0 \csc(\theta) B_{3,3} \]
\[ + \frac{\lambda^2}{2} \rho_0 \sin(\theta) B_{5,5} - \lambda^2 \alpha^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) B_{7,7} \]

\[ F_{r_{1ss}} = 3 \rho_0 \sin(\theta) B_{1,2} + \lambda^2 \rho_0 \csc(\theta) B_{3,4} \]
\[ + \lambda^2 \rho_0 \sin(\theta) B_{5,6} - \lambda^2 \alpha^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) B_{7,8} \]

\[ F_{r_{2ss}} = 3 \rho_0 \sin(\theta) B_{1,1} - \lambda^2 \alpha^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) B_{7,7} \]

\[ F_{r_{3ss}} = -\lambda \rho_0 \sin(\theta) B_{1,5} \]

\[ F_{r_{4ss}} = -\lambda \rho_0 \csc(\theta) B_{1,3} \]

\[ F_{r_{sc}} = F_{r_{0sc}} + \epsilon F_{r_{1sc}} + \epsilon F_{r_{2sc}} \delta(\theta, \phi) + \epsilon F_{r_{3sc}} \phi(\theta, \phi) + \epsilon F_{r_{4sc}} \phi(\theta, \phi) \]

\[ F_{r_{0sc}} = \frac{3}{2} \rho_0 \sin(\theta) C_{1,1} + \frac{\lambda^2}{2} \rho_0 \csc(\theta) C_{3,3} \]
\[ + \frac{\lambda^2}{2} \rho_0 \sin(\theta) C_{5,5} - \lambda^2 \alpha^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) C_{7,7} \]

\[ F_{r_{1sc}} = 3 \rho_0 \sin(\theta) C_{1,2} + \lambda^2 \rho_0 \csc(\theta) C_{3,4} \]
\[ + \lambda^2 \rho_0 \sin(\theta) C_{5,6} - \lambda^2 \alpha^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) C_{7,8} \]

\[ F_{r_{2sc}} = 3 \rho_0 \sin(\theta) C_{1,1} - \lambda^2 \alpha^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) C_{7,7} \]

\[ F_{r_{3sc}} = -\lambda \rho_0 \sin(\theta) C_{1,5} \]

\[ F_{r_{4sc}} = -\lambda \rho_0 \csc(\theta) C_{1,3} \]
\[ F_{\theta_{cc}} = F_{\theta_{0cc}} + \epsilon F_{\theta_{1cc}} + \epsilon F_{\theta_{2cc}} \delta(\theta, \phi) + \epsilon F_{\theta_{3cc}} \delta\phi(\theta, \phi) + \epsilon F_{\theta_{4cc}} \delta\phi(\theta, \phi) \]

\[ F_{\theta_{0cc}} = \lambda \rho_0 \sin(\theta) A_{1,5} \]

\[ F_{\theta_{1cc}} = \lambda \rho_0 \sin(\theta) A_{1,6} \]

\[ F_{\theta_{2cc}} = \lambda \rho_0 \sin(\theta) A_{1,5} \]

\[ F_{\theta_{3cc}} = -\frac{1}{2} \rho_0 \sin(\theta) A_{1,1} - \frac{\lambda^2}{2} \rho_0 \csc(\theta) A_{3,3} \]

\[ - \frac{3\lambda^2}{2} \rho_0 \sin(\theta) A_{5,5} + \lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) A_{7,7} \]

\[ F_{\theta_{4}} = -\lambda^2 \rho_0 \csc(\theta) A_{3,5} \]

\[ F_{\phi_{ss}} = F_{\phi_{0ss}} + \epsilon F_{\phi_{1ss}} + \epsilon F_{\phi_{2ss}} \delta(\theta, \phi) + \epsilon F_{\phi_{3ss}} \delta\phi(\theta, \phi) + \epsilon F_{\phi_{4ss}} \delta\phi(\theta, \phi) \]

\[ F_{\theta_{0ss}} = \lambda \rho_0 \sin(\theta) B_{1,5} \]

\[ F_{\theta_{1ss}} = \lambda \rho_0 \sin(\theta) B_{1,6} \]

\[ F_{\theta_{2ss}} = \lambda \rho_0 \sin(\theta) B_{1,5} \]

\[ F_{\theta_{3ss}} = -\frac{1}{2} \rho_0 \sin(\theta) B_{1,1} - \frac{\lambda^2}{2} \rho_0 \csc(\theta) B_{3,3} \]

\[ - \frac{3\lambda^2}{2} \rho_0 \sin(\theta) B_{5,5} + \lambda^2 a^2 \rho_0 \frac{\omega^2}{2c_0^2} \sin(\theta) B_{7,7} \]

\[ F_{\theta_{4}} = -\lambda^2 \rho_0 \csc(\theta) B_{3,5} \]
\begin{align*}
F_{\theta_{sc}} &= F_{\theta_{0sc}} + \epsilon F_{\theta_{1sc}} + \epsilon F_{\theta_{2sc}} \delta(\theta, \phi) + \epsilon F_{\theta_{3sc}} \delta(\theta, \phi) + \epsilon F_{\theta_{4sc}} \delta(\theta, \phi) \\
F_{\theta_{0sc}} &= \lambda \rho_0 \sin(\theta) C_{1,5} \\
F_{\theta_{1sc}} &= \lambda \rho_0 \sin(\theta) C_{1,6} \\
F_{\theta_{2sc}} &= \lambda \rho_0 \sin(\theta) C_{1,5} \\
F_{\theta_{3sc}} &= \frac{1}{2} \rho_0 \sin(\theta) C_{1,1} - \frac{\lambda^2}{2} \rho_0 \csc(\theta) C_{3,3} \\
&\quad - \frac{3\lambda^2}{2} \rho_0 \sin(\theta) C_{5,5} + \lambda^2 \rho_0 A_{\omega^2} C_{7,7} \\
F_{\theta_{4sc}} &= -\lambda^2 \rho_0 \csc(\theta) C_{3,5} \\

F_{\phi_{cc}} &= F_{\phi_{0cc}} + \epsilon F_{\phi_{1cc}} + \epsilon F_{\phi_{2cc}} \delta(\theta, \phi) + \epsilon F_{\phi_{3cc}} \delta(\theta, \phi) + \epsilon F_{\phi_{4cc}} \delta(\theta, \phi) \\
F_{\phi_{0cc}} &= \lambda \rho_0 A_{1,3} \\
F_{\phi_{1cc}} &= \lambda \rho_0 A_{1,4} + \lambda \rho_0 A_{3,2} \\
F_{\phi_{2cc}} &= \lambda \rho_0 A_{1,3} \\
F_{\phi_{3cc}} &= -\lambda^2 \rho_0 A_{3,5} \\
F_{\phi_{4cc}} &= -\frac{1}{2} \rho_0 A_{1,1} - \frac{3\lambda^2}{2} \rho_0 \csc^2(\theta) A_{3,3} - \frac{\lambda^2}{2} \rho_0 A_{5,5} + \lambda^2 \rho_0 A_{\omega^2} A_{7,7} \\

F_{\phi_{ss}} &= F_{\phi_{0ss}} + \epsilon F_{\phi_{1ss}} + \epsilon F_{\phi_{2ss}} \delta(\theta, \phi) + \epsilon F_{\phi_{3ss}} \delta(\theta, \phi) + \epsilon F_{\phi_{4ss}} \delta(\theta, \phi) \\
F_{\phi_{0ss}} &= \lambda \rho_0 B_{1,3} \\
F_{\phi_{1ss}} &= \lambda \rho_0 B_{1,4} + \lambda \rho_0 B_{3,2} \\
F_{\phi_{2ss}} &= \lambda \rho_0 B_{1,3} \\
F_{\phi_{3ss}} &= -\lambda^2 \rho_0 B_{3,5} \\
F_{\phi_{4ss}} &= -\frac{1}{2} \rho_0 B_{1,1} - \frac{3\lambda^2}{2} \rho_0 \csc^2(\theta) B_{3,3} - \frac{\lambda^2}{2} \rho_0 B_{5,5} + \lambda^2 \rho_0 A_{\omega^2} B_{7,7}
\end{align*}
\[ F_{\phi sc} = F_{\phi 0 sc} + \epsilon F_{\phi 1 sc} + \epsilon F_{\phi 2 sc} \delta(\theta, \phi) + \epsilon F_{\phi 3 sc} \delta(\theta, \phi) + \epsilon F_{\phi 4 sc} \delta(\theta, \phi) \]

\[ F_{\phi 0 sc} = \lambda \rho_0 C_{1,3} \]

\[ F_{\phi 1 sc} = \lambda \rho_0 C_{1,4} + \lambda \rho_0 C_{3,2} \]

\[ F_{\phi 2} = \lambda \rho_0 C_{1,3} \]

\[ F_{\phi 3} = -\lambda^2 \rho_0 C_{3,5} \]

\[ F_{\phi 4} = -\frac{1}{2} \rho_0 C_{1,1} - \frac{3 \lambda^2}{2} \rho_0 \csc^2(\theta) C_{3,3} - \frac{\lambda^2}{2} \rho_0 C_{5,5} + \lambda^2 a^2 \rho_0 \frac{\omega^2}{2 c_0^2} C_{7,7} \]
Finally, $F_x$, $F_y$, and $F_z$ can be rewritten as:

$$F_x \approx \sum_{0 \leq j, k \leq 2} \int_0^{2\pi} \int_0^\pi \left\{ -F_r \sin(\theta) \cos(\phi) - F_\theta \cos(\theta) \cos(\phi) + F_\phi \sin(\phi) \right\} d\theta d\phi$$

$$2 \int_0^{2\pi} \int_0^\pi \left\{ -F_{rcc} \cos^2(kh) - F_{rsc} \sin^2(kh) - F_{rsc} \sin(2kh) \right\} \sin(\theta) \cos(\phi) d\theta d\phi$$

$$- \int_0^{2\pi} \int_0^\pi \left\{ F_{\theta cc} \cos^2(kh) + F_{\theta ss} \sin^2(kh) + F_{\theta sc} \sin(2kh) \right\} \cos(\theta) \cos(\phi) d\theta d\phi$$

$$+ \int_0^{2\pi} \int_0^\pi \left\{ F_{\phi cc} \cos^2(kh) + F_{\phi ss} \sin^2(kh) + F_{\phi sc} \sin(2kh) \right\} \sin(\phi) d\theta d\phi$$

$$\approx \int_0^{2\pi} \int_0^\pi \left\{ -F_{rcc} \sin(\theta) \cos(\phi) - F_{\theta cc} \cos(\theta) \cos(\phi) + F_{\phi cc} \sin(\phi) \right\} \cos^2(kh) d\theta d\phi$$

$$+ \int_0^{2\pi} \int_0^\pi \left\{ -F_{rsc} \sin(\theta) \cos(\phi) - F_{\theta sc} \cos(\theta) \cos(\phi) + F_{\phi sc} \sin(\phi) \right\} \sin^2(kh) d\theta d\phi$$

$$+ \int_0^{2\pi} \int_0^\pi \left\{ -F_{rsc} \sin(\theta) \cos(\phi) - F_{\theta sc} \cos(\theta) \cos(\phi) + F_{\phi sc} \sin(\phi) \right\} \sin(2kh) d\theta d\phi$$

$$F_x \approx F_{xcc} \cos^2(kh) + F_{xss} \sin^2(kh) + F_{xsc} \sin(2kh)$$

where,

$$F_{xcc} = F_{x0cc} + \epsilon F_{x1cc}$$

$$F_{xss} = F_{x0ss} + \epsilon F_{x1ss}$$

$$F_{xsc} = F_{x0sc} + \epsilon F_{x1sc}$$

$$F_{x0cc} = \int_0^{2\pi} \int_0^\pi \left\{ -F_{r0cc} \sin(\theta) \cos(\phi) - F_{\theta 0cc} \cos(\theta) \cos(\phi) + F_{\phi 0cc} \sin(\phi) \right\} d\theta d\phi$$

$$F_{x1cc} = \int_0^{2\pi} \int_0^\pi \left\{ -F_{r1cc} \sin(\theta) \cos(\phi) - F_{\theta 1cc} \cos(\theta) \cos(\phi) + F_{\phi 1cc} \sin(\phi) \right\} d\theta d\phi$$

$$+ \int_0^{2\pi} \int_0^\pi \left\{ -F_{r2cc} \sin(\theta) \cos(\phi) - F_{\theta 2cc} \cos(\theta) \cos(\phi) + F_{\phi 2cc} \sin(\phi) \right\} \delta(\theta, \phi) d\theta d\phi$$

$$+ \int_0^{2\pi} \int_0^\pi \left\{ -F_{r3cc} \sin(\theta) \cos(\phi) - F_{\theta 3cc} \cos(\theta) \cos(\phi) + F_{\phi 3cc} \sin(\phi) \right\} \delta(\theta, \phi) d\theta d\phi$$

$$+ \int_0^{2\pi} \int_0^\pi \left\{ -F_{r4cc} \sin(\theta) \cos(\phi) - F_{\theta 4cc} \cos(\theta) \cos(\phi) + F_{\phi 4cc} \sin(\phi) \right\} \delta(\theta, \phi) d\theta d\phi$$
\[ F_{x0ss} = \int_{0}^{2\pi} \int_{0}^{\pi} \{-F_{r0ss} \sin(\theta) \cos(\phi) - F_{\theta0ss} \cos(\theta) \cos(\phi) + F_{\phi0ss} \sin(\phi)\} d\theta d\phi \]
\[ F_{x1ss} = \int_{0}^{2\pi} \int_{0}^{\pi} \{-F_{r1ss} \sin(\theta) \cos(\phi) - F_{\theta1ss} \cos(\theta) \cos(\phi) + F_{\phi1ss} \sin(\phi)\} d\theta d\phi \]
\[ + \int_{0}^{2\pi} \int_{0}^{\pi} \{-F_{r2ss} \sin(\theta) \cos(\phi) - F_{\theta2ss} \cos(\theta) \cos(\phi) + F_{\phi2ss} \sin(\phi)\} \delta(\theta, \phi) d\theta d\phi \]
\[ + \int_{0}^{2\pi} \int_{0}^{\pi} \{-F_{r3ss} \sin(\theta) \cos(\phi) - F_{\theta3ss} \cos(\theta) \cos(\phi) + F_{\phi3ss} \sin(\phi)\} \delta(\theta, \phi) d\theta d\phi \]
\[ + \int_{0}^{2\pi} \int_{0}^{\pi} \{-F_{r4ss} \sin(\theta) \cos(\phi) - F_{\theta4ss} \cos(\theta) \cos(\phi) + F_{\phi4ss} \sin(\phi)\} \delta(\theta, \phi) d\theta d\phi \]
\[ F_{x0sc} = \int_{0}^{2\pi} \int_{0}^{\pi} \{-F_{r0sc} \sin(\theta) \cos(\phi) - F_{\theta0sc} \cos(\theta) \cos(\phi) + F_{\phi0sc} \sin(\phi)\} d\theta d\phi \]
\[ F_{x1sc} = \int_{0}^{2\pi} \int_{0}^{\pi} \{-F_{r1sc} \sin(\theta) \cos(\phi) - F_{\theta1sc} \cos(\theta) \cos(\phi) + F_{\phi1sc} \sin(\phi)\} d\theta d\phi \]
\[ + \int_{0}^{2\pi} \int_{0}^{\pi} \{-F_{r2sc} \sin(\theta) \cos(\phi) - F_{\theta2sc} \cos(\theta) \cos(\phi) + F_{\phi2sc} \sin(\phi)\} \delta(\theta, \phi) d\theta d\phi \]
\[ + \int_{0}^{2\pi} \int_{0}^{\pi} \{-F_{r3sc} \sin(\theta) \cos(\phi) - F_{\theta3sc} \cos(\theta) \cos(\phi) + F_{\phi3sc} \sin(\phi)\} \delta(\theta, \phi) d\theta d\phi \]
\[ + \int_{0}^{2\pi} \int_{0}^{\pi} \{-F_{r4sc} \sin(\theta) \cos(\phi) - F_{\theta4sc} \cos(\theta) \cos(\phi) + F_{\phi4sc} \sin(\phi)\} \delta(\theta, \phi) d\theta d\phi \]

Similarly,

\[ F_y \approx \sum_{0 \leq j, k \leq 2} \int_{0}^{2\pi} \int_{0}^{\pi} \{-F_r \sin(\theta) \sin(\phi) - F_\theta \cos(\theta) \sin(\phi) - F_\phi \cos(\phi)\} d\theta d\phi \]
\[ F_y \approx F_{ycc} \cos^2(kh) + F_{yss} \sin^2(kh) + F_{ysc} \sin(2kh) \]

where,

\[ F_{ycc} = F_{y0cc} + \epsilon F_{y1cc} \]
\[ F_{yss} = F_{y0ss} + \epsilon F_{y1ss} \]
\[ F_{ysc} = F_{y0sc} + \epsilon F_{y1sc} \]
\[ F_{y0cc} = \int_0^{2\pi} \int_0^\pi \left\{ -F_{r0cc} \sin(\theta) \sin(\phi) - F_{\theta0cc} \cos(\theta) \sin(\phi) - F_{\phi0cc} \cos(\phi) \right\} d\theta d\phi \]

\[ F_{y1cc} = \int_0^{2\pi} \int_0^\pi \left\{ -F_{r1cc} \sin(\theta) \sin(\phi) - F_{\theta1cc} \cos(\theta) \sin(\phi) - F_{\phi1cc} \cos(\phi) \right\} d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^\pi \left\{ -F_{r2cc} \sin(\theta) \sin(\phi) - F_{\theta2cc} \cos(\theta) \sin(\phi) - F_{\phi2cc} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^\pi \left\{ -F_{r3cc} \sin(\theta) \sin(\phi) - F_{\theta3cc} \cos(\theta) \sin(\phi) - F_{\phi3cc} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^\pi \left\{ -F_{r4cc} \sin(\theta) \sin(\phi) - F_{\theta4cc} \cos(\theta) \sin(\phi) - F_{\phi4cc} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]

\[ F_{y0ss} = \int_0^{2\pi} \int_0^\pi \left\{ -F_{r0ss} \sin(\theta) \sin(\phi) - F_{\theta0ss} \cos(\theta) \sin(\phi) - F_{\phi0ss} \cos(\phi) \right\} d\theta d\phi \]

\[ F_{y1ss} = \int_0^{2\pi} \int_0^\pi \left\{ -F_{r1ss} \sin(\theta) \sin(\phi) - F_{\theta1ss} \cos(\theta) \sin(\phi) - F_{\phi1ss} \cos(\phi) \right\} d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^\pi \left\{ -F_{r2ss} \sin(\theta) \sin(\phi) - F_{\theta2ss} \cos(\theta) \sin(\phi) - F_{\phi2ss} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^\pi \left\{ -F_{r3ss} \sin(\theta) \sin(\phi) - F_{\theta3ss} \cos(\theta) \sin(\phi) - F_{\phi3ss} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^\pi \left\{ -F_{r4ss} \sin(\theta) \sin(\phi) - F_{\theta4ss} \cos(\theta) \sin(\phi) - F_{\phi4ss} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]

\[ F_{y0sc} = \int_0^{2\pi} \int_0^\pi \left\{ -F_{r0sc} \sin(\theta) \sin(\phi) - F_{\theta0sc} \cos(\theta) \sin(\phi) - F_{\phi0sc} \cos(\phi) \right\} d\theta d\phi \]

\[ F_{y1sc} = \int_0^{2\pi} \int_0^\pi \left\{ -F_{r1sc} \sin(\theta) \sin(\phi) - F_{\theta1sc} \cos(\theta) \sin(\phi) - F_{\phi1sc} \cos(\phi) \right\} d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^\pi \left\{ -F_{r2sc} \sin(\theta) \sin(\phi) - F_{\theta2sc} \cos(\theta) \sin(\phi) - F_{\phi2sc} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^\pi \left\{ -F_{r3sc} \sin(\theta) \sin(\phi) - F_{\theta3sc} \cos(\theta) \sin(\phi) - F_{\phi3sc} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]

\[ + \int_0^{2\pi} \int_0^\pi \left\{ -F_{r4sc} \sin(\theta) \sin(\phi) - F_{\theta4sc} \cos(\theta) \sin(\phi) - F_{\phi4sc} \cos(\phi) \right\} \delta(\theta, \phi) d\theta d\phi \]
\[
F_z \approx \sum_{0 \leq j, k \leq 2} \int_0^{2\pi} \int_0^\pi \{-F_r \cos(\theta) + F_\theta \sin(\theta)\} d\theta d\phi
\]
\[
\approx F_{zc} \cos^2(kh) + F_{zs} \sin^2(kh) + F_{zs} \sin(2kh)
\]

where,
\[
F_{zc} = F_{z0} + \epsilon F_{z1c}
\]
\[
F_{zs} = F_{z0} + \epsilon F_{z1s}
\]
\[
F_{zs} = F_{z0} + \epsilon F_{z1s}
\]
\[
F_{z0} = \int_0^{2\pi} \int_0^\pi \{-F_{r0} \cos(\theta) + F_{\theta0} \sin(\theta)\} d\theta d\phi
\]
\[
F_{z1c} = \int_0^{2\pi} \int_0^\pi \{-F_{r1c} \cos(\theta) + F_{\theta1c} \sin(\theta)\} d\theta d\phi
\]
\[
+ \int_0^{2\pi} \int_0^\pi \{-F_{r2c} \cos(\theta) + F_{\theta2c} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi
\]
\[
+ \int_0^{2\pi} \int_0^\pi \{-F_{r3c} \cos(\theta) + F_{\theta3c} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi
\]
\[
+ \int_0^{2\pi} \int_0^\pi \{-F_{r4c} \cos(\theta) + F_{\theta4c} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi
\]
\[
F_{z0s} = \int_0^{2\pi} \int_0^\pi \{-F_{r0s} \cos(\theta) + F_{\theta0s} \sin(\theta)\} d\theta d\phi
\]
\[
F_{z1s} = \int_0^{2\pi} \int_0^\pi \{-F_{r1s} \cos(\theta) + F_{\theta1s} \sin(\theta)\} d\theta d\phi
\]
\[
+ \int_0^{2\pi} \int_0^\pi \{-F_{r2s} \cos(\theta) + F_{\theta2s} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi
\]
\[
+ \int_0^{2\pi} \int_0^\pi \{-F_{r3s} \cos(\theta) + F_{\theta3s} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi
\]
\[
+ \int_0^{2\pi} \int_0^\pi \{-F_{r4s} \cos(\theta) + F_{\theta4s} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi
\]
\[
F_{z0c} = \int_0^{2\pi} \int_0^\pi \{-F_{r0c} \cos(\theta) + F_{\theta0c} \sin(\theta)\} d\theta d\phi
\]
\[
F_{z1c} = \int_0^{2\pi} \int_0^\pi \{-F_{r1c} \cos(\theta) + F_{\theta1c} \sin(\theta)\} d\theta d\phi
\]
\[
+ \int_0^{2\pi} \int_0^\pi \{-F_{r2c} \cos(\theta) + F_{\theta2c} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi
\]
\[
+ \int_0^{2\pi} \int_0^\pi \{-F_{r3c} \cos(\theta) + F_{\theta3c} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi
\]
\[
+ \int_0^{2\pi} \int_0^\pi \{-F_{r4c} \cos(\theta) + F_{\theta4c} \sin(\theta)\} \delta(\theta, \phi) d\theta d\phi
\]
APPENDIX D

ACOUSTIC FORCE FOR PARTICULAR PERTURBATION

We noticed that, from Equations (2.50) and (2.52) $A_{n,m}^{(0)}$ and $B_{n,m}^{(0)}$ exists only for $m = 0$. For particular value of $\delta(\theta, \phi)$ say,

$$\delta(\theta, \phi) = Y_{0}^{0}(\theta, \phi) + \frac{1}{2} (Y_{2}^{2}(\theta, \phi) + Y_{2}^{-2}(\theta, \phi)) \quad (D.1)$$

Equation (2.54) yields $B_{nm}^{(1)}$ exits only for $n = 2$ and $m = 2$.

Therefore, only the following terms exists from Equation (C.7).

$$\alpha_{1,n}^{0} = \Re\left\{ ak^{*} B_{n0}^{(0)} j_{n}^{'}(ak^{*}) e^{i\omega t} \right\}$$
$$\beta_{1,n}^{0} = \Im\left\{ ak^{*} B_{n0}^{(0)} j_{n}^{'}(ak^{*}) e^{i\omega t} \right\}$$
$$\alpha_{3,n}^{0} = 0 = \beta_{3,n}^{0}$$
$$\alpha_{5,n}^{0} = \alpha_{7,n}^{0} = \Re\left\{ B_{n0}^{(0)} j_{n}(ak^{*}) e^{i\omega t} \right\}$$
$$\beta_{5,n}^{0} = \beta_{7,n}^{0} = \Im\left\{ B_{n0}^{(0)} j_{n}(ak^{*}) e^{i\omega t} \right\}$$
$$\alpha_{2,n}^{0} = \Re\left\{ ak^{*} \left( B_{n0}^{(1)} j_{n}^{'}(ak^{*}) + ak^{*} \delta(\theta, \phi) B_{n0}^{(0)} j_{n}^{''}(ak^{*}) \right) e^{i\omega t} \right\}$$
$$\beta_{2,n}^{0} = \Im\left\{ ak^{*} \left( B_{n0}^{(1)} j_{n}^{'}(ak^{*}) + ak^{*} \delta(\theta, \phi) B_{n0}^{(0)} j_{n}^{''}(ak^{*}) \right) e^{i\omega t} \right\}$$
$$\alpha_{4,n}^{0} = 0 = \beta_{4,n}^{0}$$
$$\alpha_{6,n}^{0} = \alpha_{8,n}^{0} = \Re\left\{ \left( B_{n0}^{(1)} j_{n}(ak^{*}) + ak^{*} \delta(\theta, \phi) B_{n0}^{(0)} j_{n}^{'}(ak^{*}) \right) e^{i\omega t} \right\}$$
$$\beta_{6,n}^{0} = \beta_{8,n}^{0} = \Im\left\{ \left( B_{n0}^{(1)} j_{n}(ak^{*}) + ak^{*} \delta(\theta, \phi) B_{n0}^{(0)} j_{n}^{'}(ak^{*}) \right) e^{i\omega t} \right\}$$

Writing $\alpha_{i,n}^{0}$ and $\beta_{i,n}^{0}$ $\forall i \in \{2, 4, 6\}$ as below in order to simplify calculations.

$$\alpha_{i,n}^{0} = \alpha_{i1,n}^{0} + \alpha_{i2,n}^{0} \quad \beta_{i,n}^{0} = \beta_{i1,n}^{0} + \beta_{i2,n}^{0}$$

127
\[ \alpha_{21,n}^0 = \Re \left\{ a k^* \bar{B}_n^{(1)} j_n' (a k^*) e^{i \omega t} \right\} \]
\[ \alpha_{22,n}^0 = \Re \left\{ (a k^*)^2 \delta(\theta, \phi) \bar{B}_n^{(0)} j_n'' (a k^*) e^{i \omega t} \right\} \]
\[ \beta_{21,n}^0 = \Im \left\{ a k^* \bar{B}_n^{(1)} j_n' (a k^*) e^{i \omega t} \right\} \]
\[ \beta_{22,n}^0 = \Im \left\{ (a k^*)^2 \delta(\theta, \phi) \bar{B}_n^{(0)} j_n'' (a k^*) e^{i \omega t} \right\} \]
\[ \alpha_{61,n}^0 = \alpha_{62,n}^0 = \Re \left\{ \bar{B}_n^{(1)} j_n (a k^*) e^{i \omega t} \right\} \]
\[ \beta_{61,n}^0 = \beta_{62,n}^0 = \Im \left\{ \bar{B}_n^{(1)} j_n (a k^*) e^{i \omega t} \right\} \]
\[ \alpha_{82,n}^0 = \alpha_{81,n}^0 = \Re \left\{ a k^* \delta(\theta, \phi) \bar{B}_n^{(0)} j_n' (a k^*) e^{i \omega t} \right\} \]
\[ \beta_{81,n}^0 = \beta_{82,n}^0 = \Im \left\{ a k^* \delta(\theta, \phi) \bar{B}_n^{(0)} j_n' (a k^*) e^{i \omega t} \right\} \]
\[ \alpha_{2,2}^{\pm 2} = \Re \left\{ a k^* \bar{B}_{2 \pm 2}^{(1)} j_2' (a k^*) e^{\pm i \phi e^{i \omega t}} \right\} \]
\[ \beta_{2,2}^{\pm 2} = \Im \left\{ a k^* \bar{B}_{2 \pm 2}^{(1)} j_2' (a k^*) e^{\pm i \phi e^{i \omega t}} \right\} \]
\[ \alpha_{4,2}^{\pm 2} = \Re \left\{ 2 i \bar{B}_{2 \pm 2}^{(1)} j_2 (a k^*) e^{\pm i \phi e^{i \omega t}} \right\} \]
\[ \beta_{4,2}^{\pm 2} = \Im \left\{ 2 i \bar{B}_{2 \pm 2}^{(1)} j_2 (a k^*) e^{\pm i \phi e^{i \omega t}} \right\} \]
\[ \alpha_{8,2}^{\pm 2} = \alpha_{8,2}^0 = \Re \left\{ \bar{B}_{2 \pm 2}^{(1)} j_2 (a k^*) e^{\pm i \phi e^{i \omega t}} \right\} \]
\[ \beta_{8,2}^{\pm 2} = \beta_{8,2}^0 = \Im \left\{ \bar{B}_{2 \pm 2}^{(1)} j_2 (a k^*) e^{\pm i \phi e^{i \omega t}} \right\} \]

Therefore reduces forces to:

\[ F_x \approx F_{xcc} \cos^2(kh) + F_{xss} \sin^2(kh) + F_{xsc} \sin(2kh) \]
\[ F_y \approx F_{ycc} \cos^2(kh) + F_{yss} \sin^2(kh) + F_{ysc} \sin(2kh) \]
\[ F_z \approx F_{zcc} \cos^2(kh) + F_{zss} \sin^2(kh) + F_{zsc} \sin(2kh) \]

\[ F_{xcc} = F_{x0cc} + \epsilon F_{x1cc}, \quad F_{xss} = F_{x0ss} + \epsilon F_{x1ss}, \quad F_{xsc} = F_{x0sc} + \epsilon F_{x1sc} \]
\[ F_{ycc} = F_{y0cc} + \epsilon F_{y1cc}, \quad F_{yss} = F_{y0ss} + \epsilon F_{y1ss}, \quad F_{ysc} = F_{y0sc} + \epsilon F_{y1sc} \]
\[ F_{zcc} = F_{z0cc} + \epsilon F_{z1cc}, \quad F_{zss} = F_{z0ss} + \epsilon F_{z1ss}, \quad F_{zsc} = F_{z0sc} + \epsilon F_{z1sc} \]
where,

\[ F_{x_{0c}} = 0 \]

\[ F_{x_{1c}} = -\frac{9}{16} \rho_0 \pi \int_0^{2\pi} \langle \alpha_{1,0}^0 \alpha_{2,2}^{-2} \rangle \cos(\phi) d\phi + \frac{9}{64} \rho_0 \pi \int_0^{2\pi} \langle \alpha_{1,2}^0 \alpha_{2,2}^{-2} \rangle \cos(\phi) d\phi \]

\[ - \frac{27}{2} \rho_0 \pi \int_0^{2\pi} \langle \alpha_{1,0}^0 \alpha_{2,2}^{-2} \rangle \cos(\phi) d\phi + \frac{27}{8} \rho_0 \pi \int_0^{2\pi} \langle \alpha_{1,2}^0 \alpha_{2,2}^{-2} \rangle \cos(\phi) d\phi \]

\[ - 12 \rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{1,0}^0 \alpha_{22,0}^{-2} \rangle \sin^2(\theta) \cos(\phi) d\theta d\phi \]

\[ - 6 \rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{1,0}^0 \alpha_{22,2}^{-2} \rangle \left( -1 + 3 \cos^2(\theta) \right) \sin^2(\theta) \cos(\phi) d\theta d\phi \]

\[ - 6 \rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{1,2}^0 \alpha_{22,0}^{-2} \rangle \left( -1 + 3 \cos^2(\theta) \right) \sin^2(\theta) \cos(\phi) d\theta d\phi \]

\[ - 3 \rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{1,2}^0 \alpha_{22,2}^{-2} \rangle \left( 1 - 3 \cos^2(\theta) \right)^2 \sin^2(\theta) \cos(\phi) d\theta d\phi \]

\[ + \frac{3}{16} \lambda^2 \rho_0 \pi \int_0^{2\pi} \langle \alpha_{5,2}^0 \alpha_{6,2}^{-2} \rangle \cos(\phi) d\phi + \frac{9}{2} \lambda^2 \rho_0 \pi \int_0^{2\pi} \langle \alpha_{5,2}^0 \alpha_{6,2}^{-2} \rangle \cos(\phi) d\phi \]

\[ - 36 \lambda^2 \rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{5,2}^0 \alpha_{62,2}^{-2} \rangle \cos^2(\theta) \sin^4(\theta) \cos(\phi) d\theta d\phi \]

\[ + \frac{9 \lambda^2 \omega^2 \rho_0}{16 c_0^2} \int_0^{2\pi} \langle \alpha_{7,0}^0 \alpha_{8,2}^{-2} \rangle \cos(\phi) d\phi - \frac{9 \lambda^2 \omega^2 \rho_0}{64 c_0^2} \int_0^{2\pi} \langle \alpha_{7,2}^0 \alpha_{8,2}^{-2} \rangle \cos(\phi) d\phi \]

\[ + \frac{9 \lambda^2 \omega^2 \rho_0}{2 c_0^2} \int_0^{2\pi} \langle \alpha_{7,0}^0 \alpha_{8,2}^{-2} \rangle \cos(\phi) d\phi - \frac{9 \lambda^2 \omega^2 \rho_0}{8 c_0^2} \int_0^{2\pi} \langle \alpha_{7,2}^0 \alpha_{8,2}^{-2} \rangle \cos(\phi) d\phi \]

\[ + \frac{9 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{7,0}^0 \alpha_{82,0}^{-2} \rangle \sin^2(\theta) \cos(\phi) d\theta d\phi \]

\[ + \frac{9 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{7,0}^0 \alpha_{82,2}^{-2} \rangle \left( -1 + 3 \cos^2(\theta) \right) \sin^2(\theta) \cos(\phi) d\theta d\phi \]

\[ + \frac{9 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{7,2}^0 \alpha_{82,0}^{-2} \rangle \left( -1 + 3 \cos^2(\theta) \right) \sin^2(\theta) \cos(\phi) d\theta d\phi \]

\[ + \frac{9 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{7,2}^0 \alpha_{82,2}^{-2} \rangle \left( 1 - 3 \cos^2(\theta) \right)^2 \sin^2(\theta) \cos(\phi) d\theta d\phi \]

\[ - 4 \lambda \rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{1,0}^0 \alpha_{62,0}^{-2} \rangle \cos(\theta) \sin(\theta) \cos(\phi) d\theta d\phi \]

\[ - 2 \lambda \rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{1,0}^0 \alpha_{62,2}^{-2} \rangle \left( -1 + 3 \cos^2(\theta) \right) \sin(\theta) \cos(\theta) \cos(\phi) d\theta d\phi \]

\[ - 2 \lambda \rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{1,2}^0 \alpha_{62,0}^{-2} \rangle \left( -1 + 3 \cos^2(\theta) \right) \sin(\theta) \cos(\theta) \cos(\phi) d\theta d\phi \]
\[-\lambda \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \alpha_{1,0}^{\alpha_{62,2}} \right)^2 (1 - 3 \cos^2(\theta))^2 \sin(\theta) \cos(\theta) d\theta d\phi \\
+ \frac{1}{4} \lambda \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \alpha_{1,0}^{\alpha_{4,2}} \right)^2 \sin(\phi) d\phi \cdot \frac{1}{32} \lambda \rho_0 \int_0^{2\pi} \left( \alpha_{1,2}^{\alpha_{4,2}} \right)^2 \sin(\phi) d\phi \\
+ 6 \lambda \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \alpha_{1,0}^{\alpha_{4,2}} \right)^2 \sin(\phi) d\phi \cdot \frac{3}{4} \lambda \rho_0 \int_0^{2\pi} \left( \alpha_{1,2}^{\alpha_{4,2}} \right)^2 \sin(\phi) d\phi \]

\[F_{x0ss} = 0\]

\[F_{x1ss} = -12 \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \beta_{1,1}^{\beta_{62,1}} \right)^2 \sin^2(\theta) \cos^2(\theta) \cos(\phi) d\theta d\phi \\
- 4 \lambda^2 \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \beta_{5,1}^{\beta_{62,1}} \right)^2 \sin^4(\theta) \cos(\phi) d\theta d\phi \\
+ \frac{4a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^{\pi} \left( \beta_{7,1}^{\beta_{62,1}} \right)^2 \cos^2(\theta) \sin^2(\theta) \cos(\phi) d\theta d\phi \\
- 4 \lambda \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \beta_{1,1}^{\beta_{62,1}} \right)^2 \cos^3(\theta) \sin(\theta) \cos(\phi) d\theta d\phi \]

\[F_{x0sc} = 0\]

\[F_{x1sc} = 6 \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \alpha_{1,0}^{\alpha_{22,1}} \right)^2 \cos(\theta) \sin^2(\theta) \cos(\phi) d\theta d\phi \\
- 3 \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \alpha_{1,2}^{\alpha_{22,1}} \right)^2 (-1 + 3 \cos^2(\theta)) \cos(\theta) \sin^2(\theta) \cos(\phi) d\theta d\phi \\
- 6 \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \beta_{1,1}^{\beta_{22,0}} \right)^2 \cos(\theta) \sin^2(\theta) \cos(\phi) d\theta d\phi \\
- 3 \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \beta_{1,1}^{\beta_{22,2}} \right)^2 (-1 + 3 \cos^2(\theta)) \cos(\theta) \sin^2(\theta) \cos(\phi) d\theta d\phi \\
- 6 \rho_0 \lambda^2 \int_0^{2\pi} \int_0^{\pi} \left( \alpha_{5,1}^{\alpha_{62,2}} \right)^2 \cos(\theta) \sin^4(\theta) \cos(\phi) d\theta d\phi \\
- 6 \lambda^2 \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \beta_{5,1}^{\beta_{62,2}} \right)^2 \cos(\theta) \sin^4(\theta) \cos(\phi) d\theta d\phi \\
+ \frac{2a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^{\pi} \left( \alpha_{7,0}^{\alpha_{82,1}} \right)^2 \cos(\theta) \sin^2(\theta) \cos(\phi) d\theta d\phi \\
+ \frac{a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^{\pi} \left( \alpha_{7,2}^{\alpha_{82,1}} \right)^2 (-1 + 3 \cos^2(\theta)) \cos(\theta) \sin^2(\theta) \cos(\phi) d\theta d\phi \]
\[
\begin{align*}
+ \frac{2a^2 \lambda^2 \rho_0}{c_0^2} \int_{0}^{2\pi} \int_{0}^{\pi} \langle \beta_{7,1}^{0} \alpha_{82,0}^{0} \rangle \cos(\theta) \sin^2(\theta) \cos(\phi) d\theta d\phi \\
+ \frac{a^2 \lambda^2 \rho_0}{c_0^2} \int_{0}^{2\pi} \int_{0}^{\pi} \langle \beta_{7,1}^{0} \alpha_{82,2}^{0} \rangle (-1 + 3 \cos^2(\theta)) \cos(\theta) \sin(\theta)^2 \cos(\phi) d\theta d\phi \\
- \frac{1}{15} \lambda \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \beta_{1,1}^{0} \alpha_{62,0}^{0} \rangle \cos(\phi) d\phi - \frac{8}{5} \lambda \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \beta_{1,1}^{0} \alpha_{62,2}^{0} \rangle \cos(\phi) d\phi \\
- 2\lambda \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{1,1}^{0} \beta_{62,0}^{0} \rangle \cos^2(\theta) \sin(\theta) \cos(\phi) d\theta d\phi \\
- \lambda \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{1,1}^{0} \beta_{62,2}^{0} \rangle (-1 + 3 \cos^2(\theta)) \sin(\theta) \cos^2(\theta) \cos(\phi) d\theta d\phi \\
+ 2 \lambda \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \beta_{1,1}^{0} \alpha_{62,0}^{0} \rangle \cos^2(\theta) \sin(\theta) \cos(\phi) d\theta d\phi \\
- \lambda \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \beta_{1,1}^{0} \alpha_{62,2}^{0} \rangle (-1 + 3 \cos^2(\theta)) \cos^2(\theta) \sin(\theta) \cos(\phi) d\theta d\phi 
\end{align*}
\]

\[F_{y_{0cc}} = 0\]

\[F_{y_{1cc}} = - \frac{9}{16} \pi \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{1,0}^{0} \alpha_{2,2}^{0} \rangle \sin(\Phi) d\Phi + \frac{9}{64} \pi \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{1,2}^{0} \alpha_{2,2}^{-2} \rangle \sin(\Phi) d\Phi \\
- \frac{27}{2} \pi \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{1,0}^{0} \alpha_{2,2}^{2} \rangle \sin(\Phi) d\Phi + \frac{27}{8} \pi \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{1,2}^{0} \alpha_{2,2}^{2} \rangle \sin(\Phi) d\Phi \\
- 12 \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{1,0}^{0} \alpha_{22,0}^{0} \rangle \sin^2(\theta) \sin(\phi) d\Phi d\phi \\
- 6 \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{1,0}^{0} \alpha_{22,2}^{0} \rangle (-1 + 3 \cos^2(\theta)) \sin^2(\theta) \sin(\phi) d\Phi d\phi \\
- 6 \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{1,2}^{0} \alpha_{22,0}^{0} \rangle (-1 + 3 \cos^2(\theta)) \sin^2(\theta) \sin(\phi) d\Phi d\phi \\
- 3 \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{1,2}^{0} \alpha_{22,2}^{0} \rangle (1 - 3 \cos^2(\theta))^2 \sin^2(\theta) \sin(\phi) d\Phi d\phi \\
+ \frac{3}{16} \pi \lambda^2 \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{5,2}^{0} \alpha_{62,2}^{0} \rangle \sin(\Phi) d\Phi + \frac{9}{2} \pi \lambda^2 \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{5,2}^{0} \alpha_{62,2}^{2} \rangle \sin(\Phi) d\Phi \\
- 36 \lambda^2 \rho_0 \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{5,2}^{0} \alpha_{62,2}^{0} \rangle \cos^2(\theta) \sin^4(\theta) \sin(\phi) d\Phi d\phi \\
+ \frac{3a^2 \pi \lambda^2 \rho_0}{16c_0^2} \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{7,0}^{0} \alpha_{82,2}^{0} \rangle \sin(\Phi) d\Phi - \frac{3a^2 \pi \lambda^2 \rho_0}{64c_0^2} \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{7,2}^{0} \alpha_{82,2}^{0} \rangle \sin(\Phi) d\Phi \\
+ \frac{9a^2 \pi \lambda^2 \rho_0}{2c_0^2} \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{7,0}^{0} \alpha_{82,2}^{0} \rangle \sin(\Phi) d\Phi - \frac{9a^2 \pi \lambda^2 \rho_0}{8c_0^2} \int_{0}^{2\pi} \int_{0}^{\pi} \langle \alpha_{7,2}^{0} \alpha_{82,2}^{0} \rangle \sin(\Phi) d\Phi
\]
\[ + \frac{4a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^\pi \langle \alpha_{7,0}^0 \alpha_{82,0}^0 \rangle \sin^2 \theta \sin \phi d\theta d\phi \\
+ \frac{2a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^\pi \langle \alpha_{7,0}^0 \alpha_{82,2}^0 \rangle \sin^2 \theta \sin \phi d\theta d\phi \\
+ \frac{2a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^\pi \langle \alpha_{7,2}^0 \alpha_{82,0}^0 \rangle \sin^2 \theta \sin \phi d\theta d\phi \\
+ \frac{a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^\pi \langle \alpha_{7,2}^0 \alpha_{82,2}^0 \rangle (1 - 3 \cos^2 \theta)^2 \sin^2 \theta \sin \phi d\theta d\phi \\
- 4\lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{1,0}^0 \alpha_{62,0}^0 \rangle \cos \theta \sin \theta \sin \phi d\theta d\phi \\
- 2\lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{1,0}^0 \alpha_{62,2}^0 \rangle (-1 + 3 \cos^2 \theta) \cos \theta \sin \theta \sin \phi d\theta d\phi \\
- 2\lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{1,2}^0 \alpha_{62,0}^0 \rangle (-1 + 3 \cos^2 \theta) \cos \theta \sin \theta \sin \phi d\theta d\phi \\
- \lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{1,2}^0 \alpha_{62,2}^0 \rangle (1 - 3 \cos^2 \theta)^2 \cos \theta \sin \theta \sin \phi d\theta d\phi \\
- \frac{1}{4} \pi \lambda \rho_0 \int_0^{2\pi} \langle \alpha_{1,0}^0 \alpha_{4,2}^0 \rangle \cos \phi d\phi + \frac{1}{32} a \pi \lambda \rho_0 \int_0^{2\pi} \langle \alpha_{1,2}^0 \alpha_{4,2}^0 \rangle \cos \phi d\phi \\
- 6\pi \lambda \rho_0 \int_0^{2\pi} \langle \alpha_{1,0}^0 \alpha_{4,2}^0 \rangle \cos \phi d\phi + \frac{3}{4} a \pi \lambda \rho_0 \int_0^{2\pi} \langle \alpha_{1,2}^0 \alpha_{4,2}^0 \rangle \cos \phi d\phi \]

\[ F_{y0ss} = 0 \]

\[ F_{y1ss} = -12\rho_0 \int_0^{2\pi} \int_0^\pi \langle \beta_{1,1}^0 \beta_{22,1}^0 \rangle \cos^2 (\theta) \sin^2 (\theta) \sin (\phi) d\theta d\phi \\
- 4\lambda^2 \rho_0 \int_0^{2\pi} \int_0^\pi \langle \beta_{5,1}^0 \beta_{62,1}^0 \rangle \sin^4 (\theta) \sin (\phi) d\theta d\phi \\
+ \frac{4a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^\pi \langle \beta_{7,1}^0 \beta_{82,1}^0 \rangle \cos^2 \theta \sin^2 \theta \sin (\phi) d\theta d\phi \\
- 4\lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \beta_{1,1}^0 \beta_{62,1}^0 \rangle \cos^3 \theta \sin \theta \sin (\phi) d\theta d\phi \]

\[ F_{y0sc} = 0 \]

\[ F_{y1sc} = -6\rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{1,0}^0 \beta_{22,1}^0 \rangle \cos \theta \sin^2 \theta \sin (\phi) d\theta d\phi \\
- 3\rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{1,2}^0 \beta_{22,1}^0 \rangle (-1 + 3 \cos^2 \theta) \cos \theta \sin^2 \theta \sin (\phi) d\theta d\phi \]
- 6\rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \beta_{1,1}^0 \alpha_{22,0}^0 \rangle \cos \theta \sin^2 \theta \sin \phi d\theta d\phi
- 3\rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \beta_{1,1}^0 \alpha_{22,2}^0 \rangle \left( -1 + 3 \cos^2 \theta \right) \cos \theta \sin^2 \theta \sin \phi d\theta d\phi
- 6\rho_0 \lambda_2 \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{5,2}^0 \beta_{82,1}^0 \rangle \cos \theta \sin^4 \theta \sin \phi d\theta d\phi
- 6\lambda_2^2 \rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \beta_{5,1}^0 \alpha_{62,2}^0 \rangle \cos \theta \sin^4 \theta \sin \phi d\theta d\phi
+ \frac{2a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{7,0}^0 \beta_{82,1}^0 \rangle \cos \theta \sin^2 \theta \sin \phi d\theta d\phi
+ \frac{a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^{\pi} \langle \beta_{82,1}^0 \alpha_{7,2}^0 \rangle \left( -1 + 3 \cos^2 \theta \right) \cos \theta \sin^2 \theta \sin \phi d\theta d\phi
+ \frac{2a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{82,0}^0 \beta_{7,1}^0 \rangle \cos \theta \sin^2 \theta \sin \phi d\theta d\phi
+ \frac{a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^{\pi} \langle \beta_{7,1}^0 \alpha_{82,2}^0 \rangle \left( -1 + 3 \cos^2 \theta \right) \cos \theta \sin^2 \theta \sin \phi d\theta d\phi
- \frac{1}{15} \lambda_0 \rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \beta_{1,1}^0 \alpha_{6,2}^{-2} \rangle \sin \phi d\phi - \frac{8}{5} \lambda_0 \rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \beta_{1,1}^0 \alpha_{6,2}^2 \rangle \sin \phi d\phi
- 2\lambda_0 \rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{1,0}^0 \beta_{62,1}^0 \rangle \cos^2 \theta \sin \theta \sin \phi d\theta d\phi
- \lambda_0 \rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{1,2}^0 \beta_{62,1}^0 \rangle \left( -1 + 3 \cos^2 \theta \right) \cos^2 \theta \sin \theta \sin \phi d\theta d\phi
- 2\lambda_0 \rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \beta_{1,1}^0 \alpha_{62,0}^0 \rangle \cos^2 \theta \sin \theta \sin \phi d\theta d\phi
- \lambda_0 \rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \beta_{1,1}^0 \alpha_{62,2}^0 \rangle \left( -1 + 3 \cos^2 \theta \right) \cos^2 \theta \sin \theta \sin \phi d\theta d\phi
\nonumber
F_{z0cc} = 0
\nonumber
F_{z1cc} = -6\rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{1,0}^0 \alpha_{22,0}^0 \rangle \sin 2\theta d\theta d\phi
- 6\rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{1,0}^0 \alpha_{22,2}^0 \rangle \left( -1 + 3 \cos^2 \theta \right) \cos \theta \sin \theta d\theta d\phi
- 6\rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{1,2}^0 \alpha_{22,2}^0 \rangle \left( -1 + 3 \cos^2 \theta \right) \cos \theta \sin \theta d\theta d\phi
- 3\rho_0 \int_0^{2\pi} \int_0^{\pi} \langle \alpha_{1,3}^0 \alpha_{22,2}^0 \rangle \left( 1 - 3 \cos^2 \theta \right)^2 \cos \theta \sin \theta d\theta d\phi
\[-36 \rho_0 \lambda^2 \int_0^{2\pi} \int_0^\pi \langle \alpha_{5,2}^0 \alpha_{6,2,2}^0 \rangle \cos^3 \theta \sin^3 \theta d\theta d\phi \]
\[+ 4a^2 \lambda^2 \omega^2 \rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{7,0}^0 \alpha_{8,2,0}^0 \rangle \cos \theta \sin \theta d\theta d\phi \]
\[+ 2a^2 \lambda^2 \omega^2 \rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{7,2}^0 \alpha_{8,2,0}^0 \rangle (-1 + 3 \cos^2 \theta) \cos \theta \sin \theta d\theta d\phi \]
\[+ \frac{a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^\pi \langle \alpha_{7,2}^0 \alpha_{8,2,0}^0 \rangle (1 - 3 \cos^2 \theta)^2 \cos \theta \sin \theta d\theta d\phi \]
\[+ \frac{3}{16} \pi \lambda \rho_0 \int_0^{2\pi} \langle \alpha_{1,0}^0 \alpha_{6,2}^{0-2} \rangle d\phi - \frac{3}{64} \pi \lambda \rho_0 \int_0^{2\pi} \langle \alpha_{1,2}^0 \alpha_{6,2}^{-2} \rangle d\phi \]
\[+ \frac{9}{2} \pi \lambda \rho_0 \int_0^{2\pi} \langle \alpha_{1,0}^0 \alpha_{6,2}^2 \rangle d\phi - \frac{9}{8} \pi \lambda \rho_0 \int_0^{2\pi} \langle \alpha_{1,2}^0 \alpha_{6,2}^2 \rangle d\phi \]
\[+ 4\lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{1,0}^0 \alpha_{6,2,0}^0 \rangle \sin^2 \theta d\theta d\phi \]
\[+ 2\lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{1,0}^0 \alpha_{6,2,2}^0 \rangle (-1 + 3 \cos^2 \theta) \sin^2 \theta d\theta d\phi \]
\[+ 2\lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{1,2}^0 \alpha_{6,2,0}^0 \rangle (-1 + 3 \cos^2 \theta) \sin^2 \theta d\theta d\phi \]
\[+ \lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{1,2}^0 \alpha_{6,2,2}^0 \rangle (1 - 3 \cos^2 \theta)^2 \sin^2 \theta d\theta d\phi \]
\[+ 4\pi^2 \lambda \rho_0 \langle \alpha_{1,0}^0 \alpha_{6,1,0}^0 \rangle - \frac{1}{2} \pi^2 \lambda \rho_0 \langle \alpha_{1,2}^0 \alpha_{6,1,0}^0 \rangle - \frac{1}{2} \pi^2 \lambda \rho_0 \langle \alpha_{1,0}^0 \alpha_{6,1,2}^0 \rangle \]
\[+ \frac{5}{8} \pi^2 \lambda \rho_0 \langle \alpha_{1,2}^0 \alpha_{6,1,2}^0 \rangle \]

\(F_{20ss} = 0\)

\(F_{11ss} = -12\rho_0 \int_0^{2\pi} \int_0^\pi \langle \beta_{1,1}^0 \beta_{22,1}^0 \rangle \cos^3 \theta \cos \phi d\theta d\phi \]
\[-4\lambda^2 \rho_0 \int_0^{2\pi} \int_0^\pi \langle \beta_{5,1}^0 \beta_{62,1}^0 \rangle \cos \theta \sin^3 \theta d\theta d\phi \]
\[+ \frac{4a^2 \lambda^2 \omega^2}{c_0^2} \rho_0 \int_0^{2\pi} \int_0^\pi \langle \beta_{7,1}^0 \beta_{82,1}^0 \rangle \cos^3 \theta \sin \theta d\theta d\phi \]
\[+ 4\lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \beta_{1,1}^0 \beta_{62,1}^0 \rangle \cos^2 \theta \sin^2 \theta d\theta d\phi \]
\[+ \pi^2 \lambda \rho_0 \langle \beta_{1,1}^0 \beta_{61,1}^0 \rangle \]
\[ F_{0sc} = -8\pi \rho_0 \left( \alpha_{1,0}^0 \beta_{1,1}^0 \right) - \frac{16}{5} \pi \rho_0 \left( \alpha_{1,1}^0 \beta_{1,1}^0 \right) - \frac{16}{5} \pi \lambda^2 \rho_0 \left( \alpha_{5,2}^0 \beta_{5,1}^0 \right) + \frac{8a^2 \pi \lambda^2 \omega^2 \rho_0}{3c_0^2} \left( \beta_{2,1}^0 \alpha_{2,1}^0 \right) + \frac{16a^2 \pi \lambda^2 \omega^2 \rho_0}{15c_0^2} \left( \beta_{2,1}^0 \alpha_{2,1}^0 \right) - \frac{16}{3} \pi \lambda \rho_0 \left( \alpha_{1,0}^0 \beta_{1,1}^0 \right) + \frac{16}{15} \pi \lambda \rho_0 \left( \alpha_{1,0}^0 \beta_{1,1}^0 \right) \]

\[ F_{1sc} = -\frac{1}{5} \rho_0 \int_0^{2\pi} \int_0^\pi \left( \beta_{1,1}^0 \alpha_{2,2}^0 \right) d\phi - \frac{24}{5} \rho_0 \int_0^{2\pi} \int_0^\pi \left( \beta_{1,1}^0 \alpha_{2,2}^0 \right) d\phi \]

\[-6\rho_0 \int_0^{2\pi} \int_0^\pi \left( \alpha_{1,0}^0 \beta_{2,1}^0 \right) \cos^2 \theta \sin \theta d\theta d\phi \]

\[-3\rho_0 \int_0^{2\pi} \int_0^\pi \left( \alpha_{1,0}^0 \beta_{2,1}^0 \right) \left( -1 + 3 \cos^2 \theta \right) \cos^2 \theta \sin \theta d\theta d\phi \]

\[-6\rho_0 \int_0^{2\pi} \int_0^\pi \left( \alpha_{1,0}^0 \beta_{2,1}^0 \right) \cos^2 \theta \sin \theta d\theta d\phi \]

\[-3\rho_0 \int_0^{2\pi} \int_0^\pi \left( \alpha_{1,0}^0 \beta_{2,1}^0 \right) \left( -1 + 3 \cos^2 \theta \right) \cos^2 \theta \sin \theta d\theta d\phi \]

\[-8\pi \rho_0 \left( \alpha_{21,0}^0 \beta_{1,1}^0 \right) - \frac{16}{5} \pi \rho_0 \left( \alpha_{21,2}^0 \beta_{1,1}^0 \right) - 8\pi \rho_0 \left( \alpha_{1,0}^0 \beta_{21,1}^0 \right) \]

\[-\frac{16}{5} \pi \rho_0 \left( \alpha_{1,2}^0 \beta_{21,1}^0 \right) + \frac{2}{15} \lambda \rho_0 \int_0^{2\pi} \left( \beta_{5,1}^0 \alpha_{6,2}^0 \right) d\phi + \frac{16}{5} \lambda \rho_0 \int_0^{2\pi} \left( \beta_{5,1}^0 \alpha_{6,2}^0 \right) d\phi \]

\[-6\lambda \rho_0 \int_0^{2\pi} \int_0^\pi \left( \alpha_{5,2}^0 \beta_{6,2}^0 \right) \cos^2 \theta \sin^2 \theta d\theta d\phi \]

\[-6\lambda \rho_0 \int_0^{2\pi} \int_0^\pi \left( \beta_{5,1}^0 \alpha_{6,2}^0 \right) \cos^2 \theta \sin^2 \theta d\theta d\phi \]

\[-\frac{16}{5} \pi \lambda \rho_0 \left( \alpha_{61,2}^0 \beta_{1,1}^0 \right) - \frac{16}{5} \pi \lambda \rho_0 \left( \alpha_{5,2}^0 \beta_{61,1}^0 \right) \]

\[+ \frac{a^2 \lambda^2 \omega^2 \rho_0}{15c_0^2} \int_0^{2\pi} \left( \beta_{7,1}^0 \alpha_{8,2}^0 \right) d\phi + \frac{8a^2 \lambda^2 \omega^2 \rho_0}{5c_0^2} \int_0^{2\pi} \left( \beta_{7,1}^0 \alpha_{8,2}^0 \right) d\phi \]

\[+ \frac{2a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^\pi \left( \alpha_{7,0}^0 \beta_{8,2}^0 \right) \cos^2 \theta \sin \theta d\theta d\phi \]

\[+ \frac{a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^\pi \left( \alpha_{7,2}^0 \beta_{8,2}^0 \right) \left( -1 + 3 \cos^2 \theta \right) \cos^2 \theta \sin \theta d\theta d\phi \]

\[+ \frac{2a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^\pi \left( \beta_{7,1}^0 \alpha_{8,2}^0 \right) \cos^2 \theta \sin \theta d\theta d\phi \]

\[+ \frac{a^2 \lambda^2 \omega^2 \rho_0}{c_0^2} \int_0^{2\pi} \int_0^\pi \left( \beta_{7,1}^0 \alpha_{8,2}^0 \right) \left( -1 + 3 \cos^2 \theta \right) \cos^2 \theta \sin \theta d\theta d\phi \]

\[+ \frac{8a^2 \lambda^2 \omega^2 \rho_0}{3c_0^2} \left( \beta_{7,1}^0 \alpha_{8,1,0}^0 \right) + \frac{16a^2 \pi \lambda^2 \omega^2 \rho_0}{15c_0^2} \left( \beta_{7,1}^0 \alpha_{8,1,2}^0 \right) + \frac{8a^2 \pi \lambda^2 \omega^2 \rho_0}{3c_0^2} \left( \alpha_{7,0}^0 \beta_{8,1,1}^0 \right) \]

\[+ \frac{16a^2 \pi \lambda^2 \omega^2 \rho_0}{15c_0^2} \left( \alpha_{7,0}^0 \beta_{8,1,1}^0 \right) + 2\lambda \rho_0 \int_0^{2\pi} \int_0^\pi \left( \alpha_{1,0}^0 \beta_{6,2,1}^0 \right) \cos \theta \sin^2 \theta d\theta d\phi \]
\[+ \lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{1,2}^0 \beta_{62,1}^0 \rangle \left( -1 + 3 \cos^2 \theta \right) \cos \theta \sin^2 \theta d\theta d\phi \]

\[+ 2 \lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \beta_{1,1}^0 \alpha_{62,0}^0 \rangle \cos \theta \sin^2 \theta d\theta d\phi \]

\[+ \lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \beta_{1,2}^0 \alpha_{62,2}^0 \rangle \left( -1 + 3 \cos^2 \theta \right) \cos \theta \sin^2 \theta d\theta d\phi \]

\[- 8 \sqrt{\pi} \rho_0 \langle \alpha_{1,0}^0 \beta_{1,1}^0 \rangle \frac{1}{15} \sqrt{\pi} \rho_0 \langle \alpha_{5,2}^0 \beta_{7,1}^0 \rangle - \frac{8}{5} \sqrt{\pi} \lambda \rho_0 \langle \alpha_{5,2}^0 \beta_{7,1}^0 \rangle - \frac{8}{3} \sqrt{\pi} \lambda \rho_0 \langle \alpha_{1,0}^0 \beta_{5,1}^0 \rangle + \frac{8}{15} \sqrt{\pi} \lambda \rho_0 \langle \alpha_{1,2}^0 \beta_{5,1}^0 \rangle \]

Above can be simplified to yield:

\[F_x \approx 0 \quad F_y \approx 0 \]

\[F_z \approx F_{z0sc} \sin(2kh) + \epsilon \{ F_{z1cc} \cos^2(kh) + F_{z1ss} \sin^2(kh) + F_{z1sc} \sin(2kh) \} \]

where,

\[F_{z0sc} = -8 \pi \rho_0 \langle \alpha_{1,0}^0 \beta_{1,1}^0 \rangle - \frac{16}{5} \pi \rho_0 \langle \alpha_{1,2}^0 \beta_{4,1}^0 \rangle - \frac{16}{5} \pi \lambda \rho_0 \langle \alpha_{5,2}^0 \beta_{5,1}^0 \rangle \]

\[+ \frac{8 \pi^2 \lambda^2 \omega^2 \rho_0}{3c_0^2} \langle \alpha_{7,0}^0 \beta_{7,1}^0 \rangle + \frac{16 \pi^2 \lambda^2 \omega^2 \rho_0}{15c_0^2} \langle \alpha_{7,2}^0 \beta_{7,1}^0 \rangle - \frac{16}{5} \pi \lambda \rho_0 \langle \alpha_{5,2}^0 \beta_{1,1}^0 \rangle \]

\[+ \frac{16}{3} \pi \lambda \rho_0 \langle \alpha_{1,0}^0 \beta_{5,1}^0 \rangle + \frac{16}{15} \pi \lambda \rho_0 \langle \alpha_{1,2}^0 \beta_{5,1}^0 \rangle \]

\[F_{z1cc} = 4 \lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{1,0}^0 \alpha_{62,0}^0 \rangle \sin^2 \theta d\theta d\phi \]

\[+ 2 \lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{1,0}^0 \alpha_{62,2}^0 \rangle \left( -1 + 3 \cos^2 \theta \right) \sin^2 \theta d\theta d\phi \]

\[+ 2 \lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{1,2}^0 \alpha_{62,0}^0 \rangle \left( -1 + 3 \cos^2 \theta \right) \sin^2 \theta d\theta d\phi \]

\[+ \lambda \rho_0 \int_0^{2\pi} \int_0^\pi \langle \alpha_{1,2}^0 \alpha_{62,2}^0 \rangle \left( 1 - 3 \cos^2 \theta \right) \sin^2 \theta d\theta d\phi \]

\[+ 4 \pi^2 \lambda \rho_0 \langle \alpha_{1,0}^0 \alpha_{61,0}^0 \rangle - \frac{1}{2} \pi^2 \lambda \rho_0 \langle \alpha_{1,2}^0 \alpha_{61,0}^0 \rangle \]

\[- \frac{1}{2} \pi^2 \lambda \rho_0 \langle \alpha_{1,0}^0 \alpha_{61,2}^0 \rangle + \frac{5}{8} \pi^2 \lambda \rho_0 \langle \alpha_{1,2}^0 \alpha_{61,2}^0 \rangle \]
\[ F_{1ss} = 4\lambda \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \langle \beta_{1,1}^0, \beta_{62,1}^0 \rangle \cos^2 \theta \sin^2 \theta d\theta d\phi + \pi^2 \lambda \rho_0 \left( \langle \beta_{1,1}^0, \beta_{62,1}^0 \rangle \right) \right) \]

\[ F_{1sc} = -6\lambda \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \langle \alpha_{1,0}^0, \beta_{22,1}^0 \rangle \cos^2 \theta \sin \theta d\theta d\phi \right) \]

\[-3\lambda \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \langle \alpha_{1,2}^0, \beta_{22,1}^0 \rangle \right) \left( -1 + 3 \cos^2 \theta \right) \cos^2 \theta \sin \theta d\theta d\phi \]

\[-6\lambda \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \langle \alpha_{22,0}^0, \beta_{1,1}^0 \rangle \cos^2 \theta \sin \theta d\theta d\phi \right) \]

\[-3\lambda \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \langle \beta_{1,1}^0, \alpha_{22,2}^0 \rangle \right) \left( -1 + 3 \cos^2 \theta \right) \cos^2 \theta \sin \theta d\theta d\phi \]

\[-8\pi \rho_0 \left( \langle \alpha_{21,0}^0, \beta_{1,1}^0 \rangle \right) - \frac{16}{5} \pi \rho_0 \left( \langle \alpha_{21,2}^0, \beta_{1,1}^0 \rangle \right) \]

\[-8\pi \rho_0 \left( \langle \alpha_{1,0}^0, \beta_{21,1}^0 \rangle \right) - \frac{16}{5} \pi \rho_0 \left( \langle \alpha_{1,2}^0, \beta_{21,1}^0 \rangle \right) \]

\[+ \frac{2}{15} \lambda \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \langle \beta_{5,1}^0, \alpha_{62,1}^0 \rangle \right) \cos^2 \theta \sin^3 \theta d\theta d\phi \]

\[+ \frac{6}{5} \lambda \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \langle \alpha_{5,2}^0, \beta_{62,1}^0 \rangle \right) \cos^2 \theta \sin^3 \theta d\theta d\phi \]

\[-6\lambda \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \langle \beta_{62,0}^0, \beta_{1,1}^0 \rangle \right) \cos^2 \theta \sin^3 \theta d\theta d\phi \]

\[-6\lambda \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \langle \beta_{62,0}^0, \beta_{1,1}^0 \rangle \right) \cos^2 \theta \sin^3 \theta d\theta d\phi \]

\[-16 \pi \lambda \rho_0 \left( \langle \alpha_{61,2}^0, \beta_{5,1}^0 \rangle \right) - \frac{16}{5} \pi \lambda \rho_0 \left( \langle \alpha_{5,2}^0, \beta_{61,1}^0 \rangle \right) \]

\[+ \frac{3a^2}{2} \lambda^2 \omega^2 \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \langle \beta_{7,1}^0, \alpha_{82,1}^0 \rangle \right) \cos^2 \theta \sin \theta d\theta d\phi \]

\[+ \frac{8a^2}{5} \lambda^2 \omega^2 \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \langle \beta_{7,1}^0, \alpha_{82,1}^0 \rangle \right) \left( -1 + 3 \cos^2 \theta \right) \cos^2 \theta \sin \theta d\theta d\phi \]

\[-3a^2 \lambda^2 \omega^2 \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \langle \beta_{7,1}^0, \alpha_{82,1}^0 \rangle \right) \cos^2 \theta \sin \theta d\theta d\phi \]

\[-3a^2 \lambda^2 \omega^2 \rho_0 \int_0^{2\pi} \int_0^{\pi} \left( \langle \beta_{7,1}^0, \alpha_{82,1}^0 \rangle \right) \left( -1 + 3 \cos^2 \theta \right) \cos^2 \theta \sin \theta d\theta d\phi \]

\[+ \frac{8a^2}{3} \lambda^2 \omega^2 \rho_0 \left( \langle \alpha_{81,0}^0, \beta_{7,1}^0 \rangle \right) + \frac{16a^2}{15} \lambda^2 \omega^2 \rho_0 \left( \alpha_{81,0}^0, \beta_{7,1}^0 \rangle \right) \]

\[+ \frac{8a^2}{3} \lambda^2 \omega^2 \rho_0 \left( \langle \alpha_{81,0}^0, \beta_{7,1}^0 \rangle \right) + \frac{16a^2}{15} \lambda^2 \omega^2 \rho_0 \left( \alpha_{81,0}^0, \beta_{7,1}^0 \rangle \right) \]

\[-8\sqrt{\pi} \rho_0 \left( \alpha_{1,0}^0, \beta_{1,1}^0 \rangle \right) - \frac{16}{5} \sqrt{\pi} \rho_0 \left( \alpha_{1,2}^0, \beta_{1,1}^0 \rangle \right) \]

\[+ \frac{8a^2}{3} \sqrt{\pi} \lambda^2 \omega^2 \rho_0 \left( \langle \alpha_{7,0}^0, \beta_{7,1}^0 \rangle \right) + \frac{16a^2}{15} \sqrt{\pi} \lambda^2 \omega^2 \rho_0 \left( \langle \alpha_{7,0}^0, \beta_{7,1}^0 \rangle \right) \]

\[-\frac{8}{5} \sqrt{\pi} \lambda \rho_0 \left( \alpha_{5,2}^0, \beta_{1,1}^0 \rangle \right) - \frac{8}{3} \sqrt{\pi} \lambda \rho_0 \left( \alpha_{5,2}^0, \beta_{5,1}^0 \rangle \right) + \frac{8}{15} \sqrt{\pi} \lambda \rho_0 \left( \alpha_{1,2}^0, \beta_{5,1}^0 \rangle \right) \]
\[ F_x \approx 0 \quad F_y \approx 0 \]
\[ F_z \approx F_{20sc} \sin(2kh) + \epsilon \{ F_{z1cc} \cos^2(kh) + F_{z1ss} \sin^2(kh) + F_{z1sc} \sin(2kh) \} \]

where,
\[ F_{20sc} = -\frac{4\pi \rho_0}{(ak^*)^2} \left( \Re \bar{B}_{00}^{(0)} \Re \bar{B}_{10}^{(0)} - \Im \bar{B}_{00}^{(0)} \Im \bar{B}_{10}^{(0)} \right) (-\sin(ak^*) + \cos(ak^*)ak^*) \times (2 \cos(ak^*)ak^* + \sin(ak^*) (-2 + (ak^*)^2)) \]
\[ -\frac{8\pi \rho_0}{5} \left( \Re \bar{B}_{10}^{(0)} \Re \bar{B}_{20}^{(0)} - \Im \bar{B}_{10}^{(0)} \Im \bar{B}_{20}^{(0)} \right) (j_1 (ak^*) - j_2 (ak^*) ak^*) \times (2j_2 (ak^*) - j_3 (ak^*) ak^*) \]
\[ -\frac{8\pi \lambda^2 \rho_0}{5} \left( \Re \bar{B}_{10}^{(0)} \Re \bar{B}_{20}^{(0)} - \Im \bar{B}_{10}^{(0)} \Im \bar{B}_{20}^{(0)} \right) j_1 (ak^*) j_2 (ak^*) \]
\[ +\frac{4a^2\pi \lambda^2 \omega^2 \rho_0}{3c_0^2} \left( \Re \bar{B}_{00}^{(0)} \Re \bar{B}_{10}^{(0)} - \Im \bar{B}_{00}^{(0)} \Im \bar{B}_{10}^{(0)} \right) j_0 (ak^*) j_1 (ak^*) \]
\[ +\frac{8a^2\pi \lambda^2 \omega^2 \rho_0}{15c_0^2} \left( \Re \bar{B}_{10}^{(0)} \Re \bar{B}_{20}^{(0)} - \Im \bar{B}_{10}^{(0)} \Im \bar{B}_{20}^{(0)} \right) j_1 (ak^*) j_2 (ak^*) \]
\[ -\frac{8\pi \lambda \rho_0}{5} \left( \Re \bar{B}_{10}^{(0)} \Re \bar{B}_{20}^{(0)} - \Im \bar{B}_{10}^{(0)} \Im \bar{B}_{20}^{(0)} \right) j_2 (ak^*) (j_1 (ak^*) - j_2 (ak^*) ak^*) \]
\[ -\frac{8\pi \lambda \rho_0}{3(ak^*)^3} \left( \Re \bar{B}_{00}^{(0)} \Re \bar{B}_{10}^{(0)} - \Im \bar{B}_{00}^{(0)} \Im \bar{B}_{10}^{(0)} \right) (\sin(ak^*) - \cos(ak^*)ak^*)^2 \]
\[ +\frac{8\pi \lambda \rho_0}{15} \left( \Re \bar{B}_{10}^{(0)} \Re \bar{B}_{20}^{(0)} - \Im \bar{B}_{10}^{(0)} \Im \bar{B}_{20}^{(0)} \right) j_1 (ak^*) (2j_2 (ak^*) - j_3 (ak^*) ak^*) \]
\[ F_{z1cc} = \frac{\lambda \rho_0 \pi^{3/2}}{(ak^*)^2} \left( \Re \bar{B}_{00}^{(0)} \Re \bar{B}_{10}^{(0)} + \Im \bar{B}_{00}^{(0)} \Im \bar{B}_{10}^{(0)} \right) (\sin(ak^*) - \cos(ak^*)ak^*)^2 \]
\[ -\frac{\lambda \rho_0 \pi^{3/2} ak^*}{4} \left( \Re \bar{B}_{00}^{(0)} \Re \bar{B}_{20}^{(0)} + \Im \bar{B}_{00}^{(0)} \Im \bar{B}_{20}^{(0)} \right) j_1 (ak^*) (2j_2 (ak^*) + j_3 (ak^*) ak^*) \]
\[ +\frac{5\lambda \rho_0 \pi^{3/2}}{32} \left( \Re \bar{B}_{20}^{(0)} \Re \bar{B}_{20}^{(0)} + \Im \bar{B}_{20}^{(0)} \Im \bar{B}_{20}^{(0)} \right) (2j_2 (ak^*) + j_3 (ak^*) ak^*)^2 \]
\[ +\frac{\pi^2 \lambda \rho_0}{(ak^*)^2} \left( \Re \bar{B}_{00}^{(0)} \Re \bar{B}_{00}^{(1)} + \Im \bar{B}_{00}^{(0)} \Im \bar{B}_{00}^{(1)} \right) (-1 + \cos(2ak^*) + \sin(2ak^*)ak^*) \]
\[ -\frac{\lambda \rho_0 \pi^{2}}{4} \left( \Re \bar{B}_{20}^{(0)} \Re \bar{B}_{00}^{(1)} + \Im \bar{B}_{20}^{(0)} \Im \bar{B}_{00}^{(1)} \right) j_0 (ak^*) (2j_2 (ak^*) - j_3 (ak^*) ak^*) \]
\[ +\frac{\lambda \rho_0 \pi^{2} ak^*}{4} \left( \Re \bar{B}_{00}^{(0)} \Re \bar{B}_{20}^{(1)} + \Im \bar{B}_{00}^{(0)} \Im \bar{B}_{20}^{(1)} \right) j_1 (ak^*) j_2 (ak^*) \]
\[ +\frac{5\lambda \rho_0 \pi^{2}}{16} \left( \Re \bar{B}_{20}^{(0)} \Re \bar{B}_{20}^{(1)} + \Im \bar{B}_{20}^{(0)} \Im \bar{B}_{20}^{(1)} \right) j_2 (ak^*) (2j_2 (ak^*) - j_3 (ak^*) ak^*) \]
\[ F_{\text{class}} = \frac{\lambda \rho_0 \pi^{3/2}}{4} \left( \Re \tilde{B}^{(0)}_{10} \Re \tilde{B}^{(0)}_{10} + \Im \tilde{B}^{(0)}_{10} \Im \tilde{B}^{(0)}_{10} \right) (j_1 (ak^*) - j_2 (ak^*) (ak^*)^2 \\
+ \frac{\lambda \rho_0 \pi^2}{2} \left( \Re \tilde{B}^{(0)}_{10} \Im \tilde{B}^{(0)}_{10} + \Im \tilde{B}^{(0)}_{10} \Re \tilde{B}^{(0)}_{10} \right) j_1 (ak^*) (j_1 (ak^*) - j_2 (ak^*) (ak^*) \\
F_{\text{sec}} = -\frac{2 \rho_0 \pi^{1/2}}{(ak^*)^3} \left( \Re \tilde{B}^{(0)}_{20} \Im \tilde{B}^{(0)}_{10} - \Im \tilde{B}^{(0)}_{20} \Re \tilde{B}^{(0)}_{10} \right) (-\sin(ak^*) + \cos(ak^*) (ak^*) \\
\times (\cos(ak^*) (ak^*) (-6 + (ak^*)^2) - 3 \sin(ak^*) (-2 + (ak^*)^2)) \\
+ \frac{4 \pi^{1/2} \rho_0}{5 (ak^*)^5} \left( \Re \tilde{B}^{(0)}_{20} \Im \tilde{B}^{(0)}_{10} - \Im \tilde{B}^{(0)}_{20} \Re \tilde{B}^{(0)}_{10} \right) \\
\times (\sin(ak^*) (9 - 4 (ak^*)^2) + \cos(ak^*) (ak^*) (-9 + (ak^*)^2)) \\
\times (\cos(ak^*) (ak^*) (-6 + (ak^*)^2) - 3 \sin(ak^*) (-2 + (ak^*)^2)) \\
+ \frac{2 \rho_0 \pi^{1/2}}{(ak^*)^5} \left( \Re \tilde{B}^{(0)}_{10} \Im \tilde{B}^{(0)}_{00} - \Im \tilde{B}^{(0)}_{10} \Re \tilde{B}^{(0)}_{00} \right) \\
\times (2 \cos(ak^*) (ak^*) + \sin(ak^*) (-2 + (ak^*)^2))^2 \\
- \frac{\rho_0 \pi^{1/2}}{(ak^*)^7} \left( \Re \tilde{B}^{(0)}_{10} \Im \tilde{B}^{(0)}_{20} - \Im \tilde{B}^{(0)}_{10} \Re \tilde{B}^{(0)}_{20} \right) \\
\times (2 \cos(ak^*) (ak^*) + \sin(ak^*) (-2 + (ak^*)^2)) \\
\times (\cos(ak^*) (ak^*) (-36 + 5 (ak^*)^2) + \sin(ak^*) (36 - 17 (ak^*)^2 + (ak^*)^4)) \\
- \frac{4 \pi \rho_0}{(ak^*)^3} \left( \Re \tilde{B}^{(0)}_{10} \Im \tilde{B}^{(0)}_{00} - \Im \tilde{B}^{(0)}_{10} \Re \tilde{B}^{(0)}_{00} \right) (\sin(ak^*) - \cos(ak^*) (ak^*) \\
\times (2 \cos(ak^*) (ak^*) + \sin(ak^*) (-2 + (ak^*)^2)) \\
+ \frac{8 \pi \rho_0}{5} \left( \Re \tilde{B}^{(0)}_{10} \Im \tilde{B}^{(0)}_{20} - \Im \tilde{B}^{(0)}_{10} \Re \tilde{B}^{(0)}_{20} \right) (-j_1 (ak^*) + j_2 (ak^*) (ak^*) \\
\times (-2j_2 (ak^*) + j_3 (ak^*) (ak^*) \\
- \frac{4 \pi \rho_0}{(ak^*)^3} \left( \Re \tilde{B}^{(0)}_{00} \Im \tilde{B}^{(0)}_{10} - \Im \tilde{B}^{(0)}_{00} \Re \tilde{B}^{(0)}_{10} \right) (-\sin(ak^*) + \cos(ak^*) (ak^*) \\
\times (2 \cos(ak^*) (ak^*) + \sin(ak^*) (-2 + (ak^*)^2)) \\
\times (\cos(ak^*) (ak^*) (-36 + 5 (ak^*)^2) + \sin(ak^*) (36 - 17 (ak^*)^2 + (ak^*)^4)) \\
- \frac{8 \pi \rho_0}{5} \left( \Re \tilde{B}^{(0)}_{20} \Im \tilde{B}^{(0)}_{10} - \Im \tilde{B}^{(0)}_{20} \Re \tilde{B}^{(0)}_{10} \right) (-j_1 (ak^*) + j_2 (ak^*) (ak^*) \\
\times (-2j_2 (ak^*) + j_3 (ak^*) (ak^*) \\
- \frac{4 \lambda^2 \rho_0 \pi^{1/2}}{5} \left( \Re \tilde{B}^{(0)}_{20} \Im \tilde{B}^{(0)}_{10} - \Im \tilde{B}^{(0)}_{20} \Re \tilde{B}^{(0)}_{10} \right) j_2 (ak^*) (j_1 (ak^*) - j_2 (ak^*) (ak^*) \\
- \frac{4 \lambda^2 \rho_0 \pi^{1/2}}{5} \left( \Re \tilde{B}^{(0)}_{20} \Im \tilde{B}^{(0)}_{10} - \Im \tilde{B}^{(0)}_{20} \Re \tilde{B}^{(0)}_{10} \right) j_2 (ak^*) (j_1 (ak^*) - j_2 (ak^*) (ak^*)
\[-\frac{4\lambda^2 \rho_0 \lambda_1 \pi^{1/2}}{5} \left( \Re B^{(0)}_{10} \Im B^{(0)}_{20} - \Im B^{(0)}_{10} \Re B^{(0)}_{20} \right) \]
\[\times j_1 (ak^*) (-2j_2 (ak^*) + j_3 (ak^*) ak^*) \]
\[+ \frac{8\pi \lambda^2 \rho_0}{5} j_1 (ak^*) j_2 (ak^*) \left( \Re B^{(0)}_{10} \Im B^{(1)}_{20} - \Im B^{(0)}_{10} \Re B^{(1)}_{20} \right) \]
\[\times j_1 (ak^*) j_2 (ak^*) \left( \Re B^{(1)}_{20} \Im B^{(0)}_{10} - \Im B^{(1)}_{20} \Re B^{(0)}_{10} \right) \]
\[+ \frac{2\pi \lambda^2 \omega^2 \rho_0 \pi^{1/2}}{3c_0^2} \left( \Re B^{(0)}_{10} \Im B^{(0)}_{10} - \Im B^{(0)}_{10} \Re B^{(0)}_{10} \right) \]
\[\times j_0 (ak^*) \left( j_1 (ak^*) - j_2 (ak^*) ak^* \right) \]
\[+ \frac{4\pi \lambda^2 \omega^2 \rho_0}{15c_0^2} \left( \Re B^{(0)}_{20} \Im B^{(0)}_{10} - \Im B^{(0)}_{20} \Re B^{(0)}_{10} \right) \]
\[\times j_2 (ak^*) \left( j_1 (ak^*) - j_2 (ak^*) ak^* \right) \]
\[+ \frac{2\pi \lambda^2 \omega^2 \rho_0}{3c_0^2 (ak^*)^3} \left( \Re B^{(0)}_{10} \Im B^{(0)}_{00} - \Im B^{(0)}_{10} \Re B^{(0)}_{00} \right) \]
\[\times (\sin (ak^*) - \cos (ak^*) ak^*)^2 \]
\[+ \frac{4\pi \lambda^2 \omega^2 \rho_0}{15c_0^2} \left( \Re B^{(0)}_{10} \Im B^{(0)}_{20} - \Im B^{(0)}_{10} \Re B^{(0)}_{20} \right) \]
\[\times j_1 (ak^*) (-2j_2 (ak^*) + j_3 (ak^*) ak^*) \]
\[- \frac{4\pi \lambda^2 \omega^2 \rho_0}{3c_0^2} \left( \Re B^{(0)}_{10} \Im B^{(1)}_{00} - \Im B^{(0)}_{10} \Re B^{(1)}_{00} \right) j_0 (ak^*) j_1 (ak^*) \]
\[- \frac{8\pi \lambda^2 \omega^2 \rho_0}{15c_0^2} \left( \Re B^{(1)}_{10} \Im B^{(0)}_{00} - \Im B^{(1)}_{10} \Re B^{(0)}_{00} \right) j_1 (ak^*) j_2 (ak^*) \]
\[+ \frac{4\pi \lambda^2 \omega^2 \rho_0}{3c_0^2} \left( \Re B^{(0)}_{00} \Im B^{(0)}_{10} - \Im B^{(0)}_{00} \Re B^{(0)}_{10} \right) j_0 (ak^*) j_1 (ak^*) \]
\[+ \frac{8\pi \lambda^2 \omega^2 \rho_0}{15c_0^2} \left( \Re B^{(0)}_{20} \Im B^{(1)}_{10} - \Im B^{(0)}_{20} \Re B^{(1)}_{10} \right) j_1 (ak^*) j_2 (ak^*) \]
\[- \frac{4\pi \lambda^2 \omega^2 \rho_0}{(ak^*)^3} \left( \Re B^{(0)}_{00} \Im B^{(0)}_{10} - \Im B^{(0)}_{00} \Re B^{(0)}_{10} \right) (-\sin (ak^*) + \cos (ak^*) ak^*) \]
\[\times (2 \cos (ak^*) ak^* + \sin (ak^*) (-2 + (ak^*)^2)) \]
\[- \frac{8\pi \lambda^2 \omega^2 \rho_0}{5} \left( \Re B^{(0)}_{10} \Re B^{(0)}_{20} - \Im B^{(0)}_{10} \Im B^{(0)}_{20} \right) \left( j_1 (ak^*) - j_2 (ak^*) ak^* \right) \]
\[\times (2j_2 (ak^*) - j_3 (ak^*) ak^*) \]
\[ + \frac{4a^2 \pi^{1/2} \lambda^2 \omega^2 \rho_0}{3c_0^2} \left( \Re \tilde{B}_{10}^{(0)} \Re \tilde{B}_{10}^{(0)} - \Im \tilde{B}_{10}^{(0)} \Im \tilde{B}_{10}^{(0)} \right) j_0 (ak^*) j_1 (ak^*) \\
+ \frac{8a^2 \pi^{3/2} \lambda^2 \omega^2 \rho_0}{15c_0^3} \left( \Re \tilde{B}_{10}^{(0)} \Re \tilde{B}_{20}^{(0)} - \Im \tilde{B}_{10}^{(0)} \Im \tilde{B}_{20}^{(0)} \right) j_1 (ak^*) j_2 (ak^*) \\
- \frac{4\pi^{1/2} \lambda \rho_0}{5} \left( \Re \tilde{B}_{10}^{(0)} \Re \tilde{B}_{20}^{(0)} - \Im \tilde{B}_{10}^{(0)} \Im \tilde{B}_{20}^{(0)} \right) j_2 (ak^*) (j_1 (ak^*) - j_2 (ak^*) ak^*) \\
- \frac{4\pi^{3/2} \lambda \rho_0}{3(ak^*)^3} \left( \Re \tilde{B}_{00}^{(0)} \Re \tilde{B}_{10}^{(0)} - \Im \tilde{B}_{00}^{(0)} \Im \tilde{B}_{10}^{(0)} \right) (\sin(ak^*) - \cos(ak^*) ak^*)^2 \\
+ \frac{4\pi^{1/2} \lambda \rho_0}{15\omega} \left( \Re \tilde{B}_{10}^{(0)} \Re \tilde{B}_{20}^{(0)} - \Im \tilde{B}_{10}^{(0)} \Im \tilde{B}_{20}^{(0)} \right) j_1 (ak^*) (2j_2 (ak^*) - j_3 (ak^*) ak^*) \]
REFERENCES


